MATH & POLITICS

Yes-no voting & political power
OUTLINE

Yes-No voting systems
Weighted systems
Trade robustness

Political Power
Shapley-Shubik Power Index
Banzhaf Power Index
OPENING DISCUSSION

- How would you define political power?
- How much power does the president have relative to a single senator?
- How much power does a permanent member of the UN Security council have relative to a nonpermanent member?
EXAMPLES OF YES-NO (YEA-NAY) VOTING SYSTEMS

The European Economic Community

• Established in 1958 (Treaty of Rome)
• Six countries: France, Germany, Italy, Belgium, the Netherlands, Luxembourg
• France, Germany, Italy: four votes each
• Belgium and the Netherlands: two votes
• Luxembourg: one vote
• Passage: required 12 votes

UN Security Council

• Five permanent members: US, China, England, Russia, France
• Ten nonpermanent members
• Passage: requires nine votes, subject to veto by any permanent member
EXAMPLES OF YES-NO (YEA-NAY) VOTING SYSTEMS

US Federal System
- 537 voters
  - House of Representatives: 435
  - Senate: 100
  - Vice President & President
- VP: tie-breaker in Senate
- POTUS: veto power subject to override by 2/3 vote in House and Senate

Passage
- 218 or more representatives and 51 or more senators (with or without the vice president) and the president.
- 218 or more representatives and 50 senators and the vice president and the president.
- 290 or more representatives and 67 or more senators (with or without either the vice president or the president).
DEFINITIONS

• Yes-no voting system
  • Each voter responds to a single alternative (e.g. bill, amendment) with yes or no (yea or nay)
  • Set of rules specifying which collections of yes-votes yield passage

• Coalition
  • Any collection of voters

• Winning coalition
  • Passage guaranteed by a yes-vote

• Losing Coalition
  • Not a winning coalition

• Monotone yes-no voting system
  • Adding voters to a winning coalition still yields a winning coalition

• Minimal winning coalition
  • A winning coalition for which the deletion of one or more voters results in a losing coalition
WAYS TO DESCRIBE A YES-NO VOTING SYSTEM

1. Specify the number of votes each voter has and how many votes are required for passage. We’re essentially assigning *weights* to each voter and deciding on a *quota* that needs to be met. Example: European Economic Community

2. Explicitly list the winning coalitions. Example: US federal system

3. Combination of (1) and (2) with provisos that often involve veto powers. Example: UN Security Council
A WEIGHTED SYSTEM

A yes–no voting system is said to be a weighted system if it can be described by specifying real number weights for the voters and a real number quota—with no provisos or mention of veto power—such that a coalition is winning precisely when the sum of the weights of the voters in the coalition meets or exceeds the quota.

• The European Economic Community is clearly a weighted system because it is defined exactly that way.
EXPLORATION: IS THE UN SECURITY COUNCIL A WEIGHTED SYSTEM?

Can we assign weights to the permanent and nonpermanent members and establish a quota in such a way that the passage condition for the Council follows from this weighted system?
YES WE CAN...

• Let’s assume each nonpermanent member has a weight of 1
• The five permanent members have equal weight, so let’s call that $x$.
• Call the quote $q$
• If one permanent member votes no, we have a losing coalition even if all ten permanent members vote yes: $4x + 10 < q$
• A winning coalition could consist of all five permanent members and 4 nonpermanent members: $5x + 4 \geq q$
• If we assign a weight of 7 to the permanent members, the implied quota is 39.
• This does it!
THE US FEDERAL SYSTEM AND SWAP ROBUSTNESS

• Do you think the same can be done for the US federal system?
• Recall:
  • 218 or more representatives and 51 or more senators (with or without the vice president) and the president.
  • 218 or more representatives and 50 senators and the vice president and the president.
  • 290 or more representatives and 67 or more senators (with or without either the vice president or the president).

• It’s unreasonable to check every possible weight distribution and quota.
• We need a property that holds for every weighted voting system.
  • Then we need to show that it doesn’t hold for the US federal system.
  • Recall: If P implies Q, then not Q implies not P (contrapositive).
DEFINITION: A yes–no voting system is said to be swap robust if a one-for-one exchange of players (a “swap”) between two winning coalitions X and Y leaves at least one of the two coalitions winning. One of the players in the swap must belong to X but not Y, and the other must belong to Y but not X.

PROPOSITION: Every weighted voting system is swap robust.
• Assume we have a weighted voting system.
• Let X and Y be two winning coalitions such that there is a voter x who is in X but not Y and there is a voter y who is in Y but not X.
• Swap x and y and call the resulting coalitions X’ and Y’.
• Cases:
  • If x and y have the same weight, then X’ and Y’ weight the same as X and Y respectively, so X’ and Y’ are both winning coalitions.
  • If x weighs more than y, than Y’ becomes heavier than Y, so Y’ is a winning coalition.
  • If y weighs more than x, the argument is analogous to the one above.
PROPOSITION: THE US FEDERAL SYSTEM IS NOT SWAP ROBUST

• Let’s take two minimal winning coalitions.
  • X: the president, the 51 shortest senators, the 218 shortest members of the house
  • Y: the president, the 51 tallest senators, the 218 tallest members of the house

• Let x be the shortest senator and y be the tallest member of the House.
  • So, x is definitely not in Y and y is definitely not in X

• Swap x and y.

• X’ is a losing coalition because it only has 50 senators.

• Y’ is a losing coalition because it only has 217 members of the House.

• So, the US federal system is not swap robust. It follows that the US federal system is not a weighted system.
DISCUSSION

What do you think the non-weighted-ness of the US federal system says about the relative political power of its members?

Revisit: How much power does the president have relative to a single senator?

What are the benefits, advantages, problems, disadvantages, etc. of a weighted/non-weighted system?
POWER
A WEIGHTED SYSTEM DOESN’T MEAN POWER IS EASY TO DESCRIBE

• Suppose the US, Mexico, and Canada, create a three-member group similar to the European Economic Community.
  - In the EEC, France, Germany, and Italy were given four votes, Belgium and the Netherlands two, and Luxembourg one.

• In our hypothetical, maybe the US gets three votes and our neighbors each get one.

• Majority rule suggests a quota of three.

• Would we say that the US has triple the power of Canada and Mexico?

• It’s not necessarily true that x-times the votes means x-times the power!
Suppose we have a yes-no voting system with seven players. If we imagine that these players vote in sequence, there are $7!$ possible orderings for these players.

Fix one ordering. E.g., $p_3, p_4, p_1, p_2, p_7, p_6, p_5$

First, we’d have $p_3$ alone, then $p_3, p_4$, then $p_3, p_4, p_1$, so on and so forth.

A **pivotal player** is one whose addition to the growing coalition turns the coalition from a losing one into a winning one.
THE SHAPLEY-SHUBIK INDEX OF POWER

• Suppose \( p \) is a voter in a yes–no voting system and let \( X \) be the set of all voters. Then the Shapley–Shubik index of \( p \), denoted here by \( SSI(p) \), is the number given by:

\[
SSS(p) = \frac{\text{the number of orderings of } X \text{ for which } p \text{ is a pivotal voter}}{\text{the total number of orderings of the set } X}
\]

• The dominator is just \( n! \)

• For any voter \( p \), \( 0 \leq SSI(p) \leq 1 \)

• \( SSI(p_1) + SSI(p_2) + \ldots + SSI(p_n) = 1 \)
EXAMPLE

- Suppose we have a three-person weighted voted system in which p1 has fifty votes, p2 has forty-nine, and p3 has one. Assume passage requires fifty-one votes.
- Compute the Shapley-Shubik index for each voter.

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<tr>
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<th>p1</th>
<th>p2</th>
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- Shapley-Shubik index:

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THE BANZHAF INDEX OF POWER

TOTAL BANZHAF POWER: Suppose that p is a voter in a yes-no voting system. The total Banzhaf power of p, written TBP(p), is the number of coalitions C satisfying the following conditions:

1. p is a member of C
2. C is a winning coalition
3. If p is deleted from C, the resulting coalition is not a winning one.

BANZHAF POWER INDEX:

\[ BI(p_i) = \frac{TBP(p_i)}{TBP(p_1) + TBP(p_2) + \cdots + TBP(p_n)} \]
EXAMPLE

• Suppose we have a three-person weighted voted system in which p1 has fifty votes, p2 has forty-nine, and p3 has one. Assume passage requires fifty-one votes.

• Compute the Banzhaf Power Index for each voter.

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<thead>
<tr>
<th>Coalition 1</th>
<th>Coalition 2</th>
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<td>p1, p2, p3</td>
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The number of ways to order $n$ items: $n!$

The number of ways to choose $k$ items out of $n$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
Let’s compute this for a micro-government consisting of six senators, six members of the House, and a president.

- Minimal winning coalitions
  - 2/3 of each chamber
  - 1/2 of each chamber and the president

- Pivotal Player
  - Preceded by at least three members of each chamber, but by fewer than four members of at least one chamber.
SEVEN SCENARIOS

1. Three members of each chamber precede the president.
2. Three house members and four senators precede him.
3. Three house members and five senators precede him.
4. Three house members and six senators precede him.
5. Four house members and three senators precede him.
6. Five house members and three senators precede him.
7. Six house members and three senators precede him.
Scenario 1:

Step 1: Choose three of the six House members to precede the president in the ordering. This can be done in \( \binom{6}{3} \) ways.

Step 2: Choose three of the six senators to precede the president in the ordering. This can be done in \( \binom{6}{3} \) ways.

Step 3: Choose an ordering of the six people from steps 1 and 2 who will precede the president. This can be done in \( 6! \) ways.

Step 4: Choose an ordering of the six people (the remaining House members and senators) who will come after the president. This can be done in \( 6! \) ways.

\[
\binom{6}{3} \cdot \binom{6}{3} \cdot 6! \cdot 6!
\]
SHAPLEY-SHUBIK INDEX FOR THE PRESIDENT

\[
\frac{\binom{6}{3} \binom{6}{3} 6!6! + 2 \binom{6}{3} \binom{6}{4} 7!5! + 2 \binom{6}{3} \binom{6}{5} 8!4! + 2 \binom{6}{3} \binom{6}{6} 9!3!}{13!}
\]

\[
\frac{409}{1001} = 0.409
\]

In the real life US federal system, the President’s SSI is 0.16047.
BANZHAF POWER

• Banzhaf Power held by the president = 4%
• Banzhaf Power held by the Senate = 33%
• Banzhaf Power held by the House = 63%
DISCUSSION

• Do the measures of power we computed today reflect your sense of the power of the President and members of congress?

• Do you think the power of the president differs from the perceived power of the president? In what way(s)?

• What would happen if we remodeled the federal system into one that is weighted?