Math 201  
Homework 1  
Due: At the beginning of class on January 13

Book 1.1-1.3  
1.1: 14, 16, 24, 26  
1.2: 8, 12, 16, 20, 24, 26  
1.3: 6, 22, 26, 32

Logic and Symbols

(1) True or false (justify your answer):
   (a) $\forall x \in \mathbb{R}$, we have $x^4 > 0$.
   (b) $\exists y \in \mathbb{Z}$ such that $\forall x \in \mathbb{R}$, $y > x$.
   (c) $\exists a, b \in \mathbb{R}$ such that $\{x \in \mathbb{R} \mid ax^2 = b\} = \emptyset$.
   (d) If $1 > 2$, then $2 > 3$.
   (e) If $x \in \mathbb{Q}$, then $\exists y \in \mathbb{Z}$ such that $xy \in \mathbb{Z}$.
   (f) $x < 0$ if and only if $x^3 < 0$.
   (g) $\{x \in \mathbb{R} \mid x > 0\} = \{x \in \mathbb{R} \mid x^2 > 0\}$.

(2) Write the following statements using symbols and quantifiers:
   (a) There exists an integer such that if you multiply it by any real number, you get zero.
   (b) For every real number, there exists an integer that you can multiply it with to get an integer.

(3) Write the negation of each of the following statements.
   (a) Every problem has a solution.
   (b) The set $S$ contains at least two integers.
   (c) She likes dogs or dislikes cats.
   (d) If you study hard, then you will do well in this class.
   (e) Chickens have feathers if and only if 2 is not an integer.
Converse and Contrapositive. There are two additional logical statements that can be formed from a given “if-then” statement:

- The **converse** of the statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. The converse may be true or false, independent of the truth value of the original “if-then” statement. Why? Compare the truth tables for both statements:

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The last two columns do not coincide.

- The **contrapositive** of the statement $P \Rightarrow Q$ is the statement $\text{not } Q \Rightarrow \text{not } P$. The original “if-then” statement and its contrapositive have the same truth value. Why? Compare the truth tables for both statements:

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The columns corresponding to $P \Rightarrow Q$ and $\text{not } Q \Rightarrow \text{not } P$ coincide.

(4) Write the converse and the contrapositive of each of the following statements:

- (a) If $1 = 2$, then *Modern Family* is the best sitcom on television.
- (b) If $x > 0$, then $x^2 < 0$ or $x^3 > 0$.

(5) (a) Give an example of an “if-then” statement that has a truth value different from the truth value of its converse.
- (b) Give an example of an “if-then” statement that has the same truth value as its converse.

Set Theory

(6) (a) Use set-theoretic notation to define the half-open interval $(a, b]$ in the real numbers.
- (b) Write a common English description for the following set:
  
  \[ \{ x \in \mathbb{Z} \mid \exists y \in \mathbb{Z} \text{ such that } x = 2y - 1 \} \]

- (c) List all of the subsets of $S = \{\{1, 2\}, 3, 4\}$.

(7) Let $X = \{ x \in \mathbb{R} \mid \frac{1}{2} \leq x < 5 \}$, let $Y = \{ x \in \mathbb{R} \mid x = 2y + 1 \text{ for some } y \in \mathbb{Z} \}$, and let $Z = \{ x \in \mathbb{R} \mid x^2 > 0 \}$. Explicitly compute the following:

- (a) $X \cup Z$
- (b) $X \cap Y$
- (c) $X \setminus Y$
- (d) $Y \setminus Z$