Hermitian Determinantal Representations of Polynomials

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Background

- This project began as an attempt to find an alternate Hermitian representation for bivariate Real-Zero Polynomials [G,K-V,V,W, 2013] in the form of $\det(I - KZ)$ where I is an identity matrix, K is a Hermitian Matrix, and Z is a diagonal matrix of the variables.
- Essentially, it was an attempt to see if the usual form $\det(I + A_1x + A_2y)$ where A_1 and A_2 are Hermitian [G,K-V,V,W, 2013] could have A_1 and A_2 condensed into a single matrix.
- While that did not come to pass, what was found was in interesting property held by all polynomials with a certain Hermitian representation.
- This propery is currently called by its working title "Property P".

Property P

Definition

Let $p(z_1, \ldots, z_d)$ be a polynomial in d variables. Then p has "Property P" if when $z_i = a_i t$, with $a_1, \ldots, a_d > 0$ or $a_1, \ldots, a_d < 0$, then $q(t) = p(a_1 t, \ldots, a_d t)$ has only real roots.

Matrix Representation

Let K be a Hermitian matrix, I be the identity matrix of the same size as K, and

$$Z = \begin{bmatrix} z_1 I_1 & 0 & \dots & 0 \\ 0 & z_2 I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z_d I_d \end{bmatrix} = diag(z_j I_j)_{j=1}^{d}$$

$\det(I - KZ)$ is a polynomial with Property P.

Proof

- $p(a_1 t, \ldots, a_d t) = \det(I - K \begin{bmatrix} a_1 t I_1 & 0 & \ldots & 0 \\ 0 & a_2 t I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a_d t I_d \end{bmatrix}).$
- Let $\alpha_i = \sqrt{a_i}$, and $A = \begin{bmatrix} \alpha_1 I_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \alpha_d I_d \end{bmatrix}.$
- A is also Hermitian.

Proof (continued)

- $p(a_1 t, \dots, a_d t) = \det(I - K \begin{bmatrix} a_1 t I_1 & 0 & \dots & 0 \\ 0 & a_2 t I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_d t I_d \end{bmatrix})$
- $= \det(I - tKA^2)$
- $= \det(A)\det(I - tKA^2)\det(A^{-1})$
- $= \det(A(I - tKA^2)A^{-1})$
- $= \det(I - tAKA)$
- AKA is Hermitian and the function has roots equal to the inverse of the nonzero eigenvalues of AKA. These roots will always be real, completing the proof. □

Coefficients

- Each term in $p(x, y) = \det(I - KZ)$ can be derived from the principal minors of K.
- Specifically, for a polynomial of degree d, each term of degree n is equal to $(-1)^n$ times the sum of $(n-d)^{th}$ order principal minors of K.

Example

- Consider $K = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ and $Z = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$.
- $C(x) = (-1) * 1 = -1$
- $C(y) = (-1) * 4 = -4$
- $C(z) = (-1) * 6 = -6$
- $C(xy) = (-1)^2 * \det(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}) = 0$
- $C(xz) = (-1)^2 * \det(\begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}) = -3$
- $C(yz) = (-1)^2 * \det(\begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}) = -1$
- $C(xyz) = (-1)^3 * \det(K) = -1 * -1 = 1$

Converse?

- The next matter of interest was whether or not the converse of the previous proof is true.
- That is, do all polynomials with Property P have a determinantal representation of this form?
- Clearly it is true for the univariate case, but what of higher-order polynomials?
- The answer is no. Proving this requires a lemma.

Lemma

- Consider $\frac{d}{dt}(\det(x_0 + tx_1))$ where x_0 is invertible.
- $\det(x_0 + tx_1) = \det(x_0)\det(I + tx_0^{-1}x_1)$
- Therefore, when $x = 0$, $\frac{d}{dt}(\det(x_0 + tx_1)) = \det(x_0)\frac{d}{dt}(\det(I + tx_0^{-1}x_1))$
- $= \det(x_0)tr(x_0^{-1}x_1)$

The Coefficient of xy

- The coefficient of the xy term in a bivariate polynomial of this form (henceforth referred to as $C(xy)$) is equal to $tr(k_{11})tr(k_{22}) - tr(k_{12}k_{21})$.
- For a given $p(x,y)$, $C(xy) = \frac{\partial}{\partial y}(\frac{\partial}{\partial x}(p(x,y)))$ when x and y are 0.
- $p(x,y) = \det(I + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix})$
- $=\det(\begin{bmatrix} I + k_{11}x & k_{12}y \\ k_{21}x & I + k_{22}y \end{bmatrix})$
- $=\det(\begin{bmatrix} I & k_{12}y \\ 0 & I + k_{22}y \end{bmatrix} + \begin{bmatrix} k_{11} & 0 \\ k_{21} & 0 \end{bmatrix} x)$
- Using the lemma, $\frac{\partial}{\partial x}(\det(\begin{bmatrix} I & k_{12}y \\ 0 & I + k_{22}y \end{bmatrix} + \begin{bmatrix} k_{11} & 0 \\ k_{21} & 0 \end{bmatrix} x)) =$ $\det(\begin{bmatrix} I & k_{12}y \\ 0 & I + k_{22}y \end{bmatrix})(tr(\begin{bmatrix} I & k_{12}y \\ 0 & I + k_{22}y \end{bmatrix}^{-1} \begin{bmatrix} k_{11} & 0 \\ k_{21} & 0 \end{bmatrix})) =$ $\det(I + k_{22}y)tr(k_{11} - k_{12}y(I + k_{22}y)^{-1}k_{21})$. This leaves us to compute $\frac{\partial}{\partial y}(\det(I + k_{22}y)tr(k_{11} - k_{12}y(I + k_{22}y)^{-1}k_{21}))$.

The Coefficient of xy (continued)

- By the product rule, this equals $\frac{\partial}{\partial y}(\det(I + k_{22}y))tr(k_{11} - k_{12}y(I + k_{22}y)^{-1}k_{21}) + \det(I + k_{22}y)\frac{\partial}{\partial y}(k_{11} - k_{12}y(I + k_{22}y)^{-1}k_{21})$.
- Given that $y = 0$, the first term can be simplified to $tr(k_{11})tr(k_{22})$.
- $\frac{\partial}{\partial y}(tr(k_{11} - k_{12}y(I + k_{22}y)^{-1}k_{21}))$
- $= \frac{\partial}{\partial y}(tr(k_{11}) - y * tr(k_{12}y(I + k_{22}y)^{-1}k_{21}))$
- $= \frac{\partial}{\partial y}(-y * tr(k_{12}y(I + k_{22}y)^{-1}k_{21}))$
- $= -(tr(k_{12}y(I + k_{22}y)^{-1}k_{21}))$.
- $\det(I + k_{22}y) * -(tr(k_{12}y(I + k_{22}y)^{-1}k_{21})) = -tr(k_{12}k_{21})$.
- Therefore, $C(xy) = tr(k_{11})tr(k_{22}) - tr(k_{12}k_{21})$.

Implications of the Proof

- $C(xy) = tr(k_{11})tr(k_{22}) - tr(k_{12}k_{21})$
- Given that K is Hermitian, k_{12} and k_{21} are transposes of one another.
- This means that $k_{12}k_{21}$ is positive-semidefinite and therefore $tr(k_{12}k_{21})$ is either positive or zero.
- By extension, $C(xy) \leq tr(k_{11})tr(k_{22})$.

Counterexample

- Consider $x - y + 1$.
- This polynomial has all real roots, so it is Property P.
- However, it does not have a self-adjoint representation.
- This is due to the previous inequality. $C(x) = k_{11} = 1$ and $C(y) = k_{22} = -1$, so $C(xy) \leq -1$.
- However, $C(xy) = 0$, leading to a contradiction.

Addtional counterexample

- This inequality also limits the possible values of the other coefficients in a different way.
- Consider $p(x, y) = -9xy^2 + 2xy - 10y^2 + x + 2y + 1$.
- $p(t, rt) = -9r^2t^3 - 10r^2t^2 + 2rt^2 + t + 2rt + 1$
- $\Delta = 4400r^6 + 1368r^5 - 347r^4 - 164r^3 + 40r^2$
- p has real roots as long as $\Delta \geq 0$
- $\Delta \geq 0$ for all values of r, meaning that $p(x, y)$ has Property P.

Additional counterexample (continued)

- As established earlier, $C(x) = tr(k_{11})$, $C(y) = tr(k_{22})$, and $C(xy) = tr(tr(k_{11})tr(k_{22}) - tr(k_{12}k_{21}))$.
- Since $C(x) = 1$ and $C(y) = 2$, for $C(xy)$ to equal 2 $tr(k_{12}k_{21})$ must equal 0.
- Given that $tr(k_{12}k_{21})$ is equal to the sum of the squares of the elements of k_{12}, the only possible matrix that will yield a trace of 0 is a zero matrix.
- This limits the other values of the coefficients to the products of their respective terms.
- Because $C(x) = 1$ and $C(y^2) = -10$, this means that $C(xy^2) = -10$. But $C(xy^2) = -9$, leading to a contradiction.

Observations

- The coefficients of the x and y terms depend only on the matrices k_{11} and k_{22}, respectively.
- $C(xy) \leq C(x) * C(y)$.
- If $C(xy) = C(x) * C(y)$, then for all positive a, b the other mixed coefficients $C(x^a y^b) = C(x^a) * C(y^b)$.
- The range of values the coefficients of the other mixed terms can take on varies directly with the difference between $C(xy)$ and $C(x) * C(y)$.

Future Work

With all of this in place, the next step is to see if there are any additional restrictions or inequalities that must hold for a Property P polynomial to have a Hermitian Representation. One such additional way of looking it this involves Kronecker products.

The Kronecker Product view

- $(A\otimes^n)(e_1 \wedge \cdots \wedge e_n) = \det(A)(e_1 \wedge \cdots \wedge e_n)$
- $\det(A)$ is an eigenvalue of $(A\otimes^n)$ with associated eigenvector $(e_1 \wedge \cdots \wedge e_n)$
- Additionally, $(I + KZ)\otimes^n = (I + Kz_1 + Kz_2)\otimes^n$
- In the 3×3 case this can be expressed as $I + z_1(Kz_1 \vee I \vee I) + z_2(Kz_2 \vee I \vee I) + z_1 z_2(Kz_1 \vee Kz_2 \vee I) + \cdots$
- The product of any of these terms and $(e_1 \wedge e_2 \wedge e_3)$ will give the resulting coefficient for that term.
- Further research will be done using this as a starting point.

Works Cited

[Grinshpan et al.(2013)]2013arXiv1306.6655G Grinshpan, A., Kaliuzhnyi-Verbovetskyi, D. S., Vinnikov, V., & Woerdeman, H. J. 2013, arXiv:1306.6655