Homework Set 8

Due March 20

NOTE: Please hand in the book and proof problems separately.

Section 4.7
Problems 6, 10, 12, 14, 16

Section 5.4
Problems 6, 10, 20, 24, 28

The Proof Problems:

PROBLEM 8.1: Let $V$ be a finite dimensional vector space of dimension $n$ with basis $B$. Let $\mathcal{L}(V)$ be the vector space of linear transformations $T : V \rightarrow V$, and if $T \in \mathcal{L}(V)$, then let $[T]_B$ denote the matrix of $T$ relative to the basis $B$.

a. Prove that the function $[\cdot]_B : \mathcal{L}(V) \rightarrow M_{n \times n}(\mathbb{R})$, given by $T \mapsto [T]_B$, is a (vector space) isomorphism.

b. Let $T \in \mathcal{L}(V)$. Prove that $T$ is invertible if and only if $[T]_B$ is invertible.

c. Let $T \in \mathcal{L}(V)$. Prove that $\dim(\ker T) = \dim(\text{Nul}([T]_B))$.

PROBLEM 8.2: Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Let $B = \{e_1, e_2, e_3, v\}$ be a basis of $\mathbb{R}^4$, where the $e_i$ are the standard basis vectors.

a. Find a vector $v$ such that

$$[T]_B = \begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$  

Prove that your answer is correct.

b. Prove that the first three columns of $[T]_B$ do not depend on $v$. 
c. Prove that there is no choice of \( v \) such that
\[
[T]_B = \begin{bmatrix}
1 & 0 & 0 & 8 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 5 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}.
\]

**PROBLEM 8.3:** Let \( T : V \to W \) be a linear transformation from the vector space \( V \) to the vector space \( W \). Let \( B \) be a basis for \( V \) and let \( C \) be a basis for \( W \). Let \( [T]_{C \leftarrow B} \) be the matrix for \( T \) relative to the bases \( B \) and \( C \) (as defined in equation (4) on page 289). Let \( r \) be the rank of \( [T]_{C \leftarrow B} \).

a. Prove that there exists a basis \( D \) of \( W \) such that \( [T]_{D \leftarrow B} \) has exactly \( r \) nonzero rows.

b. Prove that \( r = \dim(\text{range}(T)) \).

c. Let \( V = \mathbb{R}^2 \) and \( W = \mathbb{R}^3 \) and suppose
\[
[T]_{C \leftarrow B} = \begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}.
\]
Prove that there does not exist a basis \( A \) of \( V \) such that \( [T]_{C \leftarrow A} \) has exactly two nonzero rows.