The definition of the derivative
Motivation

What if I said...

- Find the instantaneous rate of change for $f(x) = x^2$ at $x=1$.
- Now, find the instantaneous rate of change for $f(x) = x^2$ at $x=2$.
- Now, find the instantaneous rate of change for $f(x) = x^2$ at $x=2$.

How quickly would that get old?
Inst. Rate of Change at an arbitrary point $x=a$.

For $f(x)=x^2$, find the instantaneous rate of change of the function at $x=a$ (i.e. an arbitrary $x$-value).

You should end up with a formula in terms of $a$. Use that formula to find the slopes of several tangent lines to $f(x)=x^2$. 
\[
\lim_{h \to 0} \frac{(a + h)^2 - a^2}{h} = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} \\
= \lim_{h \to 0} \frac{2ah + h^2}{h} \\
= \lim_{h \to 0} \frac{h(2a + h)}{h} \\
= \lim_{h \to 0} 2a + h \\
= 2a + 0 \\
= 2a
\]
Now...

Replace a with x.

Now we have a function of x, $f'$, that gives us the slope of the tangent line to our original function $f$ at any point $x$ in its domain.

This is called the derivative of $f$. 
The Definition of the Derivative

To reiterate, $f'(x)$ is a function whose output for a given $x$-value is the slope of the tangent line to $f$ at that point. This means its output for a given $x$-value is also the instantaneous rate of change of $f$ at that point.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
Examples: Use the definition of the derivative to find the derivative of each function.

1. \( f(x) = \sqrt{x} \)

2. \( g(x) = x^3 \)

3. \( h(x) = \frac{1}{x} \)
Differentiability by example

- Consider the function $f(x) = |x|$.
- Write it as a piecewise function and sketch its graph.
- Find $f'$ using the definition of the derivative for each piece.
- Discuss the continuity of $f'$. 
Differentiability

We say that a function $f$ is differentiable at a point $x=a$ provided that $f'$ is continuous at that point.

Questions to consider:

- If $f$ is continuous at $x=a$, is it necessarily differentiable at $x=a$?
- If $f$ is differentiable at $x=a$, is it necessarily continuous at $x=a$?
Example

Consider the function \( f(x) = x^{2/3} \).

Sketch the graph.

Discuss the continuity and differentiability of \( f \).