**Problem 1.** Locate all relative maxima, relative minima, and saddle points for \( f(x, y) = x^3 - 9xy + y^3 \). (Use the second derivatives test to identify the type of each critical point.)

**Solution.** We first find critical points solving the equations \( f_x = 3x^2 - 9y = 0 \) and \( f_y = -9x + 3y^2 = 0 \). We obtain from the first equation that \( y = \frac{x^2}{3} \), and then substitute this \( y \) to the second equation: \(-9x + \frac{x^4}{3} = 0\), or equivalently, \(-27x + x^4 = 0\), i.e., \( x(x^3 - 27) = 0\). We obtain from here that either \( x = 0 \) or \( x = 3 \), and \( y = 0 \) or \( y = 3 \) respectively. Thus \( f(x, y) \) has two critical points: \((0, 0)\) and \((3, 3)\).

Now compute the second derivatives: \( f_{xx} = 6x \), \( f_{xy} = f_{yx} = -9 \), \( f_{yy} = 6y \). We also have \( D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 81 \). Next, \( D(0, 0) = -81 < 0 \), therefore \((0, 0)\) is a saddle point. \( f_{xx}(3, 3) = 18 > 0 \) and \( D(3, 3) = 243 > 0 \), therefore \((3, 3)\) is a point of relative minimum.

**Problem 2.** Find the absolute maximum and the absolute minimum of \( f(x, y) = x^2 + 2y^2 - x \) on the closed disk \( x^2 + y^2 \leq 4 \).

**Solution.** We start with finding critical points of \( f(x, y) \) inside the disk. We have \( f_x = 2x - 1 = 0 \) and \( f_y = 4y = 0 \) for \( x = \frac{1}{2} \) and \( y = 0 \). Thus, there is only one critical point inside the disk: \( P_1(\frac{1}{2}, 0) \). Next, consider the function \( f(x, y) \) restricted to the boundary of the disk, the circle \( x^2 + y^2 = 4 \). We plug in \( y^2 = 4 - x^2 \) into the expression for \( f(x, y) \) and obtain the function

\[
\phi(x) = x^2 + 2(4 - x^2) - x = x^2 + 8 - 2x^2 - x = -x^2 - x + 8.
\]

This function is defined on the closed interval \([-2, 2]\). We first find the critical points of \( \phi(x) \) inside the interval: \( \phi'(x) = -2x - 1 = 0 \) at \( x = -\frac{1}{2} \). We have that the corresponding \( y \) takes the values

\[
y = \pm \sqrt{4 - \left(-\frac{1}{2}\right)^2} = \pm \frac{\sqrt{15}}{2}.
\]

Therefore, there are two more points of the closed disk where the absolute extrema may occur: \( P_2(-\frac{1}{2}, \frac{\sqrt{15}}{2}) \) and \( P_3(-\frac{1}{2}, -\frac{\sqrt{15}}{2}) \). In addition, \( \phi(x) \) may take the extremal values at \( x = -2 \) and \( x = 2 \). The corresponding value of \( y \) is \( y = \sqrt{4 - (\pm 2)^2} = 0 \) for both points. Thus we add two more suspicious points, \( P_4(-2, 0) \) and \( P_5(2, 0) \). In order to find the absolute extrema, we evaluate \( f(x, y) \) at the points \( P_1, P_2, P_3, P_4, \) and \( P_5 \), and find the largest and smallest values. We have \( f(1, 0) = -\frac{1}{4}, f(-\frac{1}{2}, \pm \frac{\sqrt{15}}{2}) = \frac{33}{4}, f(-2, 0) = 6, f(2, 0) = 2 \). Therefore, \( f(x, y) \) has
the absolute minimum at $P_1\left(\frac{1}{2}, 0\right)$ which is equal to $-\frac{1}{4}$, and the absolute maximum at $P_2\left(-\frac{1}{2}, \frac{\sqrt{15}}{2}\right)$ and $P_3\left(-\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$ which is equal to $\frac{33}{4}$. 