Homework Set 9

Due March 27

Section 5.1
Problems 14, 20, 26, 32.

Section 5.2
Problems 2, 16, 19, 20.

Section 5.3
Problems 12, 22, 24, 28.

The Proof Problems:

PROBLEM 9.1. Let $A$ be both diagonalizable and invertible. Show that $A^{-1}$ is diagonalizable. What is the connection between the eigenvalues of $A^{-1}$ and those of $A$?

PROBLEM 9.2. Let $A$ and $B$ be two $n \times n$ matrices such that $AB = BA$, and assume that $A$ has $n$ distinct eigenvalues.

(a) If $\lambda$ is an eigenvalue of $A$, prove that $\text{dimNul}(A - \lambda I) = 1$.

(b) Prove that every eigenvector of $A$ is also an eigenvector of $B$.

PROBLEM 9.3. For a polynomial $p(x)$ and an $n \times n$ matrix $A$, let $p(A)$ denote the matrix obtained by “plugging in” $A$ for $x$. For example, if $p(x) = x^3 - 2x^2 + 3$, then $p(A) = A^3 - 2A^2 + 3I$.

(a) If $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, prove that $p(\lambda)$ is an eigenvalue of $p(A)$.
   (Hint: use the same eigenvectors that $A$ has.)

(b) If $P$ is invertible, show that the following equality of matrices holds: $p(P^{-1}AP) = P^{-1}p(A)P$.

(c) If $A$ is diagonalizable, prove that all eigenvalues of $p(A)$ are of the form $p(\lambda)$ for some eigenvalue $\lambda$ of $A$.
   (Note: This statement is true even if $A$ is not diagonalizable!)

PROBLEM 9.4. Let $A$ be a $2 \times 2$-matrix such that $A^2 = I$. Prove that $A$ is diagonalizable.
(Hint: show that either $A + I$ is not invertible or $A - I = 0$.)