SUFFICIENTLY STRONG DISPERSION REMOVES ILL-POSEDNESS IN TRUNCATED SERIES MODELS OF WATER WAVES

SHUNLIAN LIU AND DAVID M. AMBROSE

Abstract. Truncated series models of gravity water waves are popular for use in simulation. Recent work has shown that these models need not inherit the well-posedness properties of the full equations of motion (the irrotational, incompressible Euler equations). We show that if one adds a sufficiently strong dispersive term to a quadratic truncated series model, the system then has a well-posed initial value problem. Such dispersion can be relevant in certain physical contexts, such as in the case of a bending force present at the free surface, as in a hydroelastic sheet.

1. Introduction

In studying the motion of free surfaces in fluid dynamics, many approximate models have been introduced in order to make problems more tractable. It is not uncommon to consider the full equations of motion to be the irrotational, incompressible, Euler equations. These equations may be written in the Craig-Sulem-Zakharov formulation, in which the Dirichlet-to-Neumann operator plays a central role [15], [32]. Craig and Sulem introduce an approximate system by expanding the Dirichlet-to-Neumann operator as a series in powers of the height of the free surface, truncating the series, and substituting the truncation for the full operator in the equations of motion. In this way, by varying the order of truncation, one may develop a quadratic model, a cubic model, a quartic model, and so on.

The resulting truncated series models have proved popular for computing, as all terms in the evolution equations then can be expressed as polynomials in the unknowns, with spatial derivatives and Hilbert transforms present. These systems are readily computed using pseudospectral methods. Some examples of applications of this approach are [13], [14], [16], [19], [26].

The problem of gravity water waves is famously known to be well-posed, with this having been proved by Wu for a wide class of initial data [30], [31], and subsequently by other authors, including [4], [5],

[22] (this is not nearly an exhaustive list of well-posedness proofs for gravity water waves). Investigating the well-posedness of the truncated series models, however, the second author, together with J. Bona and D. Nicholls, found that the truncated series models appear to have ill-posed initial value problems [3]; specifically, a combination of analytical and numerical evidence was given for ill-posedness, but while convincing, it does not constitute a full proof.

While the truncated series models for gravity water waves appear to have ill-posed initial value problems, dispersion is well-known to offer regularizing effects [12], [21]. The dispersion stemming from gravity is fairly weak, and other physical effects can provide stronger dispersion. In the present work, we thus ask the question, How strong must dispersion be in order to have a well-posed quadratic truncated series model?

The quadratic model of gravity water waves can be written as

$$\begin{cases} \partial_t u = \Lambda v - \partial_x([H,u]\Lambda v), \\ \partial_t v = -gu + \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2. \end{cases} \tag{1.1}$$

where $\Lambda = H\partial_x$, H is the Hilbert transform, and g represents the acceleration due to gravity, a positive constant. Here, $u(x,t)$ represents the height of the free surface at horizontal position x and time t, and $v(x,t)$ is the value of the velocity potential evaluated at the point $(x, u(x,t))$ on the free surface. The commutator $[H,u]$ is the operator given by $[H,u]f = H(uf) - uH(f)$, for any function f. A discussion of the system (1.1) can be found in [2], for example. To study the effects of including stronger dispersion, we generalize (1.1) as follows:

$$\begin{cases} \partial_t u = \Lambda v - \partial_x([H,u]\Lambda v), \\ \partial_t v = -g\Lambda^p u + \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2, \end{cases} \tag{1.2}$$

with $p \geq 0$. Clearly, when $p = 0$, system (1.2) reduces to (1.1).

The view of ill-posedness of (1.2) for $p = 0$ taken in [3] is that the term $(\Lambda v)^2$ on the right-hand side of the evolution equation for v is a parabolic term of indefinite sign, which thus allows for catastrophic growth. This can be compensated for with various smoothing mechanisms; the second author, Bona, and Nicholls considered a viscous regularization in [2] (the particular viscous regularization used was related to the work of Dias, Dyachenko, and Zakharov [17]). The present work instead focuses on dispersive regularization; note that the dispersion relation for (1.2) is of order $(p+1)/2$. We specifically are interested in the question of how strong dispersion must be (i.e., how large p must be) to regularize the system. We demonstrate the existence and uniqueness of solutions in Sobolev spaces for the initial value problem associated to (1.2) for $p \geq 3$.

The physical relevance of various values of p is naturally of interest, then. In the case $p = 2$, the system (1.2) has the correct dispersion relation for the presence of surface tension, and the system may be regarded as a quadratic truncated series model for pure capillary waves. Note that in this case, $p = 2$, the question of well-posedness or ill-posedness is at present unresolved. The case $p = 4$, however, is relevant in cases in which an elastic bending is present at the free surface. A variety of results have been shown recently for such waves, known as hydroelastic waves. Well-posedness of the initial value problem for 2D hydroelastic waves has been demonstrated by the authors and Siegel [6], [24]. The equations of motion in this case are based upon the Cosserat model of elastic shells, as developed by Plotnikov and Toland [27]. A number of studies have been made of traveling hydroelastic waves [9], [18], [29]. The system (1.2) with $p = 4$ can be derived as a quadratic truncated series model for hydroelastic waves; see [23] for details.

In summary, we prove existence and uniqueness of solutions for the initial value problem for (1.2) for $p \geq 3$, which thus includes the hydroelastic case. We do this by using paradifferential calculus as developed by Bony [10] (see also [20], [25]). Our approach specifically is influenced by the use of paradifferential calculus by Alazard, Burq, and Zuily for the initial value problem of capillary-gravity water waves [1]. Our main result is the following:

Theorem 1.1. *Let* $p \geq 3$, $s > \max\{5/2, (p+1)/2\}$, *and* $(u_0, v_0) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$ *be given. Then there exists time* $T > 0$ *such that the Cauchy problem for* (1.2) *with initial data* (u_0, v_0) *has a unique solution*

$$(u, v) \in C^0([0, T]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})). \tag{1.3}$$

The plan of the paper is as follows: we provide some preliminary results in Section 2. We prove existence of solutions in Section 3, and uniqueness in Section 4. We make some concluding remarks in Section 5.

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2. Preliminaries

In this section we present lemmas which will be useful to us throughout the sequel. First we give some commutator estimates and related results. Then, we introduce fundamental concepts and results of paradifferential calculus.

2.1. The Hilbert Transform and related commutators.

The Hilbert transform on $\mathbb{R}$ may be defined via its symbol as follows:

$$\widehat{Hf}(\xi) = -i\text{sgn}(\xi)\widehat{f}(\xi) \tag{2.1}$$

Here, as usual, we denote the Fourier transform of a function g as $\hat{g}$. In the sequel, many commutators of the form $[H,\phi]f = H(\phi f) - \phi H(f)$ will arise, and we thus need appropriate estimates. Towards this end, the following lemma will be helpful.

Lemma 2.1. *Assume that $F(\xi,\eta)$ is piecewise continuous function, and let*

$$T_F(f,g)(\xi) = \int F(\xi,\eta)f(\eta)g(\xi-\eta)d\eta, \quad f,g \in C_0. \tag{2.2}$$

If there exists $M > 0$ such that either

$$\int |F(\xi,\eta)|^2 d\eta \le M^2 \quad \textit{for all } \xi \tag{2.3}$$

or

$$\int |F(\xi,\eta)|^2 d\xi \le M^2 \quad \textit{for all } \eta \tag{2.4}$$

holds, then $T_F : L^2 \times L^2 \to L^2$, with the estimate

$$\|T_F(f,g)\|_{L^2} \le M\|f\|_{L^2}\|g\|_{L^2}. \tag{2.5}$$

Proof. Case 1: We assume that the bound (2.3) holds. By the Schwartz inequality,

$$\begin{aligned}|T_F(f,g)(\xi)|^2 &= \left|\int F(\xi,\eta)f(\eta)g(\xi-\eta)d\eta\right|^2 \\ &\le \int |F(\xi,\eta)|^2 d\eta \int |f(\eta)g(\xi-\eta)|^2 d\eta \le M^2 \int |f(\eta)g(\xi-\eta)|^2 d\eta.\end{aligned}$$

We integrate with respect to ξ, using Tonelli's Theorem:

$$\int |T_F(f,g)(\xi)|^2 d\xi \le M^2 \int\int |f(\eta)g(\xi-\eta)|^2 d\eta d\xi = M^2\|f\|_{L^2}^2\|g\|_{L^2}^2.$$

This immediately implies $\|T_F(f,g)\|_{L^2} \le M\|f\|_{L^2}\|g\|_{L^2}$.

Case 2: We assume instead that the bound (2.4) holds, and we set $w(\xi) = T_F(f,g)(\xi)$. Let $h \in L^2$ be given. We compute as follows, using the triangle inequality, Tonelli's Theorem and the Schwartz inequality:

$$\begin{aligned}\left|\int w(\xi)h(\xi)d\xi\right| &= \left|\int\int h(\xi)F(\xi,\eta)f(\eta)g(\xi-\eta)d\eta d\xi\right| \\ &\leq \int |f(\eta)| \left(\int |h(\xi)F(\xi,\eta)g(\xi-\eta)|d\xi\right) d\eta \\ &\leq \|f\|_{L^2}\left(\int\left(\int h(\xi)F(\xi,\eta)g(\xi-\eta)d\xi\right)^2 d\eta\right)^{1/2}.\end{aligned}$$

Repeating the steps of the proof of Case 1, we are able to conclude

$$\left|\int w(\xi)h(\xi)d\xi\right| \leq \|f\|_{L^2}(M\|h\|_{L^2}\|g\|_{L^2}).$$

Since h was an arbitrary element of L^2, this implies that $w \in L^2$, with the estimate $\|w\|_{L^2} \leq M\|f\|_{L^2}\|g\|_{L^2}$. This completes the proof of the lemma. □

Next we define the L^2-based Sobolev spaces, H^s.

Definition 2.1. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ is the space of tempered distributions u with locally integrable Fourier transform, and with finite norm, with the norm defined as follows:

$$\|u\|^2_{H^s(\mathbb{R})} := \frac{1}{2\pi}\int \langle\xi\rangle^{2s}\left|\hat{u}(\xi)\right|^2 d\xi, \tag{2.6}$$

where $\langle\xi\rangle = \sqrt{(1+|\xi|^2)}$.

We can now state and prove a useful commutator estimate.

Proposition 2.2. *Let $s \in \mathbb{R}$ be given. Let $\sigma > 1/2$ be given. For any $\phi \in H^s(\mathbb{R})$ and $f \in H^\sigma$, the commutator $[H,\phi]f$ is in H^s, with the estimate*

$$\|[H,\phi]f\|_{H^s} \leq C\|\phi\|_{H^s}\|f\|_{H^\sigma}. \tag{2.7}$$

Proof. To compute the H^s-norm of the commutator, we first write the formula for $\langle\xi\rangle^s\widehat{[H,\phi]}f(\xi)$:

$$\begin{aligned}&\langle\xi\rangle^s\widehat{[H,\phi]}f(\xi) \qquad (2.8)\\ &= (2\pi)^{-1}\langle\xi\rangle^s\int\left(i\mathrm{sgn}(\xi)\widehat{\phi}(\xi-\eta)\widehat{f}(\eta) - i\mathrm{sgn}(\eta)\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\right)d\eta \\ &= \int\left((2\pi)^{-1}\langle\xi\rangle^s(i\mathrm{sgn}(\xi) - i\mathrm{sgn}(\eta))\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\right)d\eta.\end{aligned}$$

We multiply and divide:

$$\langle\xi\rangle^s\widehat{[H,\phi]f}(\xi)$$
$$=\int\Big((2\pi)^{-1}\langle\xi\rangle^s(i\mathrm{sgn}(\xi)-i\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s}\langle\eta\rangle^{-\sigma}\langle\xi-\eta\rangle^s\langle\eta\rangle^\sigma\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)\Big)\,d\eta.$$

Define $F(\xi,\eta)=(2\pi)^{-1}\langle\xi\rangle^s(i\mathrm{sgn}(\xi)-i\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s}\langle\eta\rangle^{-\sigma}$, and furthermore, make the auxiliary definitions $\phi_1(\xi)=\langle\xi\rangle^s\widehat{\phi}(\xi)$ and $f_1(\xi)=\langle\xi\rangle^\sigma\widehat{f}(\xi)$.

We wish to apply Lemma 2.1, so we begin to verify that condition (2.3) is satisfied:

$$\int|F(\xi,\eta)|^2d\eta=\int\left|(2\pi)^{-1}\langle\xi\rangle^s(i\mathrm{sgn}(\xi)-i\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{-s}\langle\eta\rangle^{-\sigma}\right|^2d\eta.$$

Notice that $\mathrm{sgn}(\xi)-\mathrm{sgn}(\eta)$ is nonzero only when ξ and η have opposite signs. Then, we have $|\xi-\eta|=|\xi|+|\eta|$. With this in mind, we continue:

$$\int|F(\xi,\eta)|^2d\eta\leq C\int\left|\langle\xi\rangle^s\langle|\xi|+|\eta|\rangle^{-s}\langle\eta\rangle^{-\sigma}\right|^2d\eta\leq C\int\langle\eta\rangle^{-2\sigma}d\eta\leq M^2.$$

Here, we have used the inequality $\langle\xi\rangle^s\langle|\xi|+|\eta|\rangle^{-s}\leq 1$ and the fact that the final integral converges for $\sigma>\dfrac{1}{2}$.

Therefore, we may apply Lemma 2.1, finding

$$\left\|\langle\xi\rangle^s\widehat{[H,\phi]f}(\xi)\right\|_{L^2}\leq M\|\phi_1\|_{L^2}\|f_1\|_{L^2}\leq M\|\phi\|_{H^s}\|f\|_{H^\sigma}. \tag{2.9}$$

This completes the proof of the proposition. □

We next generalize Proposition 2.2 somewhat in Proposition 2.3. The proof is similar to the proof of Proposition 2.2, so we omit it; details can be found in [23], however.

Proposition 2.3. *Let $s\in\mathbb{R}$ be given, and let $\gamma\in\mathbb{R}$ and $\sigma\in\mathbb{R}$ satisfy $s\leq\gamma$ and $s<\gamma+\sigma-1/2$. For any $\phi\in H^\gamma$ and for any $f\in H^\sigma$, we have*

$$\|[H,\phi]f\|_{H^s}\leq C\|\phi\|_{H^\gamma}\|f\|_{H^\sigma}. \tag{2.10}$$

We define the Fourier multiplier operator J through its symbol, $\widehat{Jf}(\xi)=\langle\xi\rangle\widehat{f}(\xi)$. We will make use of one more commutator estimate, which involves this operator J.

Proposition 2.4. *Let $s\geq 1$ and let $\sigma>3/2$. Let $\phi\in H^{s-1}$ and $f\in H^\sigma$. Then the following holds:*

$$\|J^s([H,\phi]f)-[H,J^s\phi]f\|_0\leq\|\phi\|_{s-1}\|f\|_\sigma. \tag{2.11}$$

Proof. To begin, we give the formula for the transform of $J^s([H,\phi]f) - [H, J^s\phi]f$:

$$\begin{aligned}&\langle\xi\rangle^s\widehat{[H,\phi]}f(\xi) - \widehat{[H,J^s\phi]}f(\xi)\\&= \int(2\pi)^{-1}(\langle\xi\rangle^s - \langle\xi-\eta\rangle^s)(i\mathrm{sgn}(\xi) - i\mathrm{sgn}(\eta))\widehat{\phi}(\xi-\eta)\widehat{f}(\eta)d\eta.\end{aligned}$$

We can then multiply and divide, using $\langle\xi-\eta\rangle^{1-s}\langle\eta\rangle^{-\sigma}\langle\xi-\eta\rangle^{s-1}\langle\eta\rangle^{\sigma}$. Similarly to before, then we define

$$F(\xi,\eta) = (2\pi)^{-1}(\langle\xi\rangle^s - \langle\xi-\eta\rangle^s)(i\mathrm{sgn}(\xi) - i\mathrm{sgn}(\eta))\langle\xi-\eta\rangle^{1-s}\langle\eta\rangle^{-\sigma},$$

and we again denote $\phi_1(\xi) = \langle\xi\rangle^{s-1}\widehat{\phi}(\xi)$ and $f_1(\xi) = \langle\xi\rangle^{\sigma}\widehat{f}(\xi)$. Since we again use Lemma 2.1, we compute the following:

$$\int|F(\xi,\eta)|^2 d\eta \leq C\int\left|(\langle|\xi|\rangle^s - \langle|\xi|+|\eta|\rangle^s)\langle|\xi|+|\eta|\rangle^{1-s}\langle\eta\rangle^{-\sigma}\right|^2 d\eta.$$

Define the function g by $g(\omega) = \langle\omega\rangle^s$, and let real numbers a and b such that $0 < a < b$ be given. Then, for some $\theta \in (0,1)$, we have $g(b) - g(a) = g'(a+\theta(b-a))(b-a)$. This implies the following:

$$\begin{aligned}g(b) - g(a) &= g'(a+\theta(b-a))(b-a)\\&= s\langle a+\theta(b-a)\rangle^{s-2}(a+\theta(b-a))(b-a)\\&\leq s\langle b\rangle^{s-1}(b-a).\end{aligned}$$

Here, to be able to use the fact that g' is non-decreasing, we have used the assumption $s \geq 1$. Continuing, we find the following:

$$\begin{aligned}&|\langle|\xi|\rangle^s - \langle|\xi|+|\eta|\rangle^s)\langle|\xi|+|\eta|\rangle^{1-s}\langle\eta\rangle^{-\sigma}\\&\leq s\langle|\xi|+|\eta|\rangle^{s-1}|\eta|\langle|\xi|+|\eta|\rangle^{1-s}\langle\eta\rangle^{-\sigma} \leq s\langle\eta\rangle^{1-\sigma}.\end{aligned}\tag{2.12}$$

Therefore when $\sigma - 1 > 1/2$, the above is uniformly bounded for all ξ. Our conclusion is

$$\begin{aligned}\|\langle\xi\rangle^s\widehat{[H,\phi]}f(\xi) - \widehat{[H,J^s\phi]}f(\xi)\|_{L^2} &\leq M\|\phi_1\|_{L^2}\|f_1\|_{L^2}\\&\leq M\|\phi\|_{H^{s-1}}\|f\|_{H^\sigma}.\end{aligned}\tag{2.13}$$

□

2.2. **Theorems of paradifferential calculus.** Our approach to proving our main theorem is to use the tools of paradifferential calculus, which was introduced by J.-M. Bony [10]. In this section we review some of the basic theory of paradifferential calculus, stating theorems without proof. We refer the reader to [1], [10], [20], [25], and [28] for the general theory. More specifically, the results we now state all may be found in either the book of Metivier [25] or the paper of Alazard, Burq, and Zuily [1].

Remark 1. We now state these results of paradifferential calculus in some generality, i.e., in d space dimensions. In the sequel, we apply all of the results with $d = 1$, as the free surface we consider is the one-dimensional boundary of a two-dimensional fluid.

For $m \in \mathbb{N}$, we denote by $W^{m,\infty}(\mathbb{R}^d)$ the space of functions in $L^\infty(\mathbb{R}^d)$ which are such that their derivatives of order up to m belong to $L^\infty(\mathbb{R}^d)$. For $\rho > 0$, $\rho \notin \mathbb{N}$, the space $W^{\rho,\infty}(\mathbb{R}^d)$ is the space of functions in $W^{[\rho],\infty}(\mathbb{R}^d)$ such that their derivatives of order $[\rho]$ belong to the Hölder spaces $W^{\rho-[\rho],\infty}(\mathbb{R}^d)$.

Now we give the definition of the symbol classes in paradifferential calculus; the reader could also refer to Definition 3.1 in [1].

Definition 2.2. Given $\rho \geq 0$ and $m \in \mathbb{R}$, $\Gamma^m_\rho(\mathbb{R}^d)$ denotes the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d \times (\mathbb{R}^d \backslash 0)$, which are C^∞ with respect to ξ for $\xi \neq 0$ such that, for all $\alpha \in \mathbb{N}^d$ and $\xi \neq 0$, the function $x \mapsto \partial^\alpha_\xi a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{R}^d)$ and there exists a constant C_α such that

$$\forall \quad |\xi| \geq 1/2, \quad \|\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}} \leq C_\alpha(1+|\xi|)^{m-|\alpha|}. \tag{2.14}$$

Remark 2. We consider symbols which are not C^∞ at the origin $\xi = 0$ since some symbols $a(x,\xi)$ we consider will be not smooth at this point (such as $a(x,\xi) = |\xi|$). This is not a problem since the low frequencies are irrelevant in the smoothness analysis and only contribute to remainders as noted in Section 6.4 of [25]. For such symbols, it is natural to introduce a cutoff function ψ so that if $a \in \Gamma^m_\rho$, then $\psi(\xi)a(x,\xi)$ is a C^∞ symbol for all ξ.

Given a symbol $a(x,\xi)$, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\psi(\eta)\hat{u}(\eta)d\eta. \tag{2.15}$$

Here, χ and ψ are fixed C^∞ functions which satisfy the following:
(i) there exist ϵ_1 and ϵ_2 such that $0 < \epsilon_1 < \epsilon_2 < 1$ and

$$\begin{cases} \psi(\eta) = 0 & |\eta| \leq 1, \\ \psi(\eta) = 1 & |\eta| \geq 2, \end{cases}$$

$$\begin{cases} \chi(\xi,\eta) = 1 & |\xi| \leq \epsilon_1|\eta|, \\ \chi(\xi,\eta) = 0 & |\xi| \geq \epsilon_2|\eta|, \end{cases}$$

(ii) for all $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$, there is $C_{\alpha,\beta}$ such that

$$\forall(\xi, \eta), \quad |\partial_\xi^\alpha \partial_\eta^\beta \chi(\xi, \eta)| \leq C_{\alpha,\beta}(1 + |\eta|)^{-|\alpha|-|\beta|}. \tag{2.16}$$

Definition 2.3. Given $\rho \geq 0$, $m \in \mathbb{R}$, and $a \in \Gamma_\rho^m(\mathbb{R}^d)$, we set

$$M_\rho^m(a; n) = \sup_{|\alpha| \leq n} \sup_{|\xi| \geq 1/2} \|(1 + |\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot, \xi)\|_{W^{\rho,\infty}}. \tag{2.17}$$

When $n = [d/2] + 1 + \lceil \rho \rceil$, $M_\rho^m(a, n)$ is simplified as $M_\rho^m(a)$, where $\lceil \rho \rceil$ is the smallest integer which is greater than ρ.

Definition 2.4. Let $m \in \mathbb{R}$. An operator T is said to be of order m if for all $\mu \in \mathbb{R}$, it is bounded from H^μ to $H^{\mu-m}$.

The following theorem appears as Theorem 3.6 in [1].

Theorem 2.5. *Let $m \in \mathbb{R}$. If $a \in \Gamma_\rho^m(\mathbb{R}^d)$, then T_a is of order m. Furthermore for all $\mu \in \mathbb{R}$, there is a constant C such that*

$$\|T_a u\|_{H^{\mu-m}} \leq C M_0^m(a) \|u\|_{H^\mu}. \tag{2.18}$$

Now, we give the two most important theorems in paradifferential calculus. The first theorem is about the composition of operators; this can be found as Theorem 6.1.4 in [25]. Let $m \in \mathbb{R}$ and $\rho > 0$.

Theorem 2.6 (Composition)**.** *For all $\mu \in \mathbb{R}$, there exists a constant C such that for $a \in \Gamma_\rho^m(\mathbb{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbb{R}^d)$ and $u \in H^{\mu+m+m'-\rho}$:*

$$\begin{aligned} &\|T_a T_b u - T_{a\sharp b} u\|_{H^\mu} \\ &\leq C(M_\rho^m(a; n) M_0^{m'}(b; n_0) + M_0^m(a; n) M_\rho^{m'}(b; n_0)) \|u\|_{H^{\mu+m+m'-\rho}}, \end{aligned} \tag{2.19}$$

with $n_0 = [d/2] + 1$, $n = n_0 + \lceil \rho \rceil$ and

$$a \sharp b = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|} \alpha!} \partial_\xi^\alpha a \partial_x^\alpha b. \tag{2.20}$$

Note the following inequality:

$$M_\rho^m(a; n) M_0^{m'}(b; n_0) + M_0^m(a; n) M_\rho^{m'}(b; n_0) \leq C M_\rho^m(a) M_\rho^{m'}(b).$$

Then, rewriting the inequality (2.19), we find

$$\|T_a T_b u - T_{a\sharp b} u\|_{H^\mu} \leq C M_\rho^m(a) M_\rho^{m'}(b) \|u\|_{H^{\mu+m+m'-\rho}}. \tag{2.21}$$

In the case that $\rho = 1$, notice that by equation (2.20), $a\sharp b = ab = b\sharp a$. Moreover, we then have

$$[T_a, T_b] \text{ is of order } m + m' - 1. \tag{2.22}$$

In particular, in the sequel, there is a useful special case. Let $a = a(\xi)$ be of order m, independent of x, and in Γ^m_ρ for any $\rho > 0$. Let $b = b(x) \in W^{1,\infty}$ be independent of ξ, so $b(x)$ is in Γ^0_1. Using (2.21) and (2.22), it follows that, for all $\mu \in \mathbb{R}$,

$$\|[T_a, T_b]\|_{H^{\mu+m-1} \to H^\mu} \leq C\|b(x)\|_{W^{1,\infty}}. \tag{2.23}$$

The second of the most important theorems of paradifferential calculus is about adjoint operators; for this, we can refer the reader to Theorem 6.2.4 in [25]. Denote by $(T_a)^*$ the adjoint operator of T_a and by $a^*(x,\xi)$the adjoint of $a(x,\xi)$.

Theorem 2.7 (Adjoint)**.** *For all $\mu \in \mathbb{R}$, there is a constant C such that for all $a \in \Gamma^m_\rho(\mathbb{R}^d)$ and $u \in H^{\mu+m-\rho}$, there holds*

$$\|(T_a)^* u - T_b u\|_{H^\mu} \leq C M^m_\rho(a)\|u\|_{H^{\mu+m-\rho}}, \tag{2.24}$$

with b given by

$$b(x,\xi) = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial^\alpha_x \partial^\alpha_\xi a^*(x,\xi). \tag{2.25}$$

Similarly to the composition theorem, the special case $\rho = 1$ is of interest; then, by equation (2.25), $b(x,\xi) = a^*(x,\xi)$. Moreover, we have that $(T_a)^* - T_{a^*}$ is of order $m-1$.

If $a = a(x)$ is independent of ξ, the paradifferential operator T_a is called a paraproduct. There are many useful results with respect to paraproducts. First, we quote a result concerning the case in which the symbol is only bounded (see Proposition 5.2.1 in [25]).

Theorem 2.8. *For all $a(x) \in L^\infty$ and for all s, there is a constant C such that*

$$\|T_a u\|_s \leq C\|a\|_{L^\infty}\|u\|_s. \tag{2.26}$$

Of course, the relationship between L^∞ and Sobolev spaces is that $a(x) \in L^\infty$ if $a \in H^r$ with $r > d/2$. One additional nice feature of paraproducts is that we may consider the case in which the symbol $a(x)$ is not in L^∞ but merely in the Sobolev space H^r with $r < d/2$. The following appears as Lemma 3.11 in [1].

Lemma 2.9. *Let $m > 0$. If $a \in H^{d/2-m}$ and $u \in H^\mu$, then*

$$\|T_a u\|_{\mu-m} \leq K\|a\|_{d/2-m}\|u\|_\mu, \tag{2.27}$$

for some constant K independent of a and u.

For the following two theorems about paraproducts, the reader can refer to Theorem 5.2.8 and Theorem 5.2.9 in [25].

Theorem 2.10. *Let r be a positive integer. There is a constant C such that for $a \in W^{r,\infty}$, the mapping $u \mapsto au - T_a u$ maps from L^2 to H^r and*

$$\|au - T_a u\|_{H^r} \leq C\|a\|_{W^{r,\infty}}\|u\|_{L^2}. \tag{2.28}$$

Theorem 2.11. *Let r be a positive integer. There is a constant C such that for $a \in W^{r,\infty}$ and $\alpha \in \mathbb{N}^d$ of length $|\alpha| \leq r$, the mapping $u \mapsto a\partial_x^\alpha u - T_a \partial_x^\alpha u$ maps from L^2 to L^2 and*

$$\|a\partial_x^\alpha u - T_a \partial_x^\alpha u\|_{L^2} \leq C\|a\|_{W^{r,\infty}}\|u\|_{L^2}. \tag{2.29}$$

Another key feature of paraproducts is that one can replace nonlinear expressions by paradifferential expressions, accounting for some smoother error terms. The reader can refer to Theorem 3.12 in [1].

Theorem 2.12. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha > d/2, \beta > d/2$. There exists a constant C such that for all $u \in H^\alpha(\mathbb{R}^d)$ and $v \in H^\beta(\mathbb{R}^d)$,*

$$\|uv - T_u v - T_v u\|_{\alpha+\beta-d/2} \leq C\|u\|_\alpha\|v\|_\beta. \tag{2.30}$$

Recall the usual estimate for the product of two functions (see Theorem 8.3.1 in [20]):

Theorem 2.13. *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha + \beta \geq 0$. If $u \in H^\alpha(\mathbb{R}^d)$ and $v \in H^\beta(\mathbb{R}^d)$, then $uv \in H^s$, where s is such that*

$$s \leq min\{\alpha, \beta\} \qquad \text{and} \qquad s \leq \alpha + \beta - d/2, \tag{2.31}$$

with the second inequality strict if α, β or $-s$ is equal to $d/2$.

3. Existence

Recall the Cauchy problem

$$\begin{cases} \partial_t u = \Lambda v - \partial_x[H, u]\Lambda v, \\ \partial_t v = -g\Lambda^p u + \dfrac{1}{2}(\Lambda v)^2 - \dfrac{1}{2}(v_x)^2, \end{cases}$$

with initial data

$$u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0,$$

where $(u_0, v_0) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$, $p \geq 3$ and $s > \max\{5/2, (p+1)/2\}$. We will follow the classic strategy to prove existence by first showing the local existence and uniqueness of a corresponding mollified system by the Picard theorem. We will then establish uniform estimates and use the continuation property of ODEs on a Banach space to show that all the solutions (for different values of the mollification parameter) exist uniformly in time. One of our pivotal techniques is to construct an intermediary variable, and then reformulate our system before mollifying.

3.1. **Reformulation of the system.** Let $u \in H^{s+(p-1)/2}, v \in H^s$. Inspecting the system, it will be difficult to do uniform estimates directly because of the presence of the nonlinear terms $\Lambda(u\Lambda v)$ and $(\Lambda v)^2$. So our first key step is to create paradifferential operators which allow us to rewrite these two terms. Given the para-products $T_{\Lambda v}, T_{v_x}$, we do some addition and subtraction to write our system as

$$\begin{cases} \partial_t u = \Lambda(v - T_{\Lambda v} u) - T_{v_x}\partial_x u + f_1(u,v), \\ \partial_t v = -g\Lambda^p u + T_{\Lambda v}\Lambda v - T_{v_x}\partial_x v + g(u,v), \end{cases} \tag{3.1}$$

where

$$f_1(u,v) = -\partial_x([H,u]\Lambda v) + \Lambda(T_{\Lambda v} u) + T_{v_x}\partial_x u, \tag{3.2}$$

$$g(u,v) = \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2 - T_{\Lambda v}\Lambda v + T_{v_x}\partial_x v. \tag{3.3}$$

Now, we will introduce an auxiliary variable w to reformulate the system (3.1). Let $w = v - T_{\Lambda v} u$, then $\partial_t w = \partial_t v - T_{\Lambda v}\partial_t u - T_{\Lambda v_t} u$. Plugging in the equations for u_t and v_t, the equation of w_t will be

$$\begin{aligned} \partial_t w = {} & -g\Lambda^p u + T_{\Lambda v}\Lambda v - T_{v_x}\partial_x v + g(u,v) \\ & - T_{\Lambda v}\left(\Lambda v - \partial_x([H,u]\Lambda v)\right) - T_{\Lambda v_t} u. \end{aligned} \tag{3.4}$$

After a cancellation, and denoting f_2 to be the remainder which is

$$f_2(u,v) = -T_{v_x}\partial_x(T_{\Lambda v} u) + T_{\Lambda v}\partial_x([H,u]\Lambda v) - T_{\Lambda v_t} u + g(u,v), \tag{3.5}$$

then the equation of w_t is

$$w_t = -g\Lambda^p u - T_{v_x}\partial_x w + f_2. \tag{3.6}$$

We conclude that if (u,v) solve the system (1.2) (or (3.1)), then (u,w) solve

$$\begin{cases} \partial_t u = \Lambda w - T_{v_x}\partial_x u + f_1(u,v), \\ \partial_t w = -g\Lambda^p u - T_{v_x}\partial_x w + f_2(u,v). \end{cases} \tag{3.7}$$

The system (3.7) will be helpful, especially when making uniform estimates.

3.2. **The approximate system and preliminary uniform estimates.** Let the mollification operator J_ϵ be defined as $J_\epsilon = (1 - \epsilon\Delta)^{-(p+1)/8}$. In the present work, the definition of paradifferential operators tells us $J_\epsilon \neq T_{J_\epsilon}$ and $J^s \neq T_{J^s}$ as in classical paradifferential calculus. (Note that we are abusing notation slightly, but we do not expect this will cause any confusion; we should more properly write $T_{\langle\xi\rangle^s}$ instead of T_{J^s}, but we believe the meaning is clear. We do this occassionally in the sequel for similar operators.) However the errors $J_\epsilon - T_{J_\epsilon}$ and $J^s - T_{J^s}$ correspond to the low frequencies which only contribute to remainders as we mentioned before. So, in the sequel, we do not distinguish them. Note that the operators T_{J_ϵ}, T_{J^s} are uniformly bounded operators of order zero.

We introduce our approximate version of system (1.2):

$$\left\{\begin{array}{l}\partial_t u = \Lambda J_\epsilon^2 v - \partial_x J_\epsilon [H, J_\epsilon u]\Lambda v,\\ \partial_t v = -g\Lambda^p J_\epsilon^2 u + \dfrac{1}{2} J_\epsilon^2 (\Lambda v)^2 - \dfrac{1}{2} J_\epsilon^2 (v_x)^2,\end{array}\right. \tag{3.8}$$

taken with the initial data $(u_0, v_0) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$.

For this approximate Cauchy problem, we have local existence and uniqueness by the Picard theorem.

Lemma 3.1. *For all $s > \max\{5/2, (p+1)/2\}$, for all $(u_0, v_0) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$, and for all $\epsilon > 0$, there exists $T_\epsilon > 0$ and (u^ϵ, v^ϵ) such that the system (3.8) with initial data (u_0, v_0) has unique solution $(u^\epsilon, v^\epsilon) \in C^1([0, T_\epsilon]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R}))$.*

Proof. Since J_ϵ is a smoothing operator, it is easy to check that the right hand side of system (3.1) is locally Lipschtiz from $H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$ to itself. By the Picard theorem, for any $\epsilon > 0$, there is $T_\epsilon > 0$ such that the system (3.8) has a unique solution $(u^\epsilon, v^\epsilon) \in C^1([0, T_\epsilon]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R}))$. □

Now, fix ϵ, and let (u, v) be the solution of the system (3.8) with initial data (u_0, v_0). As before, we let $w = v - T_{\Lambda v} u$; we see then that (u, w) satisfies the following approximate system:

$$\left\{\begin{array}{l}\partial_t u = \Lambda J_\epsilon^2 w - J_\epsilon (T_{v_x} J_\epsilon \partial_x u) + f_{1,\epsilon}(u, v),\\ \partial_t w = -g\Lambda^p J_\epsilon^2 u - J_\epsilon (T_{v_x} J_\epsilon \partial_x w) + f_{2,\epsilon}(u, v),\end{array}\right. \tag{3.9}$$

with the approximate $f_1(u, v)$, $f_2(u, v)$ and $g(u, v)$ defined as follows:

$$f_{1,\epsilon}(u, v) = -J_\epsilon(\partial_x([H, J_\epsilon u]\Lambda v)) + \Lambda J_\epsilon^2 (T_{\Lambda v} u) + J_\epsilon (T_{v_x} \partial_x J_\epsilon u),$$

$$f_{2,\epsilon}(u, v) = -J_\epsilon (T_{v_x} \partial_x J_\epsilon (T_{\Lambda v} u)) + T_{\Lambda v} \partial_x J_\epsilon ([H, J_\epsilon u]\Lambda v) - T_{\Lambda v_t} u + g_\epsilon(u, v),$$

$$g_\epsilon(u, v) = \frac{1}{2} J_\epsilon^2 (\Lambda v)^2 - \frac{1}{2} J_\epsilon^2 (v_x)^2 - T_{\Lambda v} J_\epsilon^2 \Lambda v + J_\epsilon (T_{v_x} \partial_x J_\epsilon v).$$

Now we prove that the above three approximate quantities are uniformly bounded for any $\epsilon \geq 0$; that is, the following estimates hold also without approximation ($\epsilon = 0$). Note that in the following arguments, σ will always be taken to satisfy $\sigma > 1/2$.

Lemma 3.2. *There exists a constant $C > 0$ such that for all $(u, v) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$,*

$$\|f_{1,\epsilon}(u, v)\|_{s+(p-1)/2} \leq C \|u\|_{s+(p-1)/2} \|v\|_s. \tag{3.10}$$

Proof. We rewrite $f_{1,\epsilon}(u,v)$ by addition and subtraction; using the triangle inequality, we have the following:

$$\|f_{1,\epsilon}(u,v)\|_{s+(p-1)/2} \leq T_{1,\epsilon} + T_{2,\epsilon} + T_{3,\epsilon} + T_{4,\epsilon}, \tag{3.11}$$

where

$$\begin{aligned}
T_{1,\epsilon} &= \left\|-J^{s+(p-1)/2}\partial_x J_\epsilon([H, J_\epsilon u]\Lambda v) + \partial_x J_\epsilon([H, J^{s+(p-1)/2}J_\epsilon u]\Lambda v)\right\|_0, \\
T_{2,\epsilon} &= \left\|\partial_x([H, J^{s+(p-1)/2}J_\epsilon u]\Lambda v) - \Lambda T_{\Lambda v}J^{s+(p-1)/2}J_\epsilon u - T_{\partial_x v}J^{s+(p-1)/2}J_\epsilon \partial_x u\right\|_0, \\
T_{3,\epsilon} &= \left\|\Lambda[T_{\Lambda v}, J^{s+(p-1)/2}]J_\epsilon u\right\|_0 + \left\|[T_{\partial_x v}, J^{s+(p-1)/2}]J_\epsilon \partial_x u\right\|_0, \\
T_{4,\epsilon} &= \left\|\Lambda J^{s+(p-1)/2}[J_\epsilon, T_{\Lambda v}]u\right\|_0.
\end{aligned}$$

Then we will estimate the above four terms.

First, applying the derivative and using the triangle inequality,

$$\begin{aligned}
T_{1,\epsilon} \leq{}& \left\|-J^{s+(p-1)/2}J_\epsilon([H, \partial_x J_\epsilon u]\Lambda v) + J_\epsilon([H, J^{s+(p-1)/2}\partial_x J_\epsilon u]\Lambda v)\right\|_0 \\
&+ \left\|-J^{s+(p-1)/2}J_\epsilon([H, J_\epsilon u]\Lambda \partial_x v)\right\|_0 + \left\|J_\epsilon([H, J^{s+(p-1)/2}J_\epsilon u]\Lambda \partial_x v)\right\|_0.
\end{aligned} \tag{3.12}$$

We remark that σ will be chosen at the end of the proof; for now, it is sufficient to know that we will take $\sigma > 1/2$, as mentioned previously. By Proposition 2.4, we may estimate the first term on the right-hand side of (3.12) as follows:

$$\begin{aligned}
&\left\|-J^{s+(p-1)/2}([H, J_\epsilon u_x]\Lambda v) + [H, J^{s+(p-1)/2}J_\epsilon u_x]\Lambda v\right\|_0 \\
&\qquad \leq C\|J_\epsilon u_x\|_{s+(p-1)/2-1}\|\Lambda v\|_{1+\sigma} \leq C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}.
\end{aligned} \tag{3.13}$$

And by Proposition 2.2, the second and third terms on the right-hand side of (3.12) may be estimated as follows:

$$\begin{aligned}
&\left\|-J^{s+(p-1)/2}([H, J_\epsilon u]\Lambda v_x)\right\|_0 + \left\|[H, J^{s+(p-1)/2}J_\epsilon u]\Lambda v_x\right\|_0 \\
&\qquad \leq C\|u\|_{s+(p-1)/2}\|\Lambda v_x\|_\sigma + \|J^{s+(p-1)/2}u\|_0\|\Lambda v_x\|_\sigma \\
&\qquad\qquad \leq C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}.
\end{aligned} \tag{3.14}$$

This concludes our consideration of $T_{1,\epsilon}$.

For $T_{2,\epsilon}$, we first rewrite $\partial_x([H, J^{s+(p-1)/2}J_\epsilon u]\Lambda v)$:

$$\begin{aligned}
\partial_x([H, J^{s+(p-1)/2}J_\epsilon u]\Lambda v) ={}& \Lambda((J^{s+(p-1)/2}J_\epsilon u)\Lambda v) \\
&+ (J^{s+(p-1)/2}J_\epsilon u_x)v_x + (J^{s+(p-1)/2}J_\epsilon u)v_{xx}.
\end{aligned}$$

By the definition of $T_{2,\epsilon}$ and by the triangle inequality,

(3.15)
$$T_{2,\epsilon} \leq \|\Lambda(\Lambda v J^{s+(p-1)/2} J_\epsilon u - T_{\Lambda v} J^{s+(p-1)/2} J_\epsilon u)\|_0 + \|(J^{s+(p-1)/2} J_\epsilon u) v_{xx}\|_0 + \|(J^{s+(p-1)/2} J_\epsilon u_x) v_x - T_{v_x} J^{s+(p-1)/2} J_\epsilon \partial_x u\|_0.$$

To estimate $T_{2,\epsilon}$, we estimate the three terms on the right-hand side of (3.15) individually. First, by the properties of the Hilbert transform,

$$\|\Lambda(\Lambda v J^{s+(p-1)/2} J_\epsilon u - T_{\Lambda v} J^{s+(p-1)/2} J_\epsilon u)\|_0 = \|\partial_x(\Lambda v J^{s+(p-1)/2} J_\epsilon u - T_{\Lambda v} J^{s+(p-1)/2} J_\epsilon u)\|_0.$$

So the first term on the right-hand side of (3.15) is bounded by

$$\|(\Lambda v_x) J^{s+(p-1)/2} J_\epsilon u\|_0 + \|[T_{\Lambda v}, \partial_x] J^{s+(p-1)/2} J_\epsilon u\|_0 + \|(\Lambda v - T_{\Lambda v}) \partial_x J^{s+(p-1)/2} J_\epsilon u\|_0.$$

The above three terms are less than $C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}$, by Theorem 2.13, by the fact that $[T_{\Lambda v}, \partial_x]$ is of order 0 according to (2.22), and by Theorem 2.11 respectively.

Next, the second and the third terms on the right-hand side of (3.15) are less than $C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}$ by Theorem 2.13 and Theorem 2.11 respectively. We thus conclude that

$$T_{2,\epsilon} \leq C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}.$$

Finally, we estimate $T_{3,\epsilon}$ and $T_{4,\epsilon}$. Noting that $T_{\Lambda v}$ and T_{v_x} are in Γ_1^0, by Theorem 2.6 and (2.22), we get

$$\begin{aligned} &\|[T_{\Lambda v}, J^{s+(p-1)/2}] J_\epsilon u\|_1 + \|[J_\epsilon, T_{\Lambda v}] u\|_{s+(p-1)/2+1} + \|[T_{v_x}, J^{s+(p-1)/2}] \partial_x J_\epsilon u)\|_0 \\ &\leq C\|\Lambda v\|_{W^{1,\infty}}\|u\|_{s+(p-1)/2} + C\|v_x\|_{W^{1,\infty}}\|u\|_{s+(p-1)/2} \\ &\leq C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}. \end{aligned}$$

Therefore, we conclude

$$\|f_{1,\epsilon}(u,v)\|_{s+(p-1)/2} \leq C\|u\|_{s+(p-1)/2}\|v\|_{2+\sigma}. \tag{3.16}$$

Letting $\sigma = s-2$, this completes the proof. □

Lemma 3.3. *There exists a nondecreasing function C such that for all $(u,v) \in H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})$,*

$$\|f_{2,\epsilon}(u,v)\|_s \leq C(\|u\|_{s+(p-1)/2}, \|v\|_s). \tag{3.17}$$

Proof. We first deal with $g_\epsilon(u,v)$. By addition and subtraction and the triangle inequality,

$$\|g_\epsilon(u,v)\|_s \le \left\| J_\epsilon^2 \left(\frac{1}{2} (\Lambda v)^2 - T_{\Lambda v} \Lambda v \right) \right\|_s + \left\| -\frac{1}{2} J_\epsilon^2 (v_x)^2 + J_\epsilon^2 (T_{v_x} v_x) \right\|_s + \left\| \left[J_\epsilon^2, T_{\Lambda v} \right] \Lambda v \right\|_s + \| [J_\epsilon, T_{v_x}] v_x \|_s . \tag{3.18}$$

By Theorem 2.12 (taking $\alpha = 1 + 1/2 < 1 + \sigma$ and $\beta = s - 1$ there), the first two terms on the right-hand side of (3.18) can be bounded as follows:

$$\left\| J_\epsilon^2 \left(\frac{1}{2} (\Lambda v)^2 - T_{\Lambda v} \Lambda v \right) \right\|_s + \left\| -\frac{1}{2} J_\epsilon^2 (v_x)^2 + J_\epsilon^2 (T_{v_x} v_x) \right\|_s \le C \|\Lambda v\|_{s-1} \|\Lambda v\|_{1+\sigma} + C \|v_x\|_{s-1} \|v_x\|_{1+\sigma} .$$

Again, by Theorem 2.6 and (2.22), the last two terms on the right-hand side of (3.18) can be estimated as follows:

$$\|[J_\epsilon^2, T_{\Lambda v}]\Lambda v\|_s + \|[J_\epsilon, T_{v_x}]v_x\|_s \le C\|\Lambda v\|_{s-1}\|\Lambda v\|_{1+\sigma} + C\|v_x\|_{s-1}\|v_x\|_{1+\sigma}.$$

Now we can make the conclusion that

$$\|g_\epsilon(u,v)\|_s \le C\|v\|_s\|v\|_{2+\sigma}.$$

By the triangle inequality,

$$\|f_{2,\epsilon}(u,v)\|_s \le \|J_\epsilon(T_{v_x}\partial_x J_\epsilon(T_{\Lambda v}u))\|_s + \|T_{\Lambda v}\partial_x J_\epsilon([H, J_\epsilon u]\Lambda v)\|_s + \|T_{\Lambda v_t}u\|_s + \|g_\epsilon(u,v)\|_s. \tag{3.19}$$

By Theorem 2.8 and since $p \ge 3$, the first term on the right-hand side of (3.19) can be bounded as follows:

$$\begin{aligned}\|T_{v_x}\partial_x J_\epsilon(T_{\Lambda v}u)\|_s &\le C\|v_x\|_{L^\infty}\|J_\epsilon(T_{\Lambda v}\partial_x u)\|_s + C\|v_x\|_{L^\infty}\|J_\epsilon[\partial_x, T_{\Lambda v}]u\|_s \\ &\le C\|v_x\|_{L^\infty}\|\Lambda v\|_{L^\infty}\|u_x\|_s + C\|v_x\|_{L^\infty}\|v_x\|_{W^{1,\infty}}\|u\|_s \\ &\le C\|v\|_{2+\sigma}^2\|u\|_{s+1} \le C\|v\|_{2+\sigma}^2\|u\|_{s+(p-1)/2}.\end{aligned}$$

The second term on the right-hand side of (3.19) is also less that $C\|v\|_{2+\sigma}^2\|u\|_{s+(p-1)/2}$ because by Theorem 2.8,

$$\|T_{\Lambda v}J_\epsilon\partial_x([H, J_\epsilon u]\Lambda v)\|_s \le C\|\Lambda v\|_{L^\infty}\|J_\epsilon\partial_x([H, J_\epsilon u]\Lambda v)\|_s,$$

and by Proposition 2.2

$$\|J_\epsilon\partial_x([H, J_\epsilon u]\Lambda v)\|_s \le \|[H, J_\epsilon u]\Lambda v\|_{s+1} \le C\|u\|_{s+1}\|v\|_{1+\sigma}.$$

By Lemma 2.9 , the third term on the right-hand side of (3.19) can be bounded as follows:

$$\|T_{\Lambda v_t} u\|_s \le \|\Lambda v_t\|_{1/2-(p-1)/2} \|u\|_{s+(p-1)/2}$$
$$\le C \left\| -g\Lambda^p u + \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2 \right\|_{3/2-(p-1)/2} \|u\|_{s+(p-1)/2}.$$

Finally, we make the estimates

$$\| -g\Lambda^p u\|_{3/2-(p-1)/2} \le C\|u\|_{2+p/2}$$

and

$$\left\| \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2 \right\|_{3/2-(p-1)/2} \le C\|v\|^2_{2+\sigma}.$$

Since $s > 5/2$, we have $\sigma = s-2 > 1/2$, moreover $2+p/2 < s+(p-1)/2$. Therefore, we have shown that there exists a nondecreasing function such that

$$\|f_{2,\epsilon}(u,v)\|_s \le C(\|u\|_{s+(p-1)/2}, \|v\|_s).$$

□

3.3. **Uniform estimates.** The main result of this section is the following proposition.

Proposition 3.4. *Let $s > \max\{5/2, (p+1)/2\}$. There exists a nondecreasing function $C(x)$ such that, for all $\epsilon \in (0,1]$, for (u,v) satisfying*

$$(u,v) \in C^1([0,T]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R})) \tag{3.20}$$

and such that (u,v) is the unique solution of the Cauchy problem for the approximate system (3.8) with initial data (u_0,v_0), the estimate

$$M(T) \le C(M_0) + TC(M(T)) \tag{3.21}$$

is satisfied, with the norm

$$M(T) = \|(u,v)\|_{L^\infty([0,T];H^{s+(p-1)/2}(\mathbb{R})\times H^s(\mathbb{R}))}, \tag{3.22}$$

and the definition $M_0 = \|(u_0,v_0)\|_{H^{s+(p-1)/2}(\mathbb{R})\times H^s(\mathbb{R})}$.

Instead of dealing with the original equation of (u,v) directly, we make uniform estimates by means of the corresponding energy estimates for (u,w). Now, recall if (u,v) is the solution of the approximate system, then (u,w) satisfies system (3.9).

Define the energy with respect to (u,w) :

$$E = E(u(t), w(t)) = E_0 + E_1, \tag{3.23}$$

where

$$E_0 = \frac{1}{2}\|w\|_s^2 + g\frac{1}{2}\|u\|_0^2,$$

and

$$E_1 = \frac{1}{2} g \int (J^s \Lambda^{(p-1)/2} u) J^s \Lambda^{(p-1)/2} u \ dx.$$

Our first step is to prove the following claim.

Claim 1. The energy of (u, w) satisfies $\dfrac{dE^{1/2}}{dt} \leq C(M(T))$.

The proof of Claim 1. First, we take the time derivative of $\|u\|_0^2$:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int u^2 \ dx &= \int (\Lambda J_\epsilon^2 v) u \ dx - \int (\partial_x J_\epsilon([H, J_\epsilon u]\Lambda v))\, u \ dx \\ &\leq C\|v\|_1 \|u\|_0 + C\|u\|_0 \|v\|_{1+\sigma} \|u\|_1.\end{aligned}$$

Second, we take the time derivative of $\frac{1}{2}\|w\|_s^2$:

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int (J^s w) J^s w \ dx \qquad (3.24)\\ &= \int \left(J^s \left(-g\Lambda^p J_\epsilon^2 u - J_\epsilon(T_{v_x}\partial_x J_\epsilon w) + f_{2,\epsilon}(u,v)\right)\right) J^s w \ dx \\ &= -g\int (J^s \Lambda^p J_\epsilon u)\, J^s J_\epsilon w \ dx - \int (J^s(T_{v_x}\partial_x J_\epsilon w))\, J^s J_\epsilon w \ dx \\ &\qquad + \int (J^s f_{2,\epsilon}(u,v))\, J^s w \ dx.\end{aligned}$$

Notice that first term on the right-hand side of (3.24) cannot simply be bounded in terms of the energy; it involves a total of $2s+p$ derivatives on a product of u and w. On such a product, we are able to bound at most a total of $2s + (p-1)/2$ derivatives of u and w by the energy, so this is too many. This contribution to the energy will be dealt with via a cancellation. To that end, we take the time derivative of E_1 :

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2} g \int \left(J^s \Lambda^{(p-1)/2} u\right) J^s \Lambda^{(p-1)/2} u \ dx \qquad (3.25)\\ &= g\int \left(J^s \Lambda^{(p-1)/2}(\Lambda J_\epsilon^2 w - J_\epsilon(T_{v_x}\partial_x J_\epsilon u) + f_{1,\epsilon}(u,v))\right) J^s \Lambda^{(p-1)/2} u \ dx \\ &= g\int \left(J^s \Lambda^{(p+1)/2} J_\epsilon w\right) J^s \Lambda^{(p-1)/2} J_\epsilon u \ dx \\ &\quad - g\int \left(J^s \Lambda^{(p-1)/2}(T_{v_x}\partial_x J_\epsilon u)\right) J^s \Lambda^{(p-1)/2} J_\epsilon u \ dx \\ &\qquad + g\int \left(J^s \Lambda^{(p-1)/2} f_{1,\epsilon}(u,v)\right) J^s \Lambda^{(p-1)/2} u \ dx.\end{aligned}$$

Again, the first term on the right-hand side of (3.25) has too many derivatives to be bounded in terms of the energy. In fact, this term cancels with the previous term (from (3.24)) upon adding (simply by using the self-adjointedness of the relevant operators).

Observe that to finish the proof of our Claim 1, we turn out to need to estimate four terms which remain when adding (3.24) and (3.25) (after observing the remarked-upon cancellation). Note that we must bound w in terms of (u, v); for this purpose, we may simply use the definition $w = v - T_{\Lambda v} u$ and Theorem 2.8 to find the estimate

$$\|w\|_s \leq \|v\|_s + \|v\|_s \|u\|_s. \tag{3.26}$$

By the Hölder inequality, it is obvious that

$$\left| \int \left(J^s f_{2,\epsilon}(u, v)\right) J^s w \; dx \right| \leq C \|f_{2,\epsilon}\|_s \|w\|_s,$$

and

$$\left| \int \left(J^s \Lambda^{(p-1)/2} f_{1,\epsilon}(u, v)\right) J^s \Lambda^{(p-1)/2} u \; dx \right| \\ \leq C \|f_{1,\epsilon}(u, v)\|_{s+(p-1)/2} \|u\|_{s+(p-1)/2}.$$

The requisite bound in terms of $M(T)$ follows from these inequalities, from (3.26), Lemma 3.2, and Lemma 3.3.

Now it is similar to deal with the remaining terms,

$$-\int \left(J^s (T_{v_x} \partial_x J_\epsilon w)\right) J^s J_\epsilon w \; dx$$

and

$$-g \int \left(J^s \Lambda^{(p-1)/2} (T_{v_x} \partial_x J_\epsilon u)\right) J^s \Lambda^{(p-1)/2} J_\epsilon u \; dx.$$

We only give details of one of these as an example. First, we have the following expansion:

$$J^s T_{v_x} \partial_x = J_s [T_{v_x}, \partial_x] + \partial_x [J_s, T_{v_x}] + \partial_x (T_{v_x} - T^*_{v_x}) J_s + \partial_x T^*_{v_x} J_s.$$

For these commutators, we know that $[T_{v_x}, \partial_x]$ is of order zero, with estimate $\|[T_{v_x}, \partial_x]\|_{H^s \to H^s} \leq C \|v_x\|_{W^{1,\infty}}$, and $[J^s, T_{v_x}]$ is of order $s - 1$, with estimate $\|[J_s, T_{v_x}]\|_{H^s \to H^1} \leq C \|v_x\|_{W^{1,\infty}}$; these estimates follow from Theorem 2.6 and the discussion which follows that theorem. Furthermore, we know the estimate $\|T_{v_x} - T^*_{v_x}\|_{L^2 \to H^1} \leq C \|v_x\|_{W^{1,\infty}}$ by

Theorem 2.7. Therefore, we have the following:

$$\begin{aligned}
&\int \left(J^s T_{v_x} \partial_x J_\epsilon w\right) J^s J_\epsilon w \ dx \\
&\leq C\|v_x\|_{W^{1,\infty}} \|w\|_s^2 + \int \left(\partial_x T^*_{v_x} J^s J_\epsilon w\right) J^s J_\epsilon w \ dx \\
&\leq C\|v_x\|_{W^{1,\infty}} \|w\|_s^2 - \int \left(J^s J_\epsilon w\right) T_{v_x} J^s \partial_x J_\epsilon w \ dx.
\end{aligned}$$

We introduce one more commutator of the form $[T_{v_x}, J^s]$, so that the final integral on the right-hand side matches the integral on the left-hand side; we then make the following conclusion:

$$\left| \int J^s T_{v_x} \partial_x J_\epsilon w J^s J_\epsilon w \ dx \right| \leq C\|v\|_s \|w\|_s^2.$$

Similar considerations yield the following as well:

$$\left| - \int \left(J^s \Lambda^{(p-1)/2} J_\epsilon (T_{v_x} \partial_x J_\epsilon u)\right) J^s \Lambda^{(p-1)/2} u \ dx \right| \leq C\|v\|_s \|u\|^2_{s+(p-1)/2}.$$

Thus, we have proved the claim:

$$\frac{dE}{dt} \leq C(M(T)) E^{1/2},$$

and thus

$$\frac{dE^{1/2}}{dt} \leq C(M(T)).$$

□

The proof of Proposition 3.4. By the claim, we have

$$\sup_{[0,T]} (E(t))^{1/2} \leq C(M_0) + C(M(T))T.$$

This immediately implies that

$$\|u\|_{s+(p-1)/2} \leq C(M_0) + C(M(T))T. \tag{3.27}$$

By Lemma 2.9, and using the definition of w, we have the following:

$$\begin{aligned}
&\|v\|_s \leq \|w\|_s + \|T_{\Lambda v} u\|_s \\
&\leq C(M_0) + C(M(T))T + \|v\|_{3/2-(p-1)/2} \|u\|_{s+(p-1)/2} \\
&\leq C(M_0) + C(M(T))T + \|v\|_{2-p/2} \|u\|_{s+(p-1)/2}.
\end{aligned}$$

To finish the proof, we show that $\|v\|_{2-p/2} \leq C(M_0) + C(M(T))T$:

$$\|v\|_{2-p/2} \leq \|v_0\|_{2-p/2} + \int_0^T \|v_t\|_{2-p/2}\, dt$$
$$\leq \|v_0\|_s + \int_0^T C(M(T))\, dt \leq C(M_0) + C(M(T))T.$$

□

Lemma 3.5. *There exists* $T_0 > 0$ *such that for all* $\epsilon \in (0,1]$, *we may take* $T_\epsilon \geq T_0$. *Furthermore, the sequence* $\{(u^\epsilon, v^\epsilon)\}$ *is bounded in* $C^0([0,T_0]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R}))$.

Proof. For any $\epsilon \in (0,1]$ and $T < T_\epsilon$, $(u^\epsilon, v^\epsilon) \in C^1([0,T]; H^{s+(p-1)/2} \times H^s)$ is the solution of system (1.2). Let

$$M_\epsilon(T) = \|(u^\epsilon, v^\epsilon)\|_{L^\infty([0,T];H^{s+(p-1)/2}\times H^s)}. \tag{3.28}$$

Claim 2. Let $\epsilon \in (0,1]$. There exists a constant $M_1 > 0$ and $T_0 > 0$ such that if $T \in [0, \min\{T_0, T_\epsilon\}]$, then $M_\epsilon(T) < M_1$.

The proof of Claim 2 . By Proposition 3.4, for all $T < T_\epsilon$, there exists a nondecreasing continuous function $C(x)$ such that

$$M_\epsilon(T) \leq C(M_0) + C(M_\epsilon(T))T. \tag{3.29}$$

Let $M_1 = 2C(M_0)$ and choose $0 < T_0 \leq 1$ small enough such that $C(M_0) + T_0 C(M_1) < M_1$. First, notice that $M_\epsilon(0) = M_0 \leq C(M_0) < M_1$.

We prove the claim by a contradiction argument. If the claim is not true, there exists $T \in [0, \min\{T_0, T_\epsilon\}]$ such that $M_\epsilon(T) = M_1$. On the other hand,

$$M_1 = M_\epsilon(T) = C(M_0) + TC(M_\epsilon(T)) \leq C(M_0) + T_0 C(M_1) < M_1. \tag{3.30}$$

We get the desired contradiction. □

The local existence time has thus far depended on ϵ. By the continuation theorem, a uniform existence time now follows by Claim 2. This completes the proof of Lemma 3.5. Moreover,

$$\sup_{\epsilon\in(0,1]} \sup_{T\in[0,T_0]} M_\epsilon(T) \leq M_1. \tag{3.31}$$

□

Proposition 3.6. *There exists* $0 < T_1 \leq T_0$ *such that* $\{(u^\epsilon, v^\epsilon)\}$ *is a Cauchy sequence in* $C^0([0,T_1]; H^{(p-1)/2}(\mathbb{R}) \times H^0(\mathbb{R}))$.

We omit the proof of Proposition 3.6 because the proof is entirely similar to the proof of uniqueness, which we provide in the subsequent section. Both results require estimating the norm of a difference; for Proposition 3.6, we estimate the norm of two solutions with different values of the regularization parameter. For uniqueness, by contrast, we estimate the norm of two solutions with possibly different initial data. The only difference is that the proof of Proposition 3.6 requires dealing with terms involving $J_{\epsilon_1} - J_{\epsilon_2}$. The following fact allows these to be estimated in a straightforward way.

Lemma 3.7. *For any* $0 < \epsilon_1 < \epsilon_2$ *and* $m > 0$*, we have*

$$\|J_{\epsilon_1} - J_{\epsilon_2}\|_{H^\mu \to H^{\mu-m}} \leq C\epsilon_2^m. \tag{3.32}$$

While we omit the full details of the proof of Proposition 3.6, we will provide some details after the proof of Proposition 4.1.

3.4. **Continuity in time.** In the previous subsection, we have proved the existence of a solution for the Cauchy problem in $C^0([0,T]; H^{(p-1)/2} \times H^0) \bigcap L^\infty([0,T]; H^{s+(p-1)/2} \times H^s)$. By interpolation, for any $s' < s$, it is true that $(u,v) \in C^0([0,T]; H^{s'+(p-1)/2} \times H^{s'})$. We now prove that the solution (u,v) is continuous in time in $H^{s+(p-1)/2} \times H^s$. To prove this, we will introduce the mollifier $J_\epsilon = (1-\epsilon\Delta)^{-(p+1)/4}$ which is of order $-(p+1)/2 \leq -2$ since $p \geq 3$. We know that $(J_\epsilon u, J_\epsilon v) \to (u,v)$ in $L^2([0,T]; H^{s'+(p-1)/2} \times H^{s'})$. So to prove $(u,v) \in C^0([0,T]; H^{s+(p-1)/2} \times H^s)$, by uniqueness of limits, we only need to prove that $(J_\epsilon u, J_\epsilon v)$ is a Cauchy sequence in $C^0([0,T]; H^{s+(p-1)/2} \times H^s)$. Now we first give the equation of $(J_\epsilon u, J_\epsilon v)$:

$$\begin{cases} \partial_t J_\epsilon u = \Lambda J_\epsilon v - J_\epsilon \partial_x [H,u]\Lambda v, \\ \partial_t J_\epsilon v = -g\Lambda^p J_\epsilon u + \dfrac{1}{2} J_\epsilon (\Lambda v)^2 - \dfrac{1}{2} J_\epsilon (v_x)^2, \end{cases}$$

with initial data

$$u(\cdot,0) = u_0, \quad v(\cdot,0) = v_0.$$

For any $\epsilon > 0$, because of the presence of J_ϵ, we know that $(\partial_t J_\epsilon u, \partial_t J_\epsilon v) \in L^1([0,T]; H^{s+(p-1)/2} \times H^s)$. This guarantees that for all ϵ, $(J_\epsilon u, J_\epsilon v)$ is in $C^0([0,T]; H^{s+(p-1)/2} \times H^s)$. We also shall introduce the auxiliary variable $w_\epsilon = J_\epsilon v - T_{\Lambda v}(J_\epsilon u)$. For any ϵ, $w_\epsilon \in C^0([0,T]; H^s)$ since $(J_\epsilon u, J_\epsilon v)$ is continuous in time and $v \in C^0([0,T]; H^{s'})$, and by Lemma 2.9. We then turn to prove that in the space $C^0([0,T]; H^{s+(p-1)/2} \times H^s)$, $(J_\epsilon u(t), w_\epsilon(t))$ is a Cauchy sequence. This will immediately imply the desired result for $J_\epsilon v(t)$, since $v \in C^0([0,T]; H^{s'})$ and

$$J_{\epsilon_1} v - J_{\epsilon_2} v = w_{\epsilon_1} - w_{\epsilon_2} + T_{\Lambda v}(J_{\epsilon_1} u - J_{\epsilon_2} u). \tag{3.33}$$

We rewrite the system for $(J_\epsilon u, J_\epsilon v)$:

$$\begin{cases} \partial_t J_\epsilon u = \Lambda(J_\epsilon v - T_{\Lambda v} J_\epsilon u) - T_{v_x}\partial_x J_\epsilon u + f_{1,\epsilon}(u,v), \\ \partial_t J_\epsilon v = -g\Lambda^p J_\epsilon u + J_\epsilon T_{\Lambda v}\Lambda v - J_\epsilon T_{v_x}\partial_x v + J_\epsilon g(u,v), \end{cases}$$

where $f_{1,\epsilon}$ is

$$f_{1,\epsilon} = -[J_\epsilon, T_{v_x}]\partial_x u - \Lambda[J_\epsilon, T_{\Lambda v}]u + J_\epsilon f_1(u,v).$$

Similarly to the system (3.7), the equation of $(J_\epsilon u, J_\epsilon w)$ follows:

$$\begin{cases} \partial_t J_\epsilon u = \Lambda J_\epsilon w - T_{v_x}\partial_x J_\epsilon u + f_{1,\epsilon}(u,v), \\ \partial_t w_\epsilon = -g\Lambda^p J_\epsilon u - T_{v_x}\partial_x w_\epsilon + f_{2,\epsilon}(u,v), \end{cases} \tag{3.34}$$

where

$$\begin{aligned} f_{2,\epsilon} = -[J_\epsilon, T_{v_x}]\partial_x v + [J_\epsilon, T_{v_x}\partial_x T_{\Lambda v}]u + [J_\epsilon, T_{\Lambda v}]\Lambda v \\ - [J_\epsilon, T_{\Lambda v}](\partial_x[H,u]\Lambda v) + [J_\epsilon, T_{\Lambda v_t}]u + J_\epsilon f_2(u,v). \end{aligned}$$

Now we do an energy estimate for $(J_\epsilon u, w_\epsilon)$ as before. We define the energy

$$E_\epsilon = \frac{1}{2}\|w_\epsilon\|_s^2 + \frac{g}{2}\|J_\epsilon u\|_0^2 + \frac{g}{2}\int (J^s \Lambda^{(p-1)/2} J_\epsilon u)^2 \, dx. \tag{3.35}$$

We omit the details of proof the energy estimate (the evolutionary system has the same structure as before), but the conclusion is the following bound:

$$\frac{dE_\epsilon}{dt} \le cE_\epsilon + \|f_{1,\epsilon}\|_{s+(p-1)/2}^2 + \|f_{2,\epsilon}\|_s^2.$$

By the Grönwall inequality, we then have

$$E_\epsilon \le e^{ct}\left(E_\epsilon(0) + \int_0^t \|f_{1,\epsilon}\|_{s+(p-1)/2}^2 + \|f_{2,\epsilon}\|_s^2 \, ds\right)$$

Now we define the difference energy as

$$\begin{aligned} E_{\epsilon_1,\epsilon_2} = \frac{1}{2}\|w_{\epsilon_1} - w_{\epsilon_2}\|_s^2 + \frac{g}{2}\|J_{\epsilon_1} u - J_{\epsilon_2} u\|_0^2 \\ + \frac{g}{2}\int (J^s \Lambda^{(p-1)/2}(J_{\epsilon_1} u - J_{\epsilon_2} u))^2 \, dx. \end{aligned}$$

We can make the corresponding estimate for the difference energy (again, the evolutionary system has the same structure), finding the following:

$$\|J_{\epsilon_1}u - J_{\epsilon_2}u\|^2_{s+(p-1)/2} + \|w_{\epsilon_1} - w_{\epsilon_2}\|^2_s$$
$$\leq e^{ct}\left(E_{\epsilon_1,\epsilon_2}(0) + \int_0^t \|f_{1,\epsilon_1} - f_{1,\epsilon_2}\|^2_{s+(p-1)/2} + \|f_{2,\epsilon_1} - f_{2,\epsilon_2}\|^2_s \, ds\right).$$

That is,

$$\sup_{t\in[0,T]} \left(\|J_{\epsilon_1}u - J_{\epsilon_2}u\|^2_{s+(p-1)/2} + \|w_{\epsilon_1} - w_{\epsilon_2}\|^2_s\right)$$
$$\leq e^{cT}\left(E_{\epsilon_1,\epsilon_2}(0) + \int_0^T \|f_{1,\epsilon_1} - f_{1,\epsilon_2}\|^2_{s+(p-1)/2} + \|f_{2,\epsilon_1} - f_{2,\epsilon_2}\|^2_s ds\right).$$

It is clear that $E_{\epsilon_1,\epsilon_2}(0)$ can be made small by taking ϵ_1 and ϵ_2 small. We then turn to prove that $(f_{1,\epsilon}, f_{2,\epsilon})$ is Cauchy in $L^1([0,T]; H^{s+(p-1)/2} \times H^s)$. Actually $(f_{1,\epsilon}, f_{2,\epsilon})$ goes to (f_1, f_2) as ϵ goes to zero, in the space $L^1([0,T]; H^{s+(p-1)/2} \times H^s)$. To see this, we start with the fact that $(J_\epsilon f_1, J_\epsilon f_2)$ goes to (f_1, f_2) as ϵ goes to zero; we mention that this is similar to Lemma 2.1 of [8]. So we turn to prove the other terms go to zero as ϵ goes to zero, in $L^1([0,T]; H^{s+(p-1)/2} \times H^s)$. The fact is that $[J_\epsilon, T_{v_x}]$ is uniformly bounded with order -1 and $[J_\epsilon, T_{v_x}]\partial_x u$ goes to zero as ϵ goes to zero, in $L^1([0,T]; H^{s+(p-1)/2})$; this can be seen by a density argument, and is similar to the proof of Lemma 7.1.13 of [25]. So $[J_\epsilon, T_{v_x}]\partial_x v$ goes to 0 as ϵ goes to zero in $L^1([0,T]; H^s)$. The other terms have analogous arguments. This completes the proof of continuity in time.

4. UNIQUENESS

In this section we prove the uniqueness of the solutions which we have shown to exist. This requires showing an estimate for the difference of two solutions.

Proposition 4.1. *Let* $s > \max\{5/2, (p+1)/2\}$ *and* $T_0 > 0$. *Let both* (u_1, v_1) *and* (u_2, v_2) *be in* $C^0([0,T_0]; H^{s+(p-1)/2}(\mathbb{R}) \times H^s(\mathbb{R}))$ *and satisfy system* (1.2), *with initial data* (u_0, v_0) *and* $(\tilde{u}_0, \tilde{v}_0)$, *respectively. Then the following estimate is satisfied:*

$$\|(u_1, v_1) - (u_2, v_2)\|_{L^\infty([0,T_0]; H^{(p-1)/2}(\mathbb{R}) \times H^0(\mathbb{R}))} \leq C \left\| [(u_1, v_1) - (u_2, v_2)] \Big|_{t=0} \right\|_{H^{(p-1)/2}(\mathbb{R}) \times H^0(\mathbb{R})}. \tag{4.1}$$

Proof. Define the differences $\delta u = u_1 - u_2$ and $\delta v = v_1 - v_2$. Let $0 < T \leq T_0$, and define

$$M_i = \|(u_i, v_i)\|_{L^\infty([0,T_0];H^{s+(p-1)/2}(\mathbb{R})\times H^s(\mathbb{R}))}, \tag{4.2}$$

$$N = \|(\delta u, \delta v)\|_{L^\infty([0,T];H^{(p-1)/2}(\mathbb{R})\times H^0(\mathbb{R}))}. \tag{4.3}$$

The system satisfied by $(\delta u, \delta v)$ is

$$\begin{cases} \partial_t \delta u = \Lambda \delta v - \partial_x[H, \delta u]\Lambda v_1 - \partial_x[H, u_2]\Lambda \delta v, \\ \partial_t \delta v = -g\Lambda^p \delta u + \dfrac{1}{2}(\Lambda v_1 + \Lambda v_2)\Lambda \delta v - \dfrac{1}{2}(\partial_x v_1 + \partial_x v_2)\partial_x \delta v. \end{cases} \tag{4.4}$$

We use the paradifferential operators $T_{\Lambda v_1}$ and $T_{\partial_x v_1}$ as before. Also we define an auxiliary variable $\delta w = \delta v - T_{\Lambda v_1}\delta u$ as the intermediate variable. We rewrite the first equation of system (4.4) by doing addition and subtraction:

$$\partial_t \delta u = \Lambda \delta w - T_{\partial_x v_1}\partial_x \delta u + f_1,$$

where f_1 is defined as

$$f_1 = \Lambda(T_{\Lambda v_1}\delta u) + T_{\partial_x v_1}\partial_x \delta u - \partial_x[H, \delta u]\Lambda v_1 - \partial_x[H, u_2]\Lambda \delta v.$$

Using the system (4.4) and substituting, the equation of δw is

$$\partial_t \delta w = -g\Lambda^p \delta u - T_{\partial_x v_1}\partial_x \delta w + f_2, \tag{4.5}$$

where

$$f_2 = Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \tag{4.6}$$

and

$$\begin{aligned} Q_1 &= \frac{1}{2}\Lambda(v_1 + v_2)\Lambda \delta v - T_{\Lambda v_1}\Lambda \delta v, \\ Q_2 &= -\frac{1}{2}\partial_x(v_1 + v_2)\partial_x \delta v + T_{\partial_x v_1}\partial_x \delta v, \\ Q_3 &= -T_{\partial_x v_1}\partial_x(T_{\Lambda v_1}\delta u), \\ Q_4 &= T_{\Lambda v_1}(\partial_x[H, \delta u]\Lambda v_1 + \partial_x[H, u_2]\Lambda \delta v), \\ Q_5 &= -T_{\partial_t \Lambda v_1}\delta u. \end{aligned}$$

Claim 3. Let f_1, f_2 be as above. Then they satisfy the following estimate:

$$\sup_{t\in[0,T]} \|(f_1, f_2)\|_{H^{(p-1)/2}(\mathbb{R})\times H^0(\mathbb{R})} \leq C(M_1, M_2)N. \tag{4.7}$$

The proof of claim 3. First, we rewrite $\|f_1\|_{(p-1)/2}$ by addition and subtraction:

$$\|f_1\|_{(p-1)/2} \leq T_1 + T_2 + T_3 + T_4,$$

where

$$T_1 = \|J^{(p-1)/2}\partial_x[H,\delta u]\Lambda v_1 - \partial_x[H, J^{(p-1)/2}\delta u]\Lambda v_1\|_0,$$
$$T_2 = \|\partial_x[H, J^{(p-1)/2}\delta u]\Lambda v_1 - \Lambda T_{\Lambda v_1} J^{(p-1)/2}\delta u - T_{\partial_x v_1} J^{(p-1)/2}\partial_x \delta u\|_0,$$
$$T_3 = \|\Lambda[T_{\Lambda v_1}, J^{(p-1)/2}]\delta u\|_0 + \|[T_{\partial_x v_1}, J^{(p-1)/2}]\partial_x\delta u\|_0,$$
$$T_4 = \|\partial_x[H,u_2]\Lambda\delta v\|_{(p-1)/2}.$$

The estimates for T_1, T_2, and T_3 correspond exactly to the estimates for $T_{1,\epsilon}$, $T_{2,\epsilon}$, and $T_{3,\epsilon}$ in the proof of Lemma 3.2, with $J^{(p-1)/2}$ being present instead of $J^{s+(p-1)/2}$. So,

$$T_1 + T_2 + T_3 \leq C\|\delta u\|_{(p-1)/2}\|v\|_s.$$

Now, we estimate T_4. By Proposition 2.3, since $(p-1)/2+1 < s + (p-1)/2 - 1 - 1/2$ for $s > 5/2$,

$$T_4 \leq \|[H,u_2]\Lambda\delta v\|_s \leq C\|u_2\|_{s+(p-1)/2}\|\Lambda\delta v\|_{-1} \leq C\|u_2\|_{s+(p-1)/2}\|\delta v\|_0.$$

We have proved

$$\|f_1\|_{(p-1)/2} \leq C(M_1,M_2)N. \tag{4.8}$$

For f_2, we estimate each of the Q_i for $1 \leq i \leq 5$. Firstly, the estimates for Q_1 and Q_2 are similar, and we only give the details of Q_1. By addition and subtraction and the triangle inequality, we have the following:

$$\|Q_1\|_0 \leq \frac{1}{2}\|\Lambda(v_1+v_2)\Lambda\delta v - T_{\Lambda(v_1+v_2)}\Lambda\delta v\|_0 + \frac{1}{2}\| - T_{\Lambda\delta v}\Lambda\delta v\|_0, \tag{4.9}$$

By Theorem 2.11, the first term of (4.9) is less than $\|\Lambda(v_1+v_2)\|_{W^{1,\infty}}\|H\delta v\|_{H^0}$, which is in turn bounded by $C(M_1,M_2)N$. And by Lemma 2.9, letting $\mu = m = 3/2$, the second term of (4.9) is less than $\|\Lambda\delta v\|_{3/2}\|\Lambda\delta v\|_{-1}$, which is bounded by $C(M_1,M_2)N$. So $\|Q_1\|_0 \leq C(M_1,M_2)N$ and $\|Q_2\|_0 \leq C(M_1,M_2)N$.

Secondly, by Theorem 2.8 and since $(p-1)/2 \geq 1$,

$$\|Q_3\|_0 \leq C\|\partial_x v_1\|_{L^\infty}\|T_{\Lambda v_1}\delta u\|_1 \leq C(M_1,M_2)N. \tag{4.10}$$

Thirdly, by Theorem 2.8, Proposition 2.2 (using $(p-1)/2 \geq 1$), and Proposition 2.3 (using $1 < s + (p-1)/2 - 1 - 1/2$),

$$\|Q_4\|_0 \leq C\|\Lambda v_1\|_{L^\infty}(\|[H,\delta u]\Lambda v_1\|_1 + \|[H,u_2]\Lambda\delta v\|_1) \leq C(M_1,M_2)N. \tag{4.11}$$

Finally, by Lemma 2.9,

$$\begin{aligned}\|Q_5\|_0 &\leq C\|\partial_t \Lambda v\|_{1/2-(p-1)/2}\|\delta u\|_{(p-1)/2} \\ &\leq C\| - g\Lambda^p u + \frac{1}{2}(\Lambda v)^2 - \frac{1}{2}(v_x)^2\|_{2-p/2}\|\delta u\|_{(p-1)/2} \\ &\leq C(M_1, M_2)N.\end{aligned}$$

This completes the proof of Claim 3. □

Having completed the proof of Claim 3, we resume the proof of Proposition 4.1. Since the evolutionary system for $(\delta u, \delta w)$ has the same structure as before, we may conclude

$$\begin{aligned}&\frac{d}{dt}\|(\delta u, \delta w)\|_{H^{(p-1)/2}\times H^0} \\ &\qquad \leq C(M_1, M_2)\|(\delta u, \delta w)\|_{H^{(p-1)/2}\times H^0} + C(M_1, M_2)N.\end{aligned}$$

Then, by Gronwall's Inequality,

$$\|(\delta u, \delta w)\|_{H^{(p-1)/2}\times H^0} \leq e^{C(M_1,M_2)T}(C(M_1, M_2)N_0 + C(M_1, M_2)NT).$$

Furthermore, we have

$$\begin{aligned}\|(\delta u,\delta v)\|_{H^{(p-1)/2}\times H^0} &\leq C\|\delta u\|_{H^{(p-1)/2}} + C\|\delta w\|_0 + \|v_1\|_s\|\delta u\|_{H^0} \\ &\leq C(M_1, M_2)e^{C(M_1,M_2)T}(C(M_1, M_2)N_0 + C(M_1, M_2)NT).\end{aligned}$$

The above inequality tells us that there exists $T_* > 0$ small enough such that

$$N \leq C(M_1, M_2)N_0. \tag{4.12}$$

Thus on the interval $[0, T_*]$, we conclude the uniqueness of solutions. The value of T_* may be smaller than T_0. However, we can repeat the argument to get uniqueness until time T_0, since the value of T_* only depends on M_1 and M_2. □

Now, as we have indicated previously, we remark on the proof of Proposition 3.6.

Proof. Since the proof is similar to the the proof of Proposition 4.1, we only point out the differences here. Let $0 < \epsilon_1 < \epsilon_2$ be given. Suppose (u_i, v_i) are the solutions for the approximate Cauchy problem corresponding to J_{ϵ_i} for $i = 1, 2$. Then they satisfy

$$\begin{cases}\partial_t u_i = \Lambda J_{\epsilon_i}^2 v_i - \partial_x J_{\epsilon_i}[H, J_{\epsilon_i} u_i]\Lambda v_i, \\ \partial_t v_i = -g\Lambda^p J_{\epsilon_i}^2 u_i + \frac{1}{2}J_{\epsilon_i}^2(\Lambda v_i)^2 - \frac{1}{2}J_{\epsilon_i}^2((v_i)_x)^2.\end{cases} \tag{4.13}$$

Set $\delta u = u_1 - u_2$ and $\delta v = v_1 - v_2$. Then, these satisfy the following system:

(4.14)
$$\begin{cases} \partial_t \delta u = \Lambda J_{\epsilon_1}^2 \delta v - \partial_x J_{\epsilon_1}\left([H, J_{\epsilon_1}\delta u]\Lambda v_1 - [H, J_{\epsilon_2}u_2]\Lambda \delta v\right) + R_1, \\ \partial_t \delta v = -g\Lambda^p J_{\epsilon_1}^2 \delta u + \frac{1}{2}J_{\epsilon_1}^2\left((\Lambda v_1)^2 - (\Lambda v_2)^2\right) - \frac{1}{2}J_{\epsilon_1}^2\left(((v_1)_x)^2 - ((v_2)_x)^2\right) + R_2, \end{cases}$$

where

$$R_1 = \Lambda(J_{\epsilon_1}^2 - J_{\epsilon_2}^2)v_2 - \partial_x(J_{\epsilon_1} - J_{\epsilon_2})[H, J_{\epsilon_2}u_2]\Lambda v_2 - \partial_x J_{\epsilon_1}\left[H, (J_{\epsilon_1} - J_{\epsilon_2})u_2\right]\Lambda v_1,$$

$$R_2 = -g\Lambda^p(J_{\epsilon_1}^2 - J_{\epsilon_2}^2)u_2 + \frac{1}{2}(J_{\epsilon_1}^2 - J_{\epsilon_2}^2)(\Lambda v_2)^2 - \frac{1}{2}(J_{\epsilon_1}^2 - J_{\epsilon_2}^2)((v_2)_x)^2.$$

We let $\delta w = \delta v - T_{\Lambda v_1}\delta u$, and we find that the system for δu and δw is

$$\begin{cases} \partial_t \delta u = \Lambda J_{\epsilon_1}^2 \delta w - J_{\epsilon_1} T_{v_x} J_{\epsilon_1}\partial_x \delta u + f_1 + R_1, \\ \partial_t \delta v = -g\Lambda^p J_{\epsilon_1}^2 \delta u - J_{\epsilon_1} T_{v_x} J_{\epsilon_1}\partial_x \delta w + f_2 + R_2. \end{cases} \tag{4.15}$$

The definitions of f_1 and f_2 may be inferred from (4.14) and (4.15), and are entirely analagous to the corresponding definitions in the proof of Proposition 4.1. To complete the proof of the proposition, we need to estimate $\|f_1\|_{(p-1)/2}$, $\|R_1\|_{(p-1)/2}$, $\|f_2\|_0$, and $\|R_2\|_0$. The estimates of $\|f_1\|_{(p-1)/2}$ and $\|f_2\|_0$ are again entirely analogous to the corresponding estimates in Proposition 4.1, so we omit them. For the estimates of R_1 and R_2, by Lemma 3.7, it follows that

$$\begin{aligned} \|R_1\|_{(p-1)/2} \le C\epsilon_2^{s-(p+1)/2}\left\|\Lambda v_2 - \partial_x[H, J_{\epsilon_2}u_2]\Lambda v_2\right\|_{s-1} \\ + C\left\|\left(J_{\epsilon_1} - J_{\epsilon_2}\right)u_2\right\|_{(p+1)/2} \le C\epsilon_2^{s-(p+1)/2}, \end{aligned} \tag{4.16}$$

(4.17)
$$\|R_2\|_0 \le C\epsilon_2^{s-(p+1)/2}(\|\Lambda^p u + 1/2(\Lambda v)^2 - 1/2(v_x)^2\|_{s-(p+1)/2} \le C\epsilon_2^{s-(p+1)/2}.$$

Notice that we state a requirement $s > \dfrac{p+1}{2}$ in Theorem 1.1; here, we need this condition so that the right-hand sides of (4.16) and (4.17) go to zero with ϵ_2. □

5. Conclusion

The second author, Bona, and Nicholls have previously presented strong evidence that quadratic and cubic truncated series models of gravity water waves are ill-posed [3]; for the quadratic case, the second author and Siegel are working on a full proof of ill-posedness [7]. That the initial value problems for these systems are ill-posed is in one sense surprising, since the full equations of motion for gravity water waves are well-posed. In another way, however, the ill-posedness is not so surprising, since the energy estimates needed to establish well-posedness, such as in [30], are quite subtle, and the design of the truncation scheme is unrelated to the energy balance.

The ill-posedness is caused by one particular parabolic term of indefinite sign, which leads to catastrophic backwards parabolic growth. We have shown here that a sufficiently strong leading-order dispersive term can control this growth. This strong dispersion can be relevant in some physical contexts, such as hydroelastic waves [27]. We mention that while in a sense we made use of the smoothing properties afforded by this dispersion, we did not present here an explicit result on gain of regularity for solutions of our system. It is possible to do so, similarly to the proofs of gain of regularity for capillary water waves [1], [11].

Finally, we mention that the present work, along with the corresponding works [3] and [7], leave open an important question. While the evidence of [3] and [7] is that system (1.2) is ill-posed when $p = 0$, and the main theorem of the present work is that the same system has a well-posed initial value problem for $p \geq 3$, the status of the initial value problem for values $p \in (0, 3)$ has not yet been established. This is especially relevant in the case $p = 2$, which corresponds to the presence of surface tension at the free surface. This will surely be the subject of future studies.

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Drexel University, Department of Mathematics, Philadelphia, PA 19104

Drexel University, Department of Mathematics, Philadelphia, PA 19104