Take home midterm
Due: Friday, February 24 at the beginning of class

Rules: You may consult your notes and the textbooks Vol. 1 and 2 of Stanley, Generating Functionology by Wilf, and Stanley’s notes on hyperplane arrangements; you may not consult with sources other than these or other people.

Problem 1. Let $a_n$ be the number of subsets of $[n]$ that contain no two consecutive elements. Determine the sequence $a_0, a_1, a_2, \ldots$, which begins $a_0 = 1, a_1 = 2, a_2 = 3$.

Problem 2. Let $T_n$ denote the number of set partitions of $[n]$ with an even number of blocks, all of which have even size. For example, $T_4 = 3$, corresponding to the set partitions $12|34, 13|24, 14|23$ of $[4]$. Determine the exponential generating function $T(x) := \sum_{n \geq 0} T_n x^n/n!$.

Problem 3. Let $P_n$ denote the poset of set partitions of $[n]$ ordered by refinement: $\{B_1, B_2, \ldots, B_k\} \leq \{B'_1, B'_2, \ldots, B'_{k'}\}$ if each $B_i$ is contained in some $B'_j$. Let $\hat{0} \in P_n$ be the set partition with $n$ blocks of size 1 and $\hat{1} \in P_n$ be the set partition with 1 block of size $n$. Determine $\mu(\hat{0}, \hat{1})$, where $\mu$ is the Möbius function of $P_n$.

Problem 4. Recall that the Eulerian number $A(n,k)$ is the number of permutations of $[n]$ with $k - 1$ descents. There are $n!$ sequences of integers $(b_1, \ldots, b_n)$ such that $0 \leq b_i \leq n - i$ for all $i$. Let $B(n,k)$ be the number of these sequences $(b_1, \ldots, b_n)$ such that $|\{b_1, \ldots, b_n\}| = k$. Prove that $B(n,k) = A(n,k)$.

Problem 5. Prove that
$$\prod_{i=1}^{s}(1 + x^{-1}q^i)\prod_{i=0}^{t-1}(1 + xq^i) = \sum_{j=-s}^{t} q^{(j)}\left[\begin{array}{c} s + t \\ s + j \end{array}\right] x^j.$$

Problem 6. Let $A$ be an arrangement in the $n$-dimensional vector space $V$ whose normals span a subspace $W$, and let $B$ be another arrangement in $V$ whose normals span a subspace $Y$. Suppose that $W \cap Y = \{0\}$. Show that $\chi_{A \cup B}(t) = t^{-n}\chi_A(t)\chi_B(t)$.

Problem 7. Evaluate the sum below (and prove that your answer is correct):
$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

Problem 8. A prime parking function is a sequence $a = (a_1, \ldots, a_n)$ of positive integers that contains at least $k + 1$ entries $\leq k$, for $k = 1, \ldots, n - 1$. Prove that the number of prime parking functions is $(n - 1)^{n-1}$. 
