Problem 1.1. In each case either show that $G$ is a group with the given operation or list the axioms that fail.

(a). $G = \mathbb{N}$; addition
(b). $G = \{2n \mid n \in \mathbb{Z}\}$; addition
(c). $G = \mathbb{R}$; $a \cdot b = a + b + 1$
(d). $G = \mathbb{R}$; $a \cdot b = a + b - ab$
(e). $G = \{q \in \mathbb{Q} \mid q > 0\}$; multiplication

Problem 1.2. Let $S$ be the set of all real numbers except $-1$. Define $*$ on $S$ by $a*b = a + b + ab$.

(a). Show that $(S,*)$ is a group.
(b). Find the solution to the equation $2*x*5 = 6$ in $S$.

Problem 1.3. Determine (with proof) whether or not the given set of invertible $n \times n$ matrices with real number entries is a subgroup of $GL_n(\mathbb{R})$.

(a). The $n \times n$ matrices with determinant 2
(b). The diagonal $n \times n$ matrices with no zeros on the diagonal
(c). The upper-triangular $n \times n$ matrices with no zeros on the diagonal
(d). The $n \times n$ matrices with determinant 1
(e). The $n \times n$ matrices with determinant $-1$ or $1$
(f). The set of all $n \times n$ matrices $A$ such that $A^T A = I$
PROBLEM 1.4. For the choices of $S \subset \mathbb{R}^3$ below, determine (with proof) the set $G$ of all linear transformations $T$ such that $T(S) = S$.

(a) $S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \subset \mathbb{R}^3$.

(b) $S = \{ v^1, v^2, v^3 \}$ is a basis of $\mathbb{R}^3$. Hint: the answer can be written in terms of the matrix $M = [v^1 \ v^2 \ v^3]$ with columns $v^1, v^2, v^3$ and its inverse.

PROBLEM 1.5. If $G$ is a group and $g \in G$, define $C(g) = \{ z \in G \mid zg = gz \}$. Show that $C(g)$ is a subgroup of $G$ (the centralizer of $g$ in $G$).

PROBLEM 1.6. If $(ab)^n = 1$ in a group, where $n \geq 0$, show that $(ba)^n = 1$.

PROBLEM 1.7. If $a^4 = 1$ and $ab = ba^2$ in a group, show that $a = 1$. 