Classical Inequalities

**Arithmetic-geometric means:** The arithmetic mean \((a + b)/2\) of two nonnegative numbers \(a\) and \(b\) is always larger than or equal to its geometric mean \(\sqrt{ab}\), with equality if and only if \(a = b\). This can be seen from \(a + b - 2\sqrt{ab} = (\sqrt{a} - \sqrt{b})^2 \geq 0\). The inequality generalizes to more than two numbers: for all \(a_1, a_2, \ldots, a_n \geq 0\),

\[
\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n},
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\). A weighted version involves weights \(w_1, w_2, \ldots, w_n\) not all equal to \(1/n\). Namely, given \(w_1, w_2, \ldots, w_n > 0\) with \(w_1 + w_2 + \cdots + w_n = 1\), for all \(a_1, a_2, \ldots, a_n \geq 0\),

\[
\sum_{i=1}^{n} w_i a_i \geq \prod_{i=1}^{n} a_i^{w_i},
\]

with equality if and only if \(a_1 = a_2 = \cdots = a_n\). This can be proved as follows.

Set \(G := a_1^{w_1} a_2^{w_2} \cdots a_n^{w_n}\) and \(A := w_1 a_1 + w_2 a_2 + \cdots + w_n a_n\). Assume without loss of generality that \(a_1 \leq a_2 \leq \cdots \leq a_n\). Since \(a_1 \leq G \leq a_n\), we consider the integer \(k \in \{1 : n - 1\}\) such that \(a_k \leq G \leq a_{k+1}\). Then one can write

\[
\sum_{i=1}^{k} w_i \int_{a_i}^{G} \left( \frac{1}{x} - \frac{1}{G} \right) dx + \sum_{i=k+1}^{n} w_i \int_{G}^{a_i} \left( \frac{1}{G} - \frac{1}{x} \right) dx \geq 0.
\]

It follows that

\[
\sum_{i=1}^{n} w_i \int_{G}^{a_i} \frac{dx}{G} \geq \sum_{i=1}^{n} w_i \int_{G}^{a_i} \frac{dx}{x}, \quad \text{i.e.,} \quad \frac{A}{G} - 1 \geq \sum_{i=1}^{n} w_i \ln \frac{a_i}{G} = 0,
\]

as desired. Equality throughout means equality in (2), i.e., \(a_1 = \cdots = a_k = a_{k+1} = \cdots = a_n = G\).

**Cauchy–Schwarz inequality:** For all real numbers \(a_1, \ldots, a_n, b_1, \ldots, b_n\),

\[
\left( \sum_{j=1}^{n} a_j b_j \right)^2 \leq \left( \sum_{j=1}^{n} a_j^2 \right) \left( \sum_{j=1}^{n} b_j^2 \right),
\]

with equality if and only if \(a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n\). Cauchy–Schwarz inequality extends to other situations, for instance we can replace sums by integrals and obtain, for all real-valued functions \(f, g\) that are continuous on \([a, b]\),

\[
\left( \int_{a}^{b} f(x)g(x)dx \right)^2 \leq \left( \int_{a}^{b} f(x)^2dx \right) \left( \int_{a}^{b} g(x)^2dx \right),
\]

with equality if and only if \(f = g\).
Hölder inequality: This is a generalization of Cauchy–Schwarz inequality to all \( p, q > 1 \) satisfying \( 1/p + 1/q = 1 \) rather than \( p = q = 2 \). It reads, for all real numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n, \)

\[
\sum_{j=1}^{n} a_j b_j \leq \left( \sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left( \sum_{j=1}^{n} |b_j|^q \right)^{1/q},
\]

with equality if and only if \( a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n \). The integral version reads, for all real-valued functions \( f, g \) that are continuous on \([a, b]\),

\[
\int_{a}^{b} f(x)g(x)dx \leq \left( \int_{a}^{b} |f(x)|^p dx \right)^{1/p} \left( \int_{a}^{b} |g(x)|^q dx \right)^{1/q},
\]

with equality if and only if \( f = g \). For the proof, set \( u_j = |a_j|/A \) where \( A := \left( \sum_{j=1}^{n} |a_j|^p \right)^{1/p} \) and \( v_j = |b_j|/B \) where \( B := \left( \sum_{j=1}^{n} |b_j|^q \right)^{1/q} \). Notice that is is enough to prove that \( \sum_{j=1}^{n} u_j v_j \leq 1 \), knowing that \( u_1, \ldots, u_n, v_1, \ldots, v_n \geq 0 \), \( \sum_{j=1}^{n} u_j^p = 1 \), and \( \sum_{j=1}^{n} v_j^q = 1 \). In turn, it is enough to prove that \( uv \leq u^p/p + v^q/q \) for all \( u, v \geq 0 \) — this is known as Young’s inequality. To justify the latter, rewrite it as \( 1 \leq u^{p-1}v^{-1/p} + (p-1)u^{-1}v^{(p-1)/p} \), or, with \( t := u^{-1}v^{1/(p-1)} \), as \( t^{-(p-1)} + (p-1) t - 1 \geq 0 \). This can now be seen by studying the variations of the function \( f(x) := x^{-(p-1)} + (p-1)x - 1 \) on \([0, \infty)\).

Jensen inequality: Let \( \varphi \) be a convex function on an interval \( I \) — if \( \varphi \) is twice differentiable, this means that \( \varphi''(x) \geq 0 \) for all \( x \in I \). We have seen in ‘Induction and Recurrence’ that, if \( x_1, \ldots, x_n \in I \) and if \( t_1, \ldots, t_n \geq 0 \) satisfy \( t_1 + \cdots + t_n = 1 \), then

\[
\varphi \left( \sum_{j=1}^{n} t_j x_j \right) \leq \sum_{j=1}^{n} t_j \varphi(x_j).
\]

The integral version of Jensen inequality reads

\[
\varphi \left( \frac{1}{b-a} \int_{a}^{b} f(x)dx \right) \leq \frac{1}{b-a} \int_{a}^{b} \varphi(f(x))dx
\]

for any continuous function \( f \) on \([a, b]\).

Chebyshev inequality: If \( a_1 \leq a_2 \leq \cdots \leq a_n \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \) or \( a_1 \geq a_2 \geq \cdots \geq a_n \) and \( b_1 \geq b_2 \geq \cdots \geq b_n \), then

\[
\frac{1}{n} \sum_{j=1}^{n} a_j b_j \geq \left( \frac{1}{n} \sum_{j=1}^{n} a_j \right) \left( \frac{1}{n} \sum_{j=1}^{n} b_j \right).
\]

An easy argument consists in rearranging the inequality \( \sum_{i,j=1}^{n} (a_i - a_j)(b_i - b_j) \geq 0 \). An integral version of Chebyshev inequality reads, for functions \( f, g \) both nondecreasing on \([a, b]\) or both nonincreasing on \([a, b]\),

\[
\frac{1}{b-a} \int_{a}^{b} f(x)g(x)dx \geq \left( \frac{1}{b-a} \int_{a}^{b} f(x)dx \right) \left( \frac{1}{b-a} \int_{a}^{b} g(x)dx \right).
\]
Rearrangement inequality: If \( a_1 \leq a_2 \leq \cdots \leq a_n \) and \( b_1 \leq b_2 \leq \cdots \leq b_n \) and if \( \sigma \) is a permutation of \([1 : n]\), then

\[
\sum_{j=1}^{n} a_j b_{\sigma(j)} \leq \sum_{j=1}^{n} a_j b_{j}.
\]

One can use the technique of summation by parts for the proof of the rightmost inequality in (7), say. Setting \( B_0 = 0, B'_0 = 0 \), and

\[
B_j = \sum_{i=1}^{j} b_{\sigma(i)}, \quad B'_j = \sum_{i=1}^{n} b_i, \quad j \in [1 : n],
\]

we have \( B'_j \leq B_j \) for \( j \in [1 : n-1] \) and \( B'_n = B_n \). It follows that

\[
\sum_{j=1}^{n} a_j b_{\sigma(j)} = \sum_{j=1}^{n} a_j B_j - \sum_{j=1}^{n} a_j B_{j-1} = a_n B_n + \sum_{j=1}^{n-1} (a_j - a_{j+1}) B_j \leq 0 \quad \text{for } B'_j \geq B_j
\]

\[
\leq a_n B'_n + \sum_{j=1}^{n-1} (a_j - a_{j+1}) B'_j = \sum_{j=1}^{n} a_j b_j,
\]

where the last equality is just the reversal of the summation by parts process.

1 Exercises

Ex.1: Prove the inequality between the geometric and harmonic means, namely

\[
\frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n} \leq \sqrt[n]{a_1 a_2 \cdots a_n},
\]

for all \( a_1, a_2, \ldots, a_n > 0 \).

Ex.2: For a continuous convex function \( \varphi \) on \([a, b]\), deduce (1) from (3).

Ex.3: For \( a, b, c, d, \ldots \geq 0 \), prove that

\[
\sqrt{a + b + c + d + \cdots} + \sqrt{b + c + d + \cdots} + \sqrt{c + d + \cdots} + \cdots \geq \sqrt{a + 4b + 9c + 16d + \cdots}.
\]

Ex.4: Prove the leftmost inequality of (7).

Ex.5: Deduce (1) from Jensen inequality.

Ex.6: Prove Chebyshev inequality (5) using summation by parts.

Ex.7: Let \( P(x) \) be a polynomial with positive coefficients. Prove that \( P(1/x) \geq 1/P(x) \) for all \( x > 0 \), provided \( P(1) \geq 1 \).

Ex.8: If \( f \) is a continuous real-valued function on \([0, 1]^2\), prove that

\[
\int_{0}^{1} \left( \int_{0}^{1} f(x,y) \, dx \right)^2 \, dy + \int_{0}^{1} \left( \int_{0}^{1} f(x,y) \, dy \right)^2 \, dx \leq \left( \int_{0}^{1} \int_{0}^{1} f(x,y) \, dx \, dy \right)^2 + \int_{0}^{1} \int_{0}^{1} f(x,y)^2 \, dx \, dy.
\]