MATH & POLITICS

Apportionment, the Electoral College, and Fairness
How do we apportion a finite, indivisible set of goods/objects/people among groups of different sizes or among individuals with different preferences?

How do we think about fairness in the context of apportionment?

What’s the best way to perform the apportionment of the House of Representatives?

Is there a best way?

In what ways does the current apportionment scheme impact the electoral system?
THE PROBLEM OF APPORTIONMENT

- The US House of Representatives has a fixed size: presently 435

- Article 1, Section 2 of the Constitution: Seats should be apportioned among the states “according to their respective numbers.”
  - Problem: Suppose a state has exactly 10% of the population. That state would be entitled to 43.5 representatives...

- DEFINITION: We refer to a state’s ideal allotment (e.g. 43.5 in the example above) as its quota.

- DEFINITION: The Apportionment Problem refers to the search for a method to replace every state’s quota with a whole number in a way that is as fair and equitable as possible.
WHAT WOULD YOU DO?

In your groups, devise a method to apportion the 75 seats of Rhode Island’s House of Representatives among its five counties.

<table>
<thead>
<tr>
<th>County</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bristol</td>
<td>49,875</td>
</tr>
<tr>
<td>Kent</td>
<td>166,158</td>
</tr>
<tr>
<td>Newport</td>
<td>82,888</td>
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<tr>
<td>Providence</td>
<td>626,667</td>
</tr>
<tr>
<td>Washington</td>
<td>126,979</td>
</tr>
</tbody>
</table>
**HOW DID YOU DO IT?**

What did you take into consideration?

Is your method equitable?
- Does it perhaps benefit larger states over smaller states or vice versa?

Is your method easily applicable to larger sets of states/counties?

Can you think of any other contexts in which this kind of apportionment scheme would apply?
HAMILTON’S METHOD

- Alexander Hamilton, Secretary of the Treasury, proposed the first solution to the apportionment problem following the initial U.S. census in 1792.

**Method**
- Round all quotas down to the nearest whole number.
- Allocate seats according to the rounded quotas.
- Hand out any remaining seats, one at a time, according to the size of the fractional part of the quota.
  - E.g. A state with a quota of 13.92 would get an extra seat before a state with a quota of 31.77.

- Vetoed by President Washington (first veto).
- Resurrected in 1850 and used for 40 years.
## APPLY HAMILTON’S METHOD

<table>
<thead>
<tr>
<th>County</th>
<th>Population</th>
<th>Quota</th>
<th>Rounded Quota</th>
<th>Final</th>
</tr>
</thead>
<tbody>
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<td>3.55381178</td>
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<td>166,158</td>
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<td>Newport</td>
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<td>626,667</td>
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<td>Washington</td>
<td>126,979</td>
<td>9.04780883</td>
<td>9</td>
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</tbody>
</table>

Subtotal: 1,052,567

Final Quota: 75
Hamilton’s Method seems reasonable, but is subject to several interesting, and perhaps unexpected, paradoxes.

### Alabama Paradox
- Discovered in 1880 by CW Seaton.
- After the census, Seaton computed all the apportionments for House of Representative sizes between 275 and 350.
- Micro-example: Consider a House with 10 representatives and 3 states.

<table>
<thead>
<tr>
<th>State</th>
<th>Population</th>
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<tbody>
<tr>
<td>A</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
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<tr>
<td>C</td>
<td>2</td>
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</tbody>
</table>
WHAT HAPPENS IF THE NUMBER OF SEATS IS INCREASED FROM 10 TO 11?

<table>
<thead>
<tr>
<th>State</th>
<th>Po</th>
<th>State</th>
<th>Population</th>
<th>Quota</th>
<th>Rounded</th>
<th>Final</th>
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<tbody>
<tr>
<td>A</td>
<td>6</td>
<td>A</td>
<td>6</td>
<td>4.7143</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>B</td>
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<td>4.7143</td>
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<tr>
<td>C</td>
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<td>C</td>
<td>2</td>
<td>1.5714</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Seaton discovered that while Alabama ended up with 8 seats if there were 299 seats in total, but 7 seats if there were 300 seats in total.
Compute the Hamiltonian apportionments for the following five-state nation with 50 seats in both 2000 and 2020.

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<thead>
<tr>
<th></th>
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</thead>
<tbody>
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<td>150</td>
</tr>
<tr>
<td>B</td>
<td>78</td>
<td>78</td>
</tr>
<tr>
<td>C</td>
<td>173</td>
<td>181</td>
</tr>
<tr>
<td>D</td>
<td>204</td>
<td>204</td>
</tr>
<tr>
<td>E</td>
<td>295</td>
<td>296</td>
</tr>
<tr>
<td>Total</td>
<td>900</td>
<td>909</td>
</tr>
</tbody>
</table>
### Population Paradox: State X’s Population Grows Faster Than State Y’s Population, But X Loses a Representative and Y Gains One.

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<tr>
<td>A</td>
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<td>8.3333</td>
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<td>8</td>
<td>150</td>
<td>8.2508</td>
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<td>8</td>
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<tr>
<td>B</td>
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<td>4.3333</td>
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<td>4</td>
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<td>4.2904</td>
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<td>5</td>
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<tr>
<td>C</td>
<td>173</td>
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<tr>
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<td>E</td>
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<tr>
<td>Total</td>
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<td>48</td>
<td>50</td>
<td>909</td>
<td></td>
<td>48</td>
<td>50</td>
</tr>
</tbody>
</table>
MONOTONICITY

- No state receives fewer seats than a state with less (or the same) population

QUOTA PROPERTY

- The number of seats allotted to a state never differs from its (ideal) quota by more than one.

POPULATION PROPERTY

- Following a census, no state should gain population and lose a seat while some other state loses population and gains a seat.
DIVISOR METHODS

- Choose a number $d$ (called a divisor) as the desired size of a congressional district.

- Allocate each state one seat per congressional district.
  - **Jefferson**: Divide state’s population by $d$ and **round down**.
  - **Adams**: Divide state’s population by $d$ and **round up**.
  - **Weber**: Divide state’s population by $d$ and **round** (in the usual way).
  - **Huntingdon-Hill**: Divide state’s population by $d$ and **round according to the geometric mean**.

- If the total number of seat allotted is the same as the House size, we’re done.

- If the total number of seats allotted differs from the House size, try a different $d$. 
The geometric mean of $x$ and $y$ is $\sqrt{xy}$

If we’re rounding according to the geometric mean, we’re looking at the geometric mean of the nearest whole numbers (the floor and the ceiling).

E.g. If we’re rounding a number between 2 and 3, then we round down if it’s less than 2.449489743 and round up if it’s greater than or equal to 2.449489743.
TRY IT

- Use an appropriate technology to compute the apportionments that result from each divisor method for the nation described here with 85 seats.

- What do you notice?
  - Are larger/smaller states better off with any particular method?
    - How/why?

- Can you think of any other ways to do this?
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<td>3</td>
<td>3</td>
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<td>3</td>
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<tr>
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<td>95500</td>
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<td>1</td>
</tr>
<tr>
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<td>5.372680628</td>
<td>5</td>
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<td>5</td>
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<td>3</td>
<td>2</td>
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<td>8</td>
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<td>K</td>
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<td>95500</td>
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<td>42</td>
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<td>L</td>
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<td>0</td>
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<tr>
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<td></td>
<td>80</td>
<td>92</td>
<td>82</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>85</td>
<td>85</td>
</tr>
</tbody>
</table>
MORE IMPOSSIBILITY

**THEOREM:** There is no apportionment method that satisfies the monotonicity property, the quota condition, and the population property.

**PROOF:** Assume we have an apportionment method that satisfies monotonicity and the quota property.

Consider the following situation in which there are 7 seats:

<table>
<thead>
<tr>
<th>State</th>
<th>Population</th>
<th>Quota</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3003</td>
<td>5.005</td>
</tr>
<tr>
<td>B</td>
<td>400</td>
<td>0.667</td>
</tr>
<tr>
<td>C</td>
<td>399</td>
<td>0.665</td>
</tr>
<tr>
<td>D</td>
<td>398</td>
<td>0.663</td>
</tr>
</tbody>
</table>
Given our assumption that the quota property and monotonicity are satisfied, there are only two possible apportionments:

- 5, 1, 1, 0 or 6, 1, 0, 0.
- The quota property tells us that the allotted seats can differ from the quota by at most one.
- Monotonicity says that no state can receive fewer seats than a less populous state.

Suppose that in the next census, we find that there are 1100 more people.

<table>
<thead>
<tr>
<th>State</th>
<th>Population</th>
<th>Quota</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3004 (+1)</td>
<td>3.968</td>
</tr>
<tr>
<td>B</td>
<td>1503 (+1103)</td>
<td>1.985</td>
</tr>
<tr>
<td>C</td>
<td>396 (-3)</td>
<td>0.523</td>
</tr>
<tr>
<td>D</td>
<td>397 (-1)</td>
<td>0.524</td>
</tr>
</tbody>
</table>

Because of the quota condition and monotonicity, there are only three possible apportionments:

- 4,2,0,1 and 4,1,1,1 and 3,2,1,1
- A gains population but loses a seat!
In 1911, Joseph A. Hill, then the chief statistician in the census bureau, suggested a philosophical principle—as opposed to a method—on which apportionment should be based.

- He wanted to look at per capita representation, that is, a state’s population divided by the number of seats.

- E.g. one state might have a per capita representation of 1,740,000 while another might have a per capita representation of 1,340,000 million.

- The difference, arrived at by subtracting, is 400,000. But what one really wants to look at here is the relative difference, in this case $\frac{400,000}{1,340,000} = 0.2985$ or 29.85%.

- It may happen that transferring a seat from the state with the smaller per capita representation to the one with the larger would reduce this relative difference, thus improving the equity.

Hill’s proposal was to find an apportionment method with the property that no two states could reduce the relative difference in per capita representation by such a transfer of a seat.

Edward V. Huntington, a professor of mathematics at Harvard, showed that the method now called the Hill-Huntington method does, in fact, satisfy Hill’s principle.

This is what’s used today!
DISCUSSION

What do you make of these different methods and the impossibility of a perfect method of apportionment?

Do you think Huntingdon-Hill is the best way of doing things? Do you foresee improvements in the future?
THE ELECTORAL COLLEGE

2016 Actual
THE ELECTORAL COLLEGE

- Devised in 1787
- US was sparsely populated at the time (13 states)
  - Communication and transportation were primitive
- Many regarded direct democracy as dangerous
- \[ \text{Number of electoral votes} = \text{number of House members} + 2 \text{ (Senate seats)} \]
  - Number of House members is determined using the Huntington-Hill method
- DC gets 3 votes (23rd Amendment, 1961)
- Total: \( 435 \) (House) + 100 (Senate) + 3 (DC) = 538
  - Majority: 270
AS OF THE 2010 CENSUS
PROS?

- Traditional and unique
- Believed to favor small states and rural areas.
  - Keeps big states and cities from dominating.
- Supports idea of federalism
  - Political system composed of a central (federal) government and regional governments
- Localizes logistics of voting
  - Russia needs to operate differently in 50 states.
- Encourages two-party system?
CONS?

- Not every voter has the same voting power. In social choice theory terms, the system isn’t anonymous.
- Violates majority principle.
- Fails independence of irrelevant alternatives
  - Spoiler candidates
- Encourages a two-party system?
- Focus on swing states
  - Think Banzhaf
WHEN HAS THE ELECTORAL COLLEGE DIFFERED FROM THE POPULAR VOTE?

1876
Rutherford B Hayes wins EC 185 to 184. Samuel J. Tilden wins 51.5% of popular vote (majority).

1888
Benjamin Harrison wins EC 201 to 200. Grover Cleveland wins plurality in popular vote.

2000
George W Bush wins EC 271 to 267. Al Gore wins plurality of popular vote (Supreme Court awards Florida’s 25 votes to Bush).

2016
Donald Trump wins EC 304 to 227. Hillary Clinton wins plurality 48% to Trump’s 46%, with Libertarian Gary Johnson winning 3.3%.
THE +2 EFFECT

2010 Census Population per Electoral Vote

CHALLENGING THE CONVENTIONAL WISDOM...

AND THEN CHALLENGING THAT CHALLENGE...

AND THEN CHALLENGING THAT CHALLENGE.
Recall that the **Banzhaf Power Index** is essentially the probability of a given voter (or voting bloc) changing the outcome of a vote by changing their vote.

The conventional wisdom around our current electoral system is that voters in small states have more power.

John Banzhaf challenged this conventional wisdom in his 1968 paper, “One Man, 3.312 Votes: A Mathematical Analysis of the Electoral College.”

- “People and not states” are the voters in presidential election.
- The electoral college is a 2-step process (voting method) from voters to the election of president.
  - Voters vote in their state, then—based on these votes—states vote in EC.
  - Not exactly weighted voting, but doesn’t satisfy anonymity (i.e. each voter having equal weight)
  - In theory one can still count critical voters in winning coalitions: but on national scale.
- However, the problem is computationally challenging!
“The complete analysis clearly demonstrates that the current Electoral College system falls short of even an approximation of equality in voting power. Such a disparity in favor of the citizens of New York and the other large states also repudiates the often-voiced view that the inequalities in the present system favor the residents of the less populous states.”

- John Banzhaf
In 1968, Banzhaf obtained access to early computer: IBM 360 with Fortran. He did “Monte Carlo” simulation of Banzhaf power for the 1960 Census.

2016 Dan Ullman, used Dell laptop with Matlab, for 2010 census, valid through 2020 election, published in A Mathematical Look at Politics.
What is the probability that a given voter’s vote will determine whether a candidate is elected or defeated (in a two candidate contest) if some value of p is drawn out of (say) a normal distribution with a mean of 0.5 and a standard deviation of 0.05 and all voters vote for the candidate with probability p? In particular, in a Presidential election under the Electoral College system, will that probability be the same for all voters or will that probability be a function of the size of the state the voter votes in?

Margolis argues that in fact individual voting power in Presidential elections is essentially uniform across states.

He suggests that the Banzhaf model assumes each voter is equally likely to vote for each of two candidates and that this is problematic. We should instead consider a range of probabilities regarding how each voter will vote.
“Margolis then points out, again quite correctly, that this model of individual voting behavior implies a distribution of aggregate election outcomes over time wildly dissimilar from that which we actually observe.”

What Margolis shows is uniform across states (under his more reasonable voting behavior assumption) is the probability that an individual voter will cast a critical vote (i.e., break what would otherwise be a tie) within a state times the electoral vote (or Electoral College voting power) of the state. But this is not the same as the probability that an individual voter will cast a critical vote in the Presidential election as a whole.

“Real-life political and geographical consideration seem to reinforce rather than attenuate the importance of big states in Presidential elections. (In the 1976 election, for example, for every additional 1 million popular votes cast, the state winner’s percentage victory margin fell on average by over 1 point; the correlation between state popular votes and winner’s victory margin was -.33.)”
DISCUSSION

What do you see as the primary pros/cons of the electoral college?
What do you make of Banzhaf’s analysis regarding the electoral college?
  - Does this pose a legitimate challenge to the conventional wisdom that small states benefit from the electoral college?
What do you think of the critiques posed by Margolis and Miller?
Do you think the electoral college favors smaller or larger states?
Given that such wildly different analyses of the electoral college are possible, how do we determine whether this system is sufficiently equitable?
FAIR DIVISION
Your parents have decided to retire early, buy a condo in Guam, and give their major assets to you and your four siblings. There’s a problem though. They have five kids, but three major assets allocate: a house in the city, a fancy sports car, and a cabin on a lake in the mountains. The new retirees have decided to leave it to you to make sense of this all. How are you and your siblings going to divide up these assets fairly?
THE METHOD OF SEALED BIDS

- Each player bids, in secret, what they think each item in question is worth.
- The player who bid the most on a given item receives that item and pays the money into a pool.
- Once the items have been awarded, any player who was not awarded an item receives, from the pool, their fair share’s worth of money.
- Any money that remains is evenly distributed among all players.
DISCUSSION

- Prove that, in the method of sealed bids, every player ends up with at least their fair share’s worth between goods and money, according, of course, to their own evaluation.

- Is this a fair way to divide up assets?

- Is this more, or less, fair than having an impartial third party make decisions about how to divide up such goods?