WELL-POSEDNESS OF A TWO-DIMENSIONAL COORDINATE-FREE MODEL FOR THE MOTION OF FLAME FRONTS

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Abstract. We study a two-dimensional coordinate-free model for the motion of flame fronts. The model specifies the normal velocity of the interface in terms of geometric information, such as the mean curvature and the Gaussian curvature of the front. As the tangential velocities do not determine the position of the interface, we choose them to maintain a favorable parameterization. We choose this to be an isothermal parameterization. After appropriately reformulating the equations of motion, we use the energy method to prove short-time well-posedness in Sobolev spaces.

1. Introduction

The Kuramoto-Sivashinsky equation is a weakly nonlinear model equation for the motion of flame fronts [29], [36]. It has been widely studied in one spatial dimension [18], [23], [24], [34], [37]. In two spatial dimensions, much work is either in the case of thin domains so that one-dimensional dynamics dominate [13], [28], [33], [35], or is computational [27]. The second author and Mazzucato have established some existence results for the two-dimensional Kuramoto-Sivashinsky equation without an assumption of thinness of the domain [11], [12].

While the Kuramoto-Sivashinsky equation has attracted much interest, as we have said, it is a weakly nonlinear model and with this comes some limitations. The biggest such limitation is that the flame front must be a graph with resepct to the horizontal coordinates. In both one and two spatial dimensions, Frankel and Sivashinsky introduced more general, coordinate-free models of the motion of flame fronts [21], [22]. In these coordinate-free models, the flame front moves according to its intrinsic geometric information, such as curvature. In addition to developing the coordinate-free models, Frankel and Sivashinsky demonstrate how the one-dimensional and two-dimensional Kuramoto-Sivashinsky equations may be derived from them.

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In the decades since these coordinate-free models were introduced, there has been some limited mathematical theory developed for them. Temperature effects were incorporated into the one-dimensional model in [20]. A number of approximations to this model were then made, including quasi-steady approximations and weakly nonlinear approximations, in a series of papers [14], [15], [16], [17]. None of these papers developed rigorous analytical theory for the full coordinate-free model of [20]. More recently, the first rigorous theory for the one-dimensional coordinate-free model of [21] was developed in [7] by the second author, Hadadifard, and Wright; there, it is demonstrated that the one-dimensional coordinate-free model is well-posed for small data, and that solutions of the coordinate-free model and solutions of the one-dimensional Kuramoto-Sivashinsky remain close if their initial conditions are close (thus this is a validation theorem for Kuramoto-Sivashinsky as a weakly nonlinear model). In the present work, we give the first analytical theory for the two-dimensional coordinate-free model of [22], proving a short-time well-posedness result for data of arbitrary size in Sobolev spaces.

We believe that there are two reasons for the dearth of rigorous theory for the full coordinate-free models of [20], [21], [22]. First, the models are not stated in evolutionary form, and instead are given as formulas for the normal velocity of the flame front. Second, even when one makes the effort to then restate the model in evolutionary form, the equations of motion for the front involve high derivatives of curvature; this means that if one were to attempt to evolve the Cartesian coordinates of the flame front, the leading-order terms in the evolution equations would be highly nonlinear. We deal with both of these difficulties by adapting ideas originating in the numerical work of Hou, Lowengrub, and Shelley for the motion of one-dimensional vortex sheets with surface tension [25], [26]. In this work, Hou, Lowengrub, and Shelley observed that only the normal velocity of the fluid interface was needed to provide for the motion of the interface, as the tangential velocity could be artificially chosen so as to enforce a preferred parameterization. They also chose to evolve geometric dependent variables such as tangent angle and arclength of the interface, as curvature is essentially linear in terms of these variables, and curvature enters the problem through the Laplace-Young jump condition for the pressure. Thus, in the one-dimensional case, the Hou, Lowengrub, Shelley work demonstrates how one might work with a normal velocity related to the curvature of an interface. The second author and Akers have adapted the numerical method of [25], [26] to the one-dimensional coordinate-free model of Frankel and Sivashinsky [21] in [2]. The second author and Masmoudi used the ideas of [25], [26] to prove well-posedness of the vortex sheet with surface tension and related problems [3], [4], [6], [8]. The second author and Masmoudi then generalized these ideas for analysis of two-dimensional fluid interface problems [5], [9], [10]. We may view the present work as the adaptation of the analysis of the second author and Masmoudi from these papers to prove well-posedness of the two-dimensional coordinate-free model of [22].

The method by which we prove well-posedness of the initial value problem for the two-dimensional coordinate-free model is to first specify tangential velocities for the flame front; recall that the normal velocity is the content of the model of [22]. We choose tangential velocities so as to maintain a favorable parameterization, and as in [5], [9], [10], [30], we choose an isothermal parameterization. Having fully specified the velocity of the flame front, we are able to write the evolution equations for the front, and to write evolution equations for related quantities. In particular, we need the evolution of the mean curvature of the front. This is because (again, as in the papers [5], [9], [10], [30] for two-dimensional fluid interface problems) we are able to make energy estimates for the mean curvature, and we can use these estimates to establish the regularity of the front itself. The energy estimates we make are not for the mean curvature of the actual front, but instead are performed in the context of an iterative scheme. We set up an iterative approximation of the equations of motion for the flame front, prove existence of solutions for the iterated equations, demonstrate bounds on the solutions (by means of the energy estimates for mean curvature) which are uniform with respect to the iteration parameter, and then pass to the limit as the iteration parameter goes to infinity, finding solutions of the original problem.

The plan of the paper is as follows. In Section 2 below we specify the model of [22] and we choose the tangential velocities for the flame front. We also state our main theorem at the end of Section 2. We explore the consequences of the equations of motion of the surface for the evolution of geometric quantities in Section 3. In Section 4, we then give some useful estimates related to commutators and to geometric quantities. In Section 5, we prove our main theorem by introducing our iterative scheme, carrying out the energy estimates for the iterates, and passing to the limit.

2. The equations of motion

We consider a two-dimensional flame front moving in three-dimensional space, with Cartesian coordinates

$$\mathbf{X}(\alpha,\beta,t) = (x(\alpha,\beta,t), y(\alpha,\beta,t), z(\alpha,\beta,t)).$$

Here, naturally, the two parameters along the surface are α and β, while t is time. We define a frame of normal and tangential vectors,

$$\hat{\mathbf{t}}^1 = \frac{\mathbf{X}_\alpha}{|\mathbf{X}_\alpha|}, \qquad \hat{\mathbf{t}}^2 = \frac{\mathbf{X}_\beta}{|\mathbf{X}_\beta|}, \qquad \hat{\mathbf{n}} = \frac{\mathbf{X}_\alpha \times \mathbf{X}_\beta}{|\mathbf{X}_\alpha \times \mathbf{X}_\beta|}. \tag{2.1}$$

The surface $\mathbf{X}$ moves according to normal velocity U and tangential velocities V_1 and V_2,

$$\mathbf{X}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2. \tag{2.2}$$

We take an initial condition for the surface $\mathbf{X}$, namely

$$\mathbf{X}(\alpha,\beta,0)=\mathbf{X}_0(\alpha,\beta). \tag{2.3}$$

The geometry we consider is that this initial surface (and then the surface at positive times) is asymptotic to the flat plane at horizontal infinity. In terms of $\mathbf{X}_0$, we have $\mathbf{X}_0(\alpha,\beta)-(\alpha,\beta,0)$ goes to zero as $(\alpha,\beta)\to\infty$.

We define the coefficients of the first fundamental form for the surface $\mathbf{X}$ as

$$E=\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha,\qquad F=\mathbf{X}_\alpha\cdot\mathbf{X}_\beta,\qquad G=\mathbf{X}_\beta\cdot\mathbf{X}_\beta.$$

The coefficients of the second fundamental form for the surface $\mathbf{X}$ are

$$L=-\mathbf{X}_\alpha\cdot\hat{\mathbf{n}}_\alpha,\qquad M=-\mathbf{X}_\alpha\cdot\hat{\mathbf{n}}_\beta=-\mathbf{X}_\beta\cdot\hat{\mathbf{n}}_\alpha,\qquad N=-\mathbf{X}_\beta\cdot\hat{\mathbf{n}}_\beta. \tag{2.4}$$

In terms of the first and second fundamental forms, the mean curvature is then

$$\kappa=\frac{EN+GL-2FM}{2(EG-F^2)}.$$

The Gaussian curvature is given by

$$q=\frac{LN-M^2}{EG-F^2}.$$

We will be choosing an isothermal parameterization for the flame front, meaning

$$E=G,\qquad F=0, \tag{2.5}$$

for all (α,β) and for all t. To enforce this parameterization, we will assume the initial surface $\mathbf{X}_0$ is parameterized accordingly, and then the tangential velocities V_1 and V_2 will be chosen so as to maintain the parameterization at positive times. The authors and Masmoudi have used this parameterization to good effect in a number of problems in interfacial fluid dynamics [5], [9], [10], [30]. The implications of this choice for V_1 and V_2 are detailed in Section 2.2 below. Using an isothermal parameterization, the mean curvature and Gaussian curvature simplify to

$$\kappa=\frac{L+N}{2E},\qquad q=\frac{LN}{E^2}. \tag{2.6}$$

Also, it will be useful to note that with an isothermal parameterization, the surface $\mathbf{X}$ satisfies

$$\Delta X=2\kappa\mathbf{X}_\alpha\times\mathbf{X}_\beta. \tag{2.7}$$

We mention that with the geometry under consideration, namely that the flame front is asymptotically flat at infinity, a global isothermal parameterization may be found [19]. Thus we are not making a restrictive assumption on the class of initial data.

2.1. **The normal velocity.** The normal velocity for the flame front is developed in [22] as

$$(2.8)\quad U = -1 + (1-\sigma)\kappa - \left(1 + \frac{\sigma^2}{2}\right)\kappa^2 + \left(\frac{\sigma^3}{3} - 5\sigma^2 - 2\sigma\right)\kappa^3 + 2(\sigma^2+1)q + (20\sigma^2 + 8\sigma - 4)\kappa q - \sigma^2(\sigma+3)\Delta_S\kappa.$$

The parameter σ satisfies $\sigma > 1$; this allows the term $(1-\sigma)\kappa$ to destabilize the front at low frequencies, leading to nontrivial dynamics (as in the Kuramoto-Sivashinsky equation [27]). The operator Δ_S indicates the Laplace-Beltrami operator of the front. We use the formula for the Laplace-Beltrami operator found in the appendix of [38],

$$\Delta_S u = \frac{1}{\sqrt{EG-F^2}}\left(\frac{Eu_\beta - Fu_\alpha}{\sqrt{EG-F^2}}\right)_\beta + \frac{1}{\sqrt{EG-F^2}}\left(\frac{Gu_\alpha - Fu_\beta}{\sqrt{EG-F^2}}\right)_\alpha.$$

If we have an isothermal parameterization with $E = G$ and $F = 0$, then the Laplace-Beltrami operator simplifies to

$$\Delta_S u = \frac{u_{\alpha\alpha} + u_{\beta\beta}}{E}.$$

We define $W(\kappa, q)$ and τ as

$$(2.9)\quad W(\kappa,q) = (1-\sigma)\kappa - \left(1 + \frac{\sigma^2}{2}\right)\kappa^2 + \left(\frac{\sigma^3}{3} - 5\sigma^2 - 2\sigma\right)\kappa^3 + 2(\sigma^2+1)q + (20\sigma^2 + 8\sigma - 4)\kappa q,$$

$$\tau = \sigma^2(\sigma+3) > 0.$$

With these definitions, we may rewrite the normal velocity as

$$(2.10)\qquad U = -\tau\Delta\kappa/E + W(\kappa,q) - 1.$$

2.2. **The tangential velocities and choice of parameterization.** As we have said, while the normal velocity comes from the physical problem, the tangential velocities may be freely chosen so as to enforce a favored parameterization. That is, moving the surface tangent to itself does not change the location of the surface.

The tangential velocities may be determined by using (2.2) together with the the time derivative of (2.5), $E_t = G_t$ and $F_t = 0$. This is the same choice made for the motion of a vortex sheet in three-dimensional fluids by the second author and Masmoudi, and the calculation of the tangential velocities may be found in [9]. The result is that the tangential velocities V_1, V_2 satisfy

$$(2.11)\qquad \left(\frac{V_1}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2}{\sqrt{E}}\right)_\beta = \frac{U(L-N)}{E},$$

$$(2.12)\qquad \left(\frac{V_1}{\sqrt{E}}\right)_\beta + \left(\frac{V_2}{\sqrt{E}}\right)_\alpha = \frac{2UM}{E}.$$

Then, if V_1 and V_2 satisfy (2.11) and (2.12), and if the initial surface $\mathbf{X}_0$ satisfies (2.5), then at positive times the surface will satisfy (2.5). We will prove well-posedness of the initial value problem (2.2), (2.3), with V_1 and V_2 enforcing the isothermal parameterization and U given by (2.8).

2.3. **The main result.** We take $s \in \mathbb{Z}$, with $s \geq 6$. In the calculations which follow in the next several sections, we will sometimes state that our reasoning is valid because s is "sufficiently large;" this simply refers to this fact that $s \geq 6$. Let c_0 be a positive constant. We define an open subset $\mathcal{O}_{c_0} \subseteq H^{s+2}$, such that for every $\mathbf{X} \in \mathcal{O}_{c_0}$, the following condition holds:

$$E(\alpha, \beta) > c_0.$$

Theorem 2.1. *We assume that the surface $\mathbf{X}_0 \in \mathcal{O}_{c_0}$ is globally parameterized by isothermal coordinates (namely* (2.5) *holds). Then, there exists a time $T > 0$ and a unique solution $\mathbf{X} \in C\left([0,T), \overline{\mathcal{O}_{\frac{c_0}{2}}}\right)$ of the Cauchy problem*

$$\begin{cases} \mathbf{X}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2, \\ U = -\tau\Delta\kappa/E + W(\kappa, q) - 1, \\ \mathbf{X}(t=0) = \mathbf{X}_0. \end{cases}$$

Remark 1. When we say $\mathbf{X} \in H^s$, this means that $\mathbf{X}(\alpha, \beta) - (\alpha, \beta, 0)$ is actually in H^s, since the surface $\mathbf{X}$ is asymptotic to the plane at infinity.

3. Geometric identities and evolution of geometric quantities

In this section, we first give some useful geometric identities. We then study the regularity of E and $\mathbf{X}$, and find evolution equations for E and κ.

3.1. **Geometric identities.** We will frequently need to differentiate the normal and tangential vectors to the front, so formulas for these derivatives (in the context of our isothermal parameterization) will be helpful. The derivatives of the normal and tangential vectors satisfy the following:

$$\begin{aligned} \hat{\mathbf{n}}_\alpha &= -\frac{L}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{M}{E^{1/2}}\hat{\mathbf{t}}^2, \\ \hat{\mathbf{n}}_\beta &= -\frac{M}{E^{1/2}}\hat{\mathbf{t}}^1 - \frac{N}{E^{1/2}}\hat{\mathbf{t}}^2, \\ \hat{\mathbf{t}}^1_\alpha &- -\frac{E_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}}, \\ \hat{\mathbf{t}}^1_\beta &= \frac{E_\alpha}{2E}\hat{\mathbf{t}}^2 + \frac{M}{E^{1/2}}\hat{\mathbf{n}}, \\ \hat{\mathbf{t}}^2_\alpha &= \frac{E_\beta}{2E}\hat{\mathbf{t}}^1 + \frac{M}{E^{1/2}}\hat{\mathbf{n}}, \\ \hat{\mathbf{t}}^2_\beta &= -\frac{E_\alpha}{2E}\hat{\mathbf{t}}^1 + \frac{N}{E^{1/2}}\hat{\mathbf{n}}. \end{aligned}$$

These formulas are all directly verifiable by using (2.1), (2.5), and the definitions of the first and second fundamental forms. For instance, to compute

$\hat{\mathbf{t}}^1_\alpha$, we first note that $\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{t}}^1 = 0$, so that

$$\hat{\mathbf{t}}^1_\alpha = (\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{t}}^2)\hat{\mathbf{t}}^2 + (\hat{\mathbf{t}}^1_\alpha \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}.$$

We then substitute $\hat{\mathbf{t}}^1 = \mathbf{X}_\alpha / E^{1/2}$, and arrive at

$$\hat{\mathbf{t}}^1_\alpha = \frac{(\mathbf{X}_{\alpha\alpha} \cdot \hat{\mathbf{t}}^2)}{E^{1/2}}\hat{\mathbf{t}}^2 + \frac{(\mathbf{X}_{\alpha\alpha} \cdot \hat{\mathbf{n}})}{E^{1/2}}\hat{\mathbf{n}} = \frac{(\mathbf{X}_{\alpha\alpha} \cdot \mathbf{X}_\beta)}{E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}}$$
$$= -\frac{(\mathbf{X}_\alpha \cdot \mathbf{X}_\alpha)_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}} = -\frac{E_\beta}{2E}\hat{\mathbf{t}}^2 + \frac{L}{E^{1/2}}\hat{\mathbf{n}}.$$

The remaining formulas are similar, and we omit the details.

3.2. **Gain of regularity for E and $\mathbf{X}$.** In the estimates we will be making, we will be using the mean curvature, κ, as our primary dependent variable. We will make estimates for κ in the Sobolev space H^s. We then need to infer regularity for $\mathbf{X}$ and E. We will be able to conclude that $\mathbf{X}$ is $s+2$-times differentiable (specifically, we will say $\mathbf{X} \in H^{s+2}$ which will mean that $\mathbf{X} - (\alpha, \beta, 0)$ is actually in the space H^{s+2}). It may appear at first glance, then, that $E \in H^{s+1}$ (again, allowing for the fact that actually $E - 1$ is the quantity which decays at infinity), but we may infer higher regularity and find in fact that $E \in H^{s+2}$ as well. This gain is a consequence of the isothermal parameterization.

The gain of one derivative for E may be seen by calculating ΔE. Recalling that $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$ and $\mathbf{X}_\alpha \cdot \mathbf{X}_\beta = 0$, we have

$$\Delta E = 2(\mathbf{X}_{\alpha\beta} \cdot \mathbf{X}_{\alpha\beta}) - 2(\mathbf{X}_{\alpha\alpha} \cdot \mathbf{X}_{\beta\beta}). \tag{3.1}$$

So, if $\mathbf{X}$ is in H^{s+2} , then the right-hand side of (3.1) is in $H^s \cap L^1$. We conclude that $E - 1$ is also in H^{s+2}. We have proved the following lemma.

Lemma 3.1. *If $(\mathbf{X} - (\alpha, \beta, 0)) \in H^{s+2}$ then $E - 1$ is in H^{s+2}.*

(We remark that this gain of regularity is related to Gauss's Theorema egregium, and we also remark that there is a similar gain of regularity for E_t.) Finally, we mention that regularity of $\mathbf{X}$ may be inferred from the regularity of κ through the formula (2.7).

3.3. **Evolution of E and κ.** As we have said, we will perform energy estimates for the mean curvature, κ; as such, we must develop the evolution equation satisfied by κ. The evolution equation for κ can be inferred from (2.2), using the formula for κ in(2.6) with the definitions of the first and second fundamental coefficients. For the moment, a convenient way to write the evolution equation for the curvature is

$$(\sqrt{E}\kappa)_t = \frac{\Delta U}{2\sqrt{E}} + \frac{V_1}{\sqrt{E}}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{\sqrt{E}}(\sqrt{E}\kappa)_\beta$$
$$+ \frac{UM^2}{\sqrt{E}} + \frac{L}{2\sqrt{E}}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2\sqrt{E}}\left(\frac{V_2}{\sqrt{E}}\right)_\beta. \tag{3.2}$$

Further details of the derivation of (3.2) may be found in [9]. Of course, to fully specify κ_t, we also must have an evolution equation for E. Such an evolution equation for E may be inferred from (2.2), using the definition $E = \mathbf{X}_\alpha \cdot \mathbf{X}_\alpha$, or alternatively $E = \mathbf{X}_\beta \cdot \mathbf{X}_\beta$. We therefore have the evolution equation

$$E_t = 2\sqrt{E}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2 E_\beta}{2E}\right) = 2\sqrt{E}\left(V_{2,\beta} - \frac{UN}{\sqrt{E}} + \frac{V_1 E_\alpha}{2E}\right). \tag{3.3}$$

Since $\kappa_t = (\sqrt{E}\kappa)_t/\sqrt{E} - E_t\kappa/2E$, using (3.2) and (3.3), we conclude that the evolution of κ is given by the following:

$$\kappa_t = \frac{\Delta U}{2E} + \frac{V_1}{E}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{E}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{E} + \frac{L}{2E}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2}{\sqrt{E}}\right)_\beta - \frac{\kappa}{\sqrt{E}}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2 E_\beta}{2E}\right).$$

Using (2.10) to substitute for U, we have the evolution of κ as being

$$\kappa_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\kappa}{E}\right) + \frac{\Delta W(\kappa,q)}{2E} + \frac{V_1}{E}(\sqrt{E}\kappa)_\alpha + \frac{V_2}{E}(\sqrt{E}\kappa)_\beta + \frac{UM^2}{E} + \frac{L}{2E}\left(\frac{V_1}{\sqrt{E}}\right)_\alpha + \frac{N}{2E}\left(\frac{V_2}{\sqrt{E}}\right)_\beta - \frac{\kappa}{\sqrt{E}}\left(V_{1,\alpha} - \frac{UL}{\sqrt{E}} + \frac{V_2 E_\beta}{2E}\right).$$

4. Preliminary estimates and useful formulas

In this section we record some basic estimates which will be useful a number of times. Before doing so, we define the operator Λ to be the operator with symbol $\hat{\Lambda}(k) = |k|$. This can also be represented using the Riesz transforms H_1 and H_2, as we also have the formula $\Lambda = H_1\partial_\alpha + H_2\partial_\beta$. The Riesz transforms may either be defined in terms of their symbols or as singular integrals [32]. For any $\ell > 0$, we define the Sobolev space H^ℓ to be the space of functions for which the norm

$$\|f\|_\ell = \left(\|f\|_{L^2}^2 + \|\Lambda^\ell f\|_{L^2}^2\right)^{1/2}$$

is finite. Notice that $\Lambda^2 = -\Delta$.

The first lemma concerns commutators; as usual, the commutator notation means $[A,B]f = ABf - BAf$.

Lemma 4.1. *Let $s > 1$. If $f \in H^{s+2}$ and $g \in H^{s+1}$, then $[\Lambda^2, f]g$ is in H^s, with the estimate*

$$\|[\Lambda^2, f]g\|_s \leq c\|f\|_{s+2}\|g\|_{s+1}.$$

Proof. Notice that

$$\left[\Lambda^2, f\right]g = -\Delta(fg) + f\Delta g = -g\Delta f - 2\nabla f \cdot \nabla g.$$

When $s > 1$, we know the Sobolev space H^s is algebraic, that is

$$\|-g\Delta f\|_s \leq \|g\|_s\|\Delta f\|_s \leq \|f\|_{s+2}\|g\|_s$$

and

$$\| -2\nabla f \cdot \nabla g\|_s \le 2\|\nabla f\|_s \|\nabla g\|_s \le 2\|f\|_{s+1}\|g\|_{s+1}.$$

This completes the proof of the lemma. □

We next have another commutator estimate, which appeared in [30] and thus we omit the proof.

Lemma 4.2. *For $s > 0$, then*

$$\|[\Lambda^s, f]g\| \le C\left(\|\nabla f\|_{L^\infty}\|g\|_{s-1} + \|f\|_s\|g\|_{L^\infty}\right).$$

Next we give a standard elementary interpolation estimate.

Lemma 4.3. *For $0 < m < s$,and $f \in H^s$, then*

$$\|\Lambda^m f\| \le \|\Lambda^s f\|^{m/s}\|f\|^{1-m/s}$$

The proof of Lemma 4.3 can be found many places, such as [3]; see also [1].

For the final result of this section, we comment on the regularity of the velocities U, V_1, and V_2.

Lemma 4.4. *If $(\mathbf{X} - (\alpha,\beta,0)) \in H^{s+2}$, $(E-1) \in H^{s+2}$ and $\kappa \in H^s$, then $U + 1 \in H^{s-2}$ and $V_i \in H^{s-1}$.*

Proof. Recall that

$$U + 1 = -\tau\Delta\kappa/E + W(\kappa, q).$$

We immediately get $U + 1 \in H^{s-2}$ when κ and q are in H^s; this uses the definition of W in (2.9) and the fact that H^s is an algebra for $s > 1$.

Turning to V_1 and V_2, we apply ∂_α to (2.11) and we apply ∂_β to (2.12), and then add the results. It follows that V_1 satisfies

$$\Delta\left(\frac{V_1}{\sqrt{E}}\right) = \left(\frac{U(L-N)}{E}\right)_\alpha + \left(\frac{2UM}{E}\right)_\beta.$$

Solving this Poisson equation, we have $V_1 \in H^{s-1}$. Similarly, we have $V_2 \in H^{s-1}$ since

$$\Delta\left(\frac{V_2}{\sqrt{E}}\right) = -\left(\frac{U(L-N)}{E}\right)_\beta + \left(\frac{2UM}{E}\right)_\alpha.$$

□

5. Well-posedness

In this section we prove Theorem 2.1, showing that the two-dimensional coordinate-free model for the motion of flame fronts is well-posed. We will use an iteration method. In Section 5.1 below, we set up an iterated system of evolution equations. We then set up a general linear Cauchy problem in Section 5.2, and demonstrate energy estimates for this problem in Section 5.3. In Section 5.4 we then use the results of Sections 5.2 and 5.3 to show that the iterates in our iteration scheme obey bounds uniform with respect to the iteration parameter. We use the uniform bounds to take the limit of the sequence of iterates in Section 5.5. We then complete the proof of Theorem 2.1 in Section 5.6.

5.1. **The iterated system.** We now set up iterated evolution equations for $\mathbf{X}$ and κ. We take initial data $\mathbf{X}_0 \in H^{s+2}$, such that the surface has a global isothermal parameterization with $E_0 = \mathbf{X}_{0\alpha} \cdot \mathbf{X}_{0\alpha} = \mathbf{X}_{0\beta} \cdot \mathbf{X}_{0\beta} > c_0$, for some constant $c_0 > 0$. Of course, in taking a global isothermal paramerization for $\mathbf{X}_0$, we also have $\mathbf{X}_{0\alpha} \cdot \mathbf{X}_{0\beta} = 0$.

We initialize the iterative scheme with the initial iterate being $\mathbf{X}^0 = \mathbf{X}_0$. Then $E^0 = E_0$ as well, and $\kappa^0 = \kappa_0$, where κ_0 is the mean curvature for the initial surface $\mathbf{X}_0$. Similarly we also have the other quantities corresponding to $\mathbf{X}_0$, namely L^0, M^0, N^0, and q^0. Recall that these quantities are defined through (2.4) and (2.6), using $\mathbf{X}_0$ as the surface. We then also may define U^0 through (2.10), using κ^0, E^0, and q^0. Then (V_1^0, V_2^0) is determined by solving the following system,

$$\begin{aligned} \left(\frac{V_1^0}{\sqrt{E^0}}\right)_\alpha - \left(\frac{V_2^0}{\sqrt{E^0}}\right)_\beta &= \frac{U^0(L^0 - N^0)}{E^0}, \\ \left(\frac{V_1^0}{\sqrt{E^0}}\right)_\beta + \left(\frac{V_2^0}{\sqrt{E^0}}\right)_\alpha &= \frac{2U^0 M^0}{E^0}. \end{aligned}$$

Assume that for some $l \geq 0$ we have already constructed $(\mathbf{X}^l, \kappa^l, E^l)$. We then compute the related quantities

$$\begin{aligned} &\hat{\mathbf{t}}^{1,l} = \frac{\mathbf{X}^l_\alpha}{|\mathbf{X}^l_\alpha|}, \qquad \hat{\mathbf{t}}^{2,l} = \frac{\mathbf{X}^l_\beta}{|\mathbf{X}^l_\beta|}, \qquad \hat{\mathbf{n}}^l = \frac{\mathbf{X}^l_\alpha \times \mathbf{X}^l_\beta}{|\mathbf{X}^l_\alpha \times \mathbf{X}^l_\beta|}, \\ &L^l = \mathbf{X}^l_{\alpha\alpha} \cdot \hat{\mathbf{n}}^l, \qquad N^l = \mathbf{X}^l_{\beta\beta} \cdot \hat{\mathbf{n}}^l, \qquad M^l = \mathbf{X}^l_{\alpha\beta} \cdot \hat{\mathbf{n}}^l, \\ &q^l = \frac{L^l N^l - (M^l)^2}{(\mathbf{X}^l_\alpha \cdot \mathbf{X}^l_\alpha)(\mathbf{X}^l_\beta \cdot \mathbf{X}^l_\beta) - (\mathbf{X}^l_\alpha \cdot \mathbf{X}^l_\beta)^2}. \end{aligned}$$

To begin finding the next iterates, we construct κ^{l+1} to solve the linear Cauchy problem

$$\kappa_t^{l+1} = -\frac{\tau}{2E^l}\Delta\left(\frac{\Delta\kappa^{l+1}}{E^l}\right) + Q_1^l, \tag{5.1}$$

where Q_1^l is given by

$$\begin{aligned} Q_1^l = {}& \frac{\Delta W(\kappa^l, q^l)}{2E^l} + \frac{V_1^l}{E^l}\left(\sqrt{E^l}\kappa^l\right)_\alpha + \frac{V_2^l}{E^l}\left(\sqrt{E^l}\kappa^l\right)_\beta \\ &+ \frac{U^l (M^l)^2}{E^l} + \frac{L^l}{2E^l}\left(\frac{V_1^l}{\sqrt{E^l}}\right)_\alpha + \frac{N^l}{2E^l}\left(\frac{V_2^l}{\sqrt{E^l}}\right)_\beta - \frac{\kappa^l}{\sqrt{E^l}}\left(V^l_{1,\alpha} - \frac{U^l L^l}{\sqrt{E^l}} + \frac{V_2^l E^l_\beta}{2E^l}\right), \end{aligned} \tag{5.2}$$

and with initial condition $\kappa^{l+1}|_{t=0} = \kappa_0$. After we demonstrate that this problem is solvable, we can then define U^{l+1} as

$$U^{l+1} = -\tau\Delta\kappa^{l+1}/E^l + W(\kappa^l, q^l) - 1. \tag{5.3}$$

We next determine (V_1^{l+1}, V_2^{l+1}) by solving the system

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\alpha - \left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\beta = \frac{U^{l+1}(L^l - N^l)}{E^l} = -\tau\frac{\Delta\kappa^{l+1}(L^l - N^l)}{(E^l)^2} + f_1^l, \tag{5.4}$$

$$\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right)_\beta + \left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right)_\alpha = \frac{2U^{l+1}M^l}{E^l} = -\tau\frac{2\Delta\kappa^{l+1}M^l}{(E^l)^2} + f_2^l. \tag{5.5}$$

Here, the functions f_1^l, f_2^l are defined by

$$f_1^l = \frac{(W(\kappa^l, q^l) - 1)(L^l - N^l)}{E^l}, \tag{5.6}$$

$$f_2^l = \frac{2(W(\kappa^l, q^l) - 1)M^l}{E^l}. \tag{5.7}$$

These equations can be combined to more clearly have solvable Poisson equations,

$$\Delta\left(\frac{V_1^{l+1}}{\sqrt{E^l}}\right) = -\left(\frac{\tau\Delta\kappa^{l+1}(L^l - N^l)}{(E^l)^2}\right)_\alpha - \left(\frac{2\tau\Delta\kappa^{l+1}M^l}{(E^l)^2}\right)_\beta + (f_1^l)_\alpha + (f_2^l)_\beta, \tag{5.8}$$

$$\Delta\left(\frac{V_2^{l+1}}{\sqrt{E^l}}\right) = \left(\frac{\tau\Delta\kappa^{l+1}(L^l - N^l)}{(E^l)^2}\right)_\beta - \left(\frac{2\tau\Delta\kappa^{l+1}M^l}{(E^l)^2}\right)_\alpha - (f_1^l)_\beta + (f_2^l)_\alpha. \tag{5.9}$$

We define the next iterate of the surface, $\mathbf{X}^{l+1}$, in a few stages. To begin we let $\widetilde{\mathbf{X}}^{l+1}$ be the solution of the initial value problem

$$\widetilde{\mathbf{X}}_t^{l+1} = U^{l+1}\hat{\mathbf{n}}^l + V_1^{l+1}\hat{\mathbf{t}}^{1,l} + V_2^{l+1}\hat{\mathbf{t}}^{2,l}, \quad \widetilde{\mathbf{X}}^{l+1}|_{t=0} = \mathbf{X}_0. \tag{5.10}$$

Substituting for U^{l+1} using the equation (5.3), we can write this as

$$\widetilde{\mathbf{X}}_t^{l+1} = (-\tau\Delta\kappa^{l+1}/E^l)\hat{\mathbf{n}}^l + V_1^{l+1}\hat{\mathbf{t}}^{1,l} + V_2^{l+1}\hat{\mathbf{t}}^{2,l} + Q_2^l, \tag{5.11}$$

where Q_2^l is given by

$$Q_2^l = (W(\kappa^l, q^l) - 1)\hat{\mathbf{n}}^l. \tag{5.12}$$

We will have one more intermediate variable $\widehat{\mathbf{X}}^{l+1}$, which is given by solving the elliptic equation

$$\Delta\widehat{\mathbf{X}}^{l+1} - \widehat{\mathbf{X}}^{l+1} = 2\kappa^l\widetilde{\mathbf{X}}_\alpha^{l+1} \times \widetilde{\mathbf{X}}_\beta^{l+1} - \widetilde{\mathbf{X}}^{l+1}. \tag{5.13}$$

Note that this formula is based upon the equation (2.7). Now we are ready to construct $\mathbf{X}^{l+1}$ by solving the following elliptic equation (again influenced by (2.7)),

$$\Delta\mathbf{X}^{l+1} - \mathbf{X}^{l+1} = 2\kappa^l\widehat{\mathbf{X}}_\alpha^{l+1} \times \widehat{\mathbf{X}}_\beta^{l+1} - \widetilde{\mathbf{X}}^{l+1}. \tag{5.14}$$

Finally, we will define E^{l+1} also by solving an elliptic equation,

(5.15)
$$\Delta E^{l+1} - E^{l+1} = 2(\mathbf{X}^{l+1}_{\alpha\beta}\cdot\mathbf{X}^{l+1}_{\alpha\beta} - \mathbf{X}^{l+1}_{\alpha\alpha}\cdot\mathbf{X}^{l+1}_{\beta\beta}) - \frac{1}{2}(\mathbf{X}^{l+1}_{\alpha}\cdot\mathbf{X}^{l+1}_{\alpha} + \mathbf{X}^{l+1}_{\beta}\cdot\mathbf{X}^{l+1}_{\beta}).$$

Note that this equation is based upon (3.1) as well as the fact that in an isothermal parameterization, we have $E = \frac{1}{2}\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha + \frac{1}{2}\mathbf{X}_\beta\cdot\mathbf{X}_\beta$.

Remark 2. Note that for any of the versions of the iterated surfaces, they are not expected to be parameterized isothermally. The isothermal parameterization of the solution will be recovered after taking the limit as $l\to\infty$.

5.2. **A linear Cauchy problem.** To deal with the iterated system, we study the well-posedness of the linearized Cauchy problem for κ^{l+1} and $\widetilde{\mathbf{X}}^{l+1}$. More precisely, we will consider the linear Cauchy problems for η and $\mathbf{Y}$,where η satisfies the evolution equation

$$\eta_t = -\frac{\tau}{2E}\Delta\left(\frac{\Delta\eta}{E}\right) + Q_1, \tag{5.16}$$

with initial condition $\eta|_{t=0} = \eta_0$, and $\mathbf{Y}$ satisfies evolution equation

$$\mathbf{Y}_t = (-\tau\Delta\eta/E)\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2 + Q_2, \tag{5.17}$$

with initial condition $\mathbf{Y}|_{t=0} = \mathbf{Y}_0$. The tangential velocities V_1 and V_2 solve

$$\left(\frac{V_1}{\sqrt{E}}\right)_\alpha - \left(\frac{V_2}{\sqrt{E}}\right)_\beta = -\tau\frac{\Delta\eta(L-N)}{(E)^2} + f_1, \tag{5.18}$$

$$\left(\frac{V_1}{\sqrt{E}}\right)_\beta + \left(\frac{V_2}{\sqrt{E}}\right)_\alpha = -\tau\frac{2\Delta\eta M}{(E)^2} + f_2. \tag{5.19}$$

Here, Q_1, Q_2, f_1, and f_2 are given nonhomogenous terms. We are assuming here that $\tau > 0$ and that there is a given surface $\mathbf{X}$ and function E. The functions L, M, and N are the second fundamental coefficients of $\mathbf{X}$, and the vectors $\hat{\mathbf{t}}^i$, $\hat{\mathbf{n}}$ are the unit tangent and normal vectors to $\mathbf{X}$.

5.3. **The *a priori* estimate.** The iterative scheme which we set up involves the solution of a sequence of linear equations, all of the form (5.16)-(5.19). Bounds for the solutions of these equations are therefore fundamental to our existence proof. We now establish the needed estimates.

Theorem 5.1. *Suppose that there exists $T > 0$ such that*
$$E \in C([0,T],H^{s+2})\bigcap C^1([0,T],H^{s-2}),$$
$$\mathbf{X} \in C([0,T],H^{s+2})\bigcap C^1([0,T],H^{s-2}),$$
and
$$Q_1 \in L^2([0,T],H^{s-2}), Q_2 \in L^2([0,T],H^{s}), f_i \in L^2([0,T],H^{s-1}).$$
Assume there exists $C_0 > 0$ such that for all $t\in[0,T]$, we have $E(\cdot,t) \ge C_0 > 0$. Let initial data $\eta_0 \in H^s$ and $\mathbf{Y}_0 \in H^s$ be given. Then there exists a constant

$m > 0$ and a unique solution to the Cauchy problem $\eta \in C([0,T],H^s) \cap L^2([0,T],H^{s+2})$ and $\mathbf{Y} \in C([0,T],H^s)$ such that the bounds

$$\|\eta\|_s^2 + \int_0^t \frac{\tau m \|\Lambda^{s+2}\eta\|_0^2}{4}\, dt' \le e^{Ct}\left(\|\eta_0\|_s^2 + \int_0^t \frac{4}{m\tau}\|Q_1\|_{s-2}^2\, dt'\right), \tag{5.20}$$

$$\|\mathbf{Y}\|_s^2 \le e^{Ct}\left(\|\mathbf{Y}_0\|_s^2 + \|\eta_0\|_s^2 + C\int_0^t \|Q_1\|_{s-2}^2 + \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2\, dt'\right), \tag{5.21}$$

are satisfied for all $t \in [0,T]$.

Proof. The proof of well-posedness for the Cauchy problem for (5.16) and (5.17) follows classical steps, namely approximation, existence for the approximate problems, establishing uniform estimates, passage to the limit, and establishing further estimates for uniqueness and continuous dependence. By far the most important and most interesting of these steps is the uniform bound, and this will be the focus of our presentation here. We will give the relevant energy estimate, and omit the other details.

We will first establish the energy estimate for η . We define the energy

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1,$$

with $\mathcal{E}_0 = \frac{1}{2}\|\eta\|_0^2$ and $\mathcal{E}_1 = \frac{1}{2}\|\Lambda^s\eta\|_0^2$. To begin, we take the time derivative of $\mathcal{E}_0$:

$$\frac{d\mathcal{E}_0}{dt} = \iint \eta\eta_t\, d\alpha d\beta.$$

Since s is sufficiently large, using the evolution of η and the assumed regularity of E and Q_1, we may immediately conclude

$$\frac{d\mathcal{E}_0}{dt} \le C(\mathcal{E} + \|Q_1\|_{s-2}^2).$$

We next take the time derivative of $\mathcal{E}_1$,

$$\frac{d\mathcal{E}_1}{dt} = -\tau\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta + \iint (\Lambda^{s+2}\eta)(\Lambda^{s-2}Q_1)\, d\alpha d\beta. \tag{5.22}$$

For the second term on the right-hand side of (5.22), we have used the fact that Λ is a self-adjoint operator (which can be seen from its symbol in Fourier space).

We first deal with the first term on the right-hand side of (5.22). Recalling that that $\Delta = -\Lambda^2$, and pulling $\frac{1}{E}$ through Λ^2 (incurring a commutator), we have

$$\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right) d\alpha d\beta = \iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Lambda^2}{2E}\left(\frac{\Lambda^2\eta}{E}\right)\right) d\alpha d\beta$$

$$= \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{\Lambda^4\eta}{2E^2}\right) d\alpha d\beta + \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2, \frac{1}{E}\right]\Lambda^2\eta\right) d\alpha d\beta.$$

We then incur another commutator, this time pulling $\frac{1}{2E^2}$ through Λ^{s-2}; this yields

$$\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right)\,d\alpha d\beta$$
$$= \iint \frac{1}{2E^2}(\Lambda^{s+2}\eta)^2\,d\alpha d\beta + \iint (\Lambda^{s+2}\eta)\left[\Lambda^{s-2},\frac{1}{2E^2}\right](\Lambda^4\eta)\,d\alpha d\beta$$
$$+ \iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2,\frac{1}{E}\right]\Lambda^2\eta\right)\,d\alpha d\beta.$$

By Lemma 4.2, for sufficiently large s, we have

$$\left\|\left[\Lambda^{s-2},\frac{1}{2E^2}\right]\Lambda^4\eta\right\|_0 \le c\left(\left\|\nabla\left(\frac{1}{2E^2}\right)\right\|_{L^\infty}\|\Lambda^4\eta\|_{s-3} + \left\|\frac{1}{2E^2}\right\|_{s-2}\|\Lambda^4\eta\|_{L^\infty}\right)$$
$$\le c\|\eta\|_{s+1} \le c(\|\eta\|_0 + \|\Lambda^{s+1}\eta\|_0).$$

And, by Lemma 4.1, we have

$$\left\|\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2,\frac{1}{E}\right]\Lambda^2\eta\right)\right\|_0 \le c\|\Lambda^2\eta\|_{s-1} \le c(\|\eta\|_0 + \|\Lambda^{s+1}\eta\|_0).$$

By Lemma 4.3, $\|\Lambda^{s+1}\eta\|_0 \le c\|\Lambda^{s+2}\eta\|_0^{\frac{s+1}{s+2}}\|\eta\|_0^{\frac{1}{s+2}}$. By first applying Hölder's inequality and then Young's inequality (with parameter $n > 0$ to be chosen), we have

$$\left|\iint (\Lambda^{s+2}\eta)\left[\Lambda^{s-2},\frac{1}{2E^2}\right](\Lambda^4\eta)\,d\alpha d\beta\right|$$
$$\le c\|\eta\|_0^{\frac{1}{s+2}}\|\Lambda^{s+2}\eta\|_0^{\frac{2s+3}{s+2}} + c\|\eta\|_0\|\Lambda^{s+2}\eta\|_0 \le \frac{\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E}.$$

In just the same way, we have

$$\left|\iint (\Lambda^{s+2}\eta)\Lambda^{s-2}\left(\frac{1}{2E}\left[\Lambda^2,\frac{1}{E}\right]\Lambda^2\eta\right)\,d\alpha d\beta\right| \le \frac{\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E}.$$

We make our first conclusion that

$$-\tau\iint (\Lambda^s\eta)\Lambda^s\left(\frac{\Delta}{2E}\left(\frac{\Delta\eta}{E}\right)\right)\,d\alpha d\beta$$
$$\le -\tau\iint \frac{1}{2E^2}(\Lambda^{s+2}\eta)^2\,d\alpha d\beta + \frac{2\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + C\tau\mathcal{E}.$$

For the second term on the right-hand side of (5.22), by Young's inequality, we have

$$\left|\iint \Lambda^{s+2}\eta\Lambda^{s-2}Q_1 d\alpha d\beta\right| \le \frac{\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + \frac{n\|Q_1\|_{s-2}^2}{\tau}.$$

Now we make the conclusion that

$$\frac{d\mathcal{E}}{dt} \le -\tau\iint \frac{1}{4E^2}(\Lambda^{s+2}\eta)^2\,d\alpha d\beta + \frac{3\tau\|\Lambda^{s+2}\eta\|_0^2}{n} + C\mathcal{E} + n\|Q_1\|_{s-2}^2/\tau.$$

We know $E > 0$, and $E \in L^\infty$ when $s > 1$. Then there exists $m > 0$ such that $-\frac{1}{4E^2} \le -m$. Now we take $n = 4/m$. Then

$$\frac{d\mathcal{E}}{dt} + \frac{\tau m \|\Lambda^{s+2}\eta\|_0^2}{4} \le C\mathcal{E} + \frac{4}{m\tau}\|Q_1\|_{s-2}^2.$$

By Gronwall's inequality, it follows that

$$\mathcal{E}(t) + \int_0^t e^{C(t-t')}\frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\ dt' \le e^{Ct}\left(\mathcal{E}(0) + \int_0^t \frac{4}{m\tau}e^{-Ct'}\|Q_1\|_{s-2}^2\ dt'\right)$$

Moreover, since $e^{C(t-t')} \ge 1$ and $e^{-Ct'} \le 1$, we have

$$\mathcal{E}(t) + \int_0^t \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\ dt' \le e^{Ct}\left(\mathcal{E}(0) + \int_0^t \frac{4}{m\tau}\|Q_1\|_{s-2}^2\ dt'\right).$$

Now we will do the energy estimate for $\mathbf{Y}$, where $\mathbf{Y}$ evolves according to (5.17). We define energy

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_0 + \widetilde{\mathcal{E}}_1,$$

with $\widetilde{\mathcal{E}}_0 = \frac{1}{2}\|\mathbf{Y}\|_0^2$ and $\widetilde{\mathcal{E}}_1 = \frac{1}{2}\|\Lambda^s\mathbf{Y}\|_0^2$. To begin, we take the time derivative of $\widetilde{\mathcal{E}}_0$:

$$\frac{d\widetilde{\mathcal{E}}_0}{dt} = \iint \mathbf{Y}\cdot\mathbf{Y}_t\ d\alpha d\beta.$$

Since s is sufficiently large, using the evolution of $\mathbf{Y}$ and the estimates of Section 4, we may immediately conclude

$$\frac{d\widetilde{\mathcal{E}}_0}{dt} \le C(\widetilde{\mathcal{E}} + \|\eta\|_s^2 + \|Q_2\|_s^2 + \|f_1\|_{s-1}^2 + \|f_2\|_{s-1}^2).$$

Next, we take the time derivative of $\widetilde{\mathcal{E}}_1$, and make a use of Young's inequality:

$$\begin{aligned}\frac{d\widetilde{\mathcal{E}}_1}{dt} &= \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s\mathbf{Y}_t\ d\alpha d\beta\\ &= \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s\big((-\tau\Delta\eta/E)\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2 + Q_2\big)\ d\alpha d\beta\\ &\le \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{8} + \iint \Lambda^s\mathbf{Y}\cdot\Lambda^s(V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2)\ d\alpha d\beta + \|Q_2\|_s^2 + C\widetilde{\mathcal{E}} + C\|\eta\|_s^2\end{aligned}$$

Since V_1 and V_2 solve the elliptic system (5.18), (5.19), for sufficiently large s we have

$$\begin{aligned}\|\Lambda^s(V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2)\|_0 &\le c\tau\|\eta\|_{s+1} + c(\|f_1\|_{s-1} + \|f_2\|_{s-1})\\ &\le c\tau(\|\eta\|_0 + \|\Lambda^{s+2}\eta\|_0) + c(\|f_1\|_{s-1} + \|f_2\|_{s-1}).\end{aligned}$$

By first applying Hölder's inequality and then Young's inequality, we have

$$\begin{aligned}&\left|\iint \Lambda^s\mathbf{Y}\cdot\Lambda^s(V_1\hat{\mathbf{t}}^1+V_2\hat{\mathbf{t}}^2)\ d\alpha d\beta\right|\\ &\qquad\le \|\Lambda^s\mathbf{Y}\|_0\left(c(\|\eta\|_0+\|\Lambda^{s+2}\eta\|_0)+c(\|f_1\|_{s-1}+\|f_2\|_{s-1})\right)\\ &\qquad\qquad\le \frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{8}+C\widetilde{\mathcal{E}}+C(\|f_1\|_{s-1}^2+\|f_2\|_{s-1}^2+\|\eta\|_s^2).\end{aligned}$$

We may now make the conclusion that

$$\frac{d\widetilde{\mathcal{E}}}{dt}\le C\widetilde{\mathcal{E}}+\frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}+C(\|Q_2\|_s^2+\|f_1\|_{s-1}^2+\|f_2\|_{s-1}^2+\|\eta\|_s^2).$$

By Grönwall's inequality, it follows

$$\widetilde{\mathcal{E}}(t)\le e^{Ct}\left(\widetilde{\mathcal{E}}(0)+\int_0^t\frac{\tau m\|\Lambda^{s+2}\eta\|_0^2}{4}\ dt'+\int_0^t C(\|Q_2\|_s^2+\|f_1\|_{s-1}^2+\|f_2\|_{s-1}^2+\|\eta\|_s^2\ dt'\right).$$

Using our estimate for η, (5.20), we have

$$\begin{aligned}\widetilde{\mathcal{E}}(t)&\le e^{Ct}\left(\widetilde{\mathcal{E}}(0)+C\int_0^t\|Q_2\|_s^2+\|f_1\|_{s-1}^2+\|f_2\|_{s-1}^2\ dt'\right)\\ &\quad+e^{Ct}\left(Ce^{Ct}\left(\|\eta_0\|_s^2+\int_0^t\frac{4}{m\tau}\|Q_1\|_{s-2}^2\ dt'\right)\right)\\ &\le e^{Ct}\left(\widetilde{\mathcal{E}}(0)+\|\eta_0\|_s^2+C\int_0^t\|Q_2\|_s^2+\|f_1\|_{s-1}^2+\|f_2\|_{s-1}^2+\|Q_1\|_{s-2}^2\ dt'\right).\end{aligned}$$

This concludes the proof of the theorem. □

Remark 3. All constants which are denoted C above may be expressed in terms of the norms of $\mathbf{X}$ and E, and the given functions Q_1, Q_2, f_1, and f_2, as well as the parameters of the problem such as τ.

5.4. **Estimates for the iteration scheme.** To prove that the sequences of iterations converge, we will need the following lemma.

Lemma 5.2. *The family of iterates* $(\mathbf{X}^l, E^l, \kappa^l)$ *are defined for all* l *and there exists* $T>0$ *and positive constants* C_0, C_1, C_2, C_3 *and* C_4 *such that for all* l,

$$E^l\ge C_0>0,\quad |\mathbf{X}^l_\alpha\times\mathbf{X}^l_\beta|\ge C_0>0, \tag{5.23}$$

$$\|\kappa^l\|_{C^0([0,T];H^s)}\le C_1, \tag{5.24}$$

$$\|\mathbf{X}^l\|_{C^0([0,T];H^{s+2})}+\|E^l\|_{C^0([0,T];H^{s+2})}\le C_2, \tag{5.25}$$

$$\|\partial_t\kappa^l\|_{C^0([0,T];H^{s-4})}\le C_3, \tag{5.26}$$

$$\|\partial_t\mathbf{X}^l\|_{C^0([0,T];H^{s-2})}+\|\partial_tE^l\|_{C^0([0,T];H^{s-2})}\le C_4. \tag{5.27}$$

Proof. We proceed induction. We take $C_0=c_0/2$. We will determine appropriate values for C_1, C_2, C_3, and C_4 as we go. Given the definition of our initial iterates, the needed bounds are satisfied for $(\mathbf{X}^0, E^0, \kappa^0)$. Assume that $(\mathbf{X}^l, E^l, \kappa^l)$ satisfies (5.23), (5.24) (5.25), (5.26) and (5.27). By the definition of Q_1^l in (5.2), applying $s-2$ derivatives to Q_1^l involves at most s-derivatives of κ^l and at most $(s+2)$-derivatives of $\mathbf{X}^l$, and at most s-derivatitves of E^l. Thus $\|Q_1^l\|_{C([0,T];H^{s-2})} \le C(C_0,C_1,C_2)$. By the result of our energy estimates, (5.20), κ^{l+1} satisfies

$$\|\kappa^{l+1}(t)\|_s^2 + \int_0^t \frac{\tau m \|\Lambda^{s+2}\kappa^{l+1}(s)\|_0^2}{4}\, dt' \\ \le e^{C(C_0,C_1,C_2)t}\|\kappa_0\|_s^2 + C(C_0,C_1,C_2)te^{C(C_0,C_1,C_2)t}.$$

Hence taking $C_1 = 2\|\kappa_0\|_s$, we may take T small enough so that

$$\|\kappa^{l+1}(t)\|_{C([0,T];H^s)}^2 + \int_0^t \frac{\tau m \|\Lambda^{s+2}\kappa^{l+1}(s)\|_0^2}{4}\, dt' \le C_1^2. \tag{5.28}$$

Inspecting the definitions of f_1^l, f_2^l, and Q_2^l in (5.6), (5.7), and (5.12), we see that we may bound these as $\|Q_2^l\|_{C([0,t];H^s)} \le C(C_0,C_1,C_2)$ and $\|f_i^l\|_{C([0,t];H^{s-1})} \le C(C_0,C_1,C_2)$. Using (5.21), the estimate of $\widetilde{\mathbf{X}}^{l+1}$ is then

$$\|\widetilde{\mathbf{X}}^{l+1}(t)\|_s^2 \le e^{C(C_0,C_1,C_2)t}(\|\kappa_0\|_s^2 + \|\mathbf{X}_0\|_s^2) + C(C_0,C_1,C_2)te^{C(C_0,C_1,C_2)t}.$$

Hence we may again choose T sufficiently small so that $\|\widetilde{\mathbf{X}}^{l+1}\|_{C([0,T];H^s)} \le 2(\|\mathbf{X}_0\|_s + \|\kappa_0\|_s)$. By the elliptic equations (5.13) and (5.14), we then have $\mathbf{X}^{l+1} \in C^0([0,T];H^{s+2})$ and

$$\|\widehat{\mathbf{X}}^{l+1}\|_{C([0,T];H^{s+1})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s),$$
$$\|\mathbf{X}^{l+1}\|_{C([0,T];H^{s+2})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s).$$

Then, by the elliptic equation (5.15), we have $E^{l+1} \in C^0([0,T];H^{s+2})$ with estimate

$$\|E^{l+1}\|_{C([0,T];H^{s+2})} \le C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s)$$

We may then take $C_2 = 2C(\|\mathbf{X}_0\|_s, \|\kappa_0\|_s)$, and we see that the estimate (5.25) holds.

Using the inductive hypotheses and the bound (5.28), we may then bound the right-hand side of (5.1), finding

$$\|\partial_t \kappa^{l+1}(t)\|_{s-4} \le C(C_0,C_1,C_2)(1 + \|\kappa^{l+1}(t)\|_s) \le C(C_0,C_1,C_2)(1+C_1).$$

We may similarly bound the right-hand side of (5.10), finding

$$\|\partial_t \widetilde{\mathbf{X}}^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0,C_1,C_2).$$

Taking C_3 such that $C_3 \ge C(C_0,C_1,C_2)(1+C_1)$, we have the estimate (5.26).

Taking the time derivative of (5.13) and (5.14), we may conclude that

$$\|\partial_t \mathbf{X}^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0,C_1,C_2,C_3).$$

Similarly, taking the time derivative of (5.15), we may conclude the bound

$$\|\partial_t E^{l+1}\|_{C([0,T];H^{s-2})} \le C(C_0,C_1,C_2,C_3).$$

We then may take C_4 to be such that $C_4 \ge C(C_0,C_1,C_2,C_3)$, getting the estimate (5.27).

Notice that when s is sufficiently large so that $H^{s-2} \subseteq L^\infty$,

$$|E^{l+1}(t)| \geq E_0 - \int_0^t \partial_t E^{l+1}(t')\ dt' \geq c_0 - tC_4$$

and

$$\begin{aligned}&|\mathbf{X}_\alpha^{l+1}(t) \times \mathbf{X}_\beta^{l+1}(t)|\\ &\quad\geq |\mathbf{X}_\alpha^{l+1}(0) \times \mathbf{X}_\beta^{l+1}(0)| - \int_0^t \partial_\tau(\mathbf{X}_\alpha^{l+1}(s) \times \mathbf{X}_\beta^{l+1})(t')\ dt' \geq c_0 - tC_4C_2.\end{aligned}$$

So, we can take T small enough such that $c_0 - TC_4 \geq C_0$ and $c_0 - TC_4C_2 \geq C_0$. This completes the proof of the lemma. □

5.5. **The limit of the iterated system.** In this section, we will prove that our iterates form Cauchy sequences; this will imply the convergence of the iterative procedure. We will prove existence of a limit in a low norm. Regularity of the limit will follow in the subsequent section, primarily using the uniform bound in the high norm. The main result of this section is the following Lemma.

Lemma 5.3. *The sequence* $(\mathbf{X}^l, \kappa^l, E^l, \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^l)$ *is Cauchy sequence respectively in the following space*

$$C^0([0,T]; H^4 \times H^2 \times H^4 \times H^2 \times H^3).$$

Proof. We denote $(\delta\mathbf{X}, \delta\kappa, \delta E, \delta\widetilde{\mathbf{X}}, \delta\widehat{\mathbf{X}})$ to equal $(\mathbf{X}^{l+1} - \mathbf{X}^l, \kappa^{l+1} - \kappa^l, E^{l+1} - E^l, \widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^{l+1} - \widehat{\mathbf{X}}^l)$. We define an energy functional

$$D_l = \frac{1}{2}\|\delta\kappa\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\kappa\|_0^2 + \frac{1}{2}\|\delta\widetilde{\mathbf{X}}\|_0^2 + \frac{1}{2}\|\Lambda^2\delta\widetilde{\mathbf{X}}\|_0^2.$$

We also denote $\delta U = U^{l+1} - U^l$, $\delta V_i = V_i^{l+1} - V_i^l$, etc. . . .

We may see from (5.13) that we have the estimate

$$\|\delta\widehat{\mathbf{X}}\|_3 \leq C(\|\delta\widetilde{\mathbf{X}}\|_2 + \|\kappa^l - \kappa^{l-1}\|_1) \leq C(D_l + D_{l-1}).$$

Using this estimate with (5.14), we have

$$\begin{aligned}\|\delta\mathbf{X}\|_4^2 &\leq \|2\kappa^l\widehat{\mathbf{X}}_\alpha^{l+1} \times \widehat{\mathbf{X}}_\beta^{l+1} - 2\kappa^{l-1}\widehat{\mathbf{X}}_\alpha^l \times \widehat{\mathbf{X}}_\beta^l\|_2^2 + \|\delta\widetilde{\mathbf{X}}\|_2^2\\ &\leq C\|\kappa^l - \kappa^{l-1}\|_2^2 + C\|\delta\widehat{\mathbf{X}}\|_3^2 + C\|\widetilde{\mathbf{X}}^{l+1} - \widetilde{\mathbf{X}}^l\|_2^2\\ &\leq C(D_l^2 + D_{l-1}^2).\end{aligned}$$

This estimate and (5.15) then imply

$$\|\delta E\|_4 \leq C\|\delta\mathbf{X}\|_4 \leq C(D_l + D_{l-1}).$$

We can write evolution equations for $\delta\kappa$ and $\delta\tilde{\mathbf{X}}$. These are

$$\delta\kappa_t = -\frac{\tau}{2E^l}\Delta\left(\frac{\Delta\delta\kappa}{E^l}\right) + A,$$

where the remainder A is defined as

$$A = -\frac{\tau}{2E^l}\Delta\left(\frac{\Delta\kappa^l}{E^l}\right) + \frac{\tau}{2E^{l-1}}\Delta\left(\frac{\Delta\kappa^l}{E^{l-1}}\right) + Q_1^l - Q_1^{l-1},$$

and

$$\delta\widetilde{\mathbf{X}}_t = \delta U\hat{\mathbf{n}}^l + (\delta V_1)\hat{\mathbf{t}}^{1,l} + (\delta V_2)\hat{\mathbf{t}}^{2,l} + B,$$

where the remainder B is defined as

$$B = U^l(\hat{\mathbf{n}}^l - \hat{\mathbf{n}}^{l-1}) + V_1^l(\hat{\mathbf{t}}^{1,l} - \hat{\mathbf{t}}^{1,l-1}) + V_2^l(\hat{\mathbf{t}}^{2,l} - \hat{\mathbf{t}}^{2,l-1}).$$

Since s is sufficiently large and we are only dealing here with a low norm, with reference to the formula (5.2), we have the following bound for A,

$$\|A\|_2^2 \leq C(D_{l-1} + D_{l-2}).$$

Similarly, we have the following estimate for B,

$$\|B\|_2^2 \leq C(D_{l-1} + D_{l-2}).$$

In taking the time derivative of D_l, we begin with the most important piece,

$$\frac{d}{dt}\frac{1}{2}\iint (\Lambda^2\delta\kappa)^2\ d\alpha d\beta = \iint (\Lambda^2\delta\kappa)(\Lambda^2\delta\kappa_t)\ d\alpha d\beta$$
$$= -\tau\iint (\Lambda^2\delta\kappa)\Lambda^2\left(\frac{1}{2E^l}\Delta\left(\frac{\Delta\delta\kappa}{E^l}\right)\right)\ d\alpha d\beta + \iint (\Lambda^2\delta\kappa)(\Lambda^2 A)\ d\alpha d\beta.$$

We rewrite the first integral on the right-hand side using the fact that Λ is self-adjoint and also incurring a commutator,

$$\frac{d}{dt}\frac{1}{2}\iint (\Lambda^2\delta\kappa)^2\ d\alpha d\beta = -\tau\iint \frac{1}{2(E^l)^2}(\Lambda^4\delta\kappa)^2\ d\alpha d\beta$$
$$-\tau\iint \frac{1}{2E^l}(\Lambda^4\delta\kappa)\left[\Delta, \frac{1}{E^l}\right](\Delta\delta\kappa)\ d\alpha d\beta + \iint (\Lambda^2\delta\kappa)(\Lambda^2 A)\ d\alpha d\beta. \tag{5.29}$$

As in our prior energy estimates, by using Young's inequality, we may control any fourth derivatives of $\delta\kappa$ appearing in the second integral on the right-hand side of (5.29) with the first integral on the right-hand side of (5.29). Other than fourth-derivatives, the remaining terms on the right-hand side of (5.29) can be bounded by $C(D_l^2 + D_{l-1}^2 + D_{l-2}^2)$. Taking the time derivative of the rest of D_l^2 is similar: all terms may either be controlled by using Young's inequality and then absorbing them by the first term on the right-hand side of (5.29), or may be controlled by bounding them by $C(D_l^2 + D_{l-1}^2 + D_{l-2}^2)$.

Thus, omitting the remaining details, we conclude that for $l \geq 2$,

$$\partial_t D_l^2 \leq C(D_l^2 + D_{l-1}^2 + D_{l-2}^2).$$

Moreover, for all l, we have $D_l(0) = 0$. Let us say also that

$$\sup_{t\in[0,T]} D_1^2(t) \leq C, \qquad \sup_{t\in[0,T]} D_2^2(t) \leq C.$$

Then we may prove by induction that for $l \geq 3$, we have

$$\sup_{t\in[0,T]} D_l^2(t) \leq 2Ce^{CT}\left(\frac{(CT)^{n-2}}{(n-2)!} + \frac{(CT)^{n-3}}{(n-3)!}\right).$$

This is sufficient to conclude that κ^l and $\widetilde{\mathbf{X}}^l$ are Cauchy sequences. The elliptic equations (5.13), (5.14), and (5.15) then imply that $\widehat{\mathbf{X}}^l$, $\mathbf{X}^l$, and E^l are also Cauchy sequences in the appropriate spaces. □

5.6. **Proof of the main theorem.** In this subsection we prove the main theorem, Theorem 2.1.

First, it remains to show that the limit of the iterates is a solution of the original system. We have proved that $(\mathbf{X}^l, \kappa^l, E^l, \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^l)$ is a Cauchy sequence in a low norm. Of course, we also have that the iterates are uniformly bounded in a high norm, $H^{s+2} \times H^s \times H^{s+2} \times H^s \times H^{s+1}$. The interpolation inequality of Lemma 4.3 then implies our sequence is Cauchy in $H^{s'+2} \times H^{s'} \times H^{s'+2} \times H^{s'} \times H^{s'+1}$, for any $s' < s$. We denote the limit of $(\mathbf{X}^l, \kappa^l, E^l, \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^l)$ as $(\mathbf{X}, \kappa, E, \widetilde{\mathbf{X}}, \widehat{\mathbf{X}})$. Then we have sufficient regularity to see that the limit satisfies the system (5.4)-(5.15) without index l and $l+1$. We then have the following equations satisfied:

$$\begin{aligned}
&\widetilde{\mathbf{X}}_t = U\hat{\mathbf{n}} + V_1\hat{\mathbf{t}}^1 + V_2\hat{\mathbf{t}}^2\\
&\Delta\widehat{\mathbf{X}} - \widehat{\mathbf{X}} = 2\kappa\widetilde{\mathbf{X}}_\alpha \times \widetilde{\mathbf{X}}_\beta - \widetilde{\mathbf{X}}.\\
&\Delta\mathbf{X} - \mathbf{X} = 2\kappa\widehat{\mathbf{X}}_\alpha \times \widehat{\mathbf{X}}_\beta - \widetilde{\mathbf{X}}.\\
&\Delta E - E = 2(\mathbf{X}_{\alpha\beta}\cdot\mathbf{X}_{\alpha\beta} - \mathbf{X}_{\alpha\alpha}\cdot\mathbf{X}_{\beta\beta}) - (\mathbf{X}_\alpha\cdot\mathbf{X}_\alpha + \mathbf{X}_\beta\cdot\mathbf{X}_\beta).
\end{aligned}$$

We must also prove that the following relations hold:

$$\widetilde{\mathbf{X}} = \widehat{\mathbf{X}} = \mathbf{X}, \quad E = \mathbf{X}_\alpha\cdot\mathbf{X}_\alpha = \mathbf{X}_\beta\cdot\mathbf{X}_\beta, \quad \mathbf{X}_\alpha\cdot\mathbf{X}_\beta = 0, \quad \kappa = \frac{L+N}{2E}. \tag{5.30}$$

We omit the details of the proof here as all of the details demonstrating (5.30) are included in Section 5.4 of [39]. (Like the present work, [39] also uses the isothermal parameterization and iterative scheme of [9]. Therefore, after taking the limit of the iterates, they demonstrate that exactly the relationships given in (5.30) hold.)

The higher regularity of the solution $\mathbf{X}$ must still be established. We have already shown that the solutions are continuous in time in a low norm, and the boundedness in a high norm together with the interpolation result Lemma 4.3 implies that $\mathbf{X} \in C([0,T]; H^{s'+2})$ for any $s' < s$. All that remains to show is that $\mathbf{X} \in C^0([0,T]; H^{s+2})$. We do not include the remaining details, but this can be done by adapting the corresponding argument for regularity of solutions for the Navier-Stokes equations in Chapter 3 of [31].

Finally, we also omit the details of the argument for uniqueness of the solution, as this requires performing an energy estimate for the difference of two solutions. This is therefore almost exactly the same as the proof that $(\mathbf{X}^l, \kappa^l, E^l, \widetilde{\mathbf{X}}^l, \widehat{\mathbf{X}}^l)$ is a Cauchy sequence, which we have given already in Section 5.5.

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