

## Exam 1: Solutions

**Problem 1.** Use De Morgan's laws to find the negation of each of the following statements.

- (a) Jan is rich and happy.
- (b) Carlos will bicycle or run tomorrow.
- (c) Mei walks or takes the bus to class.
- (d) Ibrahim is smart and hard working.

**Solution.** (a) Jan is either not rich or not happy.  
 (b) Carlos will neither bicycle nor run tomorrow.  
 (c) Mei neither walks nor takes the bus to class.  
 (d) Ibrahim is either not smart or not hard working.

**Problem 2.** Using truth tables,

- (a) show that  $p \leftrightarrow q$  and  $(p \wedge q) \vee (\neg p \wedge \neg q)$  are logically equivalent.
- (b) show that  $(p \wedge q) \rightarrow r$  and  $(p \rightarrow r) \wedge (q \rightarrow r)$  are not logically equivalent.

**Solution.** (a) We have

$p$	$q$	$p \leftrightarrow q$	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
0	0	1	0	1	1	1	1
0	1	0	0	1	0	0	0
1	0	0	0	0	1	0	0
1	1	1	1	0	0	1	1

Since  $p \leftrightarrow q$  and  $(p \wedge q) \vee (\neg p \wedge \neg q)$  take the same truth values for all  $p$  and  $q$ , they are logically equivalent.

(b) If  $p = 1, q = 0, r = 0$ , then  $(p \wedge q) \rightarrow r = 1$  and  $(p \rightarrow r) \wedge (q \rightarrow r) = 0$ . Therefore,  $(p \wedge q) \rightarrow r$  and  $(p \rightarrow r) \wedge (q \rightarrow r)$  are not logically equivalent.

**Problem 3.** Determine the truth value of each of these statements if the domain consists of all real numbers.

- (a)  $\exists x(x^3 = -1)$ .
- (b)  $\exists x(x^4 < x^2)$ .
- (c)  $\forall x((-x)^2 = x^2)$ .
- (d)  $\forall x(2x > x)$ .

**Solution.** (a) True, e.g., for  $x = -1$ .  
 (b) True, e.g., for  $x = 0.1$ .  
 (c) True:  $(-x)^2 = (-1)^2 x^2 = x^2$ .  
 (d) False, e.g., for  $x = -1$ .

**Problem 4.** Determine whether each of these statements is true or false.

- (a)  $0 \in \emptyset$ .
- (b)  $\emptyset \in \{0\}$ .
- (c)  $\{0\} \subset \emptyset$ .
- (d)  $\emptyset \subset \{0\}$ .
- (e)  $\{\emptyset\} \subseteq \{\emptyset\}$ .

**Solution.** (a) False:  $\emptyset$  has no elements.  
 (b) False:  $\emptyset$  is not an element of the set  $\{0\}$  (whose only element is 0).  
 (c) False:  $\emptyset$  has no subsets other than  $\emptyset$ .  
 (d) True:  $\emptyset$  is contained in every set.  
 (e) every set is contained in itself.

**Problem 5.** Show that if  $A$  and  $B$  are sets, then

- (a)  $A - B = A \cap \overline{B}$ .
- (b)  $(A \cap B) \cup (A \cap \overline{B}) = A$ .

**Solution.** (a)  $A - B = \{x \in A \mid x \notin B\} = \{x \in A \mid x \in \overline{B}\} = A \cap \overline{B}$ .  
 (b)  $(A \cap B) \cup (A \cap \overline{B}) = A \cap (B \cup \overline{B}) = A \cap U = A$ .

**Problem 6.** Let  $A_i = \{-i, \dots, -2, -1, 0, 1, 2, \dots, i\}$ . Find

- (a)  $\bigcup_{i=1}^n A_i$ .
- (b)  $\bigcap_{i=1}^n A_i$ .

**Solution.** (a) and (b). Clearly, we have  $A_1 \subset A_2 \subset \dots \subset A_n$ . Therefore,  
 $\bigcup_{i=1}^n A_i = A_n = \{-n, \dots, -2, -1, 0, 1, 2, \dots, n\}$ , and  $\bigcap_{i=1}^n A_i = A_1 = \{-1, 0, 1\}$ .

**Problem 7.** Find these values:

- (a)  $\lfloor 1.1 \rfloor$ .
- (b)  $\lfloor -0.1 \rfloor$ .
- (c)  $\lceil 2.99 \rceil$ .
- (d)  $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ .

**Solution.** (a) 1; (b) -1; (c) 3; (d) 1.

**Problem 8.** Determine whether each of these functions is a bijection from  $\mathbb{R}$  to  $\mathbb{R}$ .

- (a)  $f(x) = 2x + 1$ .
- (b)  $f(x) = x^2 + 1$ .
- (c)  $f(x) = x^3$ .
- (d)  $f(x) = \frac{x^2+1}{x^2+2}$ .

**Solution.** (a)  $f$  is a bijection. Since  $f$  is strictly increasing, it is an injection:  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$ . Since for every real  $y$  the equation  $2x + 1 = y$  has a solution  $x = (y - 1)/2$ , it is also a surjection.

(b)  $f$  is not a bijection. First, because  $f$  is not an injection:  $f(-x) = f(x)$  for every  $x$ . Second, because  $f$  is not a surjection: the values of  $f$  cover only the interval  $[1, \infty)$ .

(c)  $f$  is a bijection. Since  $f$  is strictly increasing, it is an injection. Since for every real  $y$  the equation  $x^3 = y$  has a solution  $x = \sqrt[3]{y}$ , it is also a surjection.

(d)  $f$  is not a bijection. First, because  $f$  is not an injection:  $f(-x) = f(x)$  for every  $x$ . Second, because  $f$  is not a surjection: the values of  $f(x) = \frac{x^2+1}{x^2+2} = 1 - \frac{1}{x^2+2}$  cover only the interval  $[1/2, 1)$ .

**Problem 9.** (a) Show that

$$\sum_{j=1}^n (a_j - a_{j+1}) = a_1 - a_{n+1},$$

where  $\{a_j\}$  is a sequence of real numbers. This type of sum is called **telescoping**.

(b) Use the identity

$$\frac{1}{j(j+1)} = \frac{1}{j} - \frac{1}{j+1}$$

and part (a) to compute

$$\sum_{j=1}^n \frac{1}{j(j+1)}.$$

**Solution.** (a) Due to cancellations, we have

$$\begin{aligned} & \sum_{j=1}^n (a_j - a_{j+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_{n-2} - a_{n-1}) + (a_{n-1} - a_n) + (a_n - a_{n+1}) \\ &= a_1 - a_{n+1}. \end{aligned}$$

(b)

$$\sum_{j=1}^n \frac{1}{j(j+1)} = \sum_{j=1}^n \left( \frac{1}{j} - \frac{1}{j+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

**Problem 10.** Show that

(a)  $2^x + 17$  is  $O(3^x)$ .

(b)  $\frac{x^2+1}{x+1}$  is  $O(x)$ .

**Solution.** (a) For every  $x \geq k = 1$  we have  $2^x + 17 < 3^x + 18 \leq 3^x + 6 \cdot 3^x = 7 \cdot 3^x$ , i.e.,  $2^x + 17 \leq C \cdot 3^x$  with  $C = 7$ . Therefore,  $2^x + 17$  is  $O(3^x)$ .

(b) For every  $x \geq k = 1$  we have  $\frac{x^2+1}{x+1} \leq \frac{x^2+x}{x+1} = \frac{x(x+1)}{x+1} = x$ , i.e.,  $\frac{x^2+1}{x+1} \leq Cx$  with  $C = 1$ . Therefore,  $\frac{x^2+1}{x+1}$  is  $O(x)$ .

**Problem 11 (extra credit).** Describe an algorithm that takes as input a list of  $n$  integers and finds the number of negative integers in the list.

**Solution.** Let  $a_1, \dots, a_n$  be the integers in our list. Set  $N := 0$  (the initial value for the number of negative integers in the list). For  $k := 1$  to  $n$ , we proceed as follows. If  $a_k < 0$ , then we set  $N := N + 1$ , otherwise  $N$  is unchanged. Then after all  $n$  steps we end up with  $N$  equal the number of negative integers in the list.