Functions

Let $X$ and $Y$ be sets. A function $f : X \to Y$ is a map which assigns a unique element $f(x) \in Y$ to each element $x \in X$. The domain of $f$ is the set $X$; the codomain of $f$ is the set $Y$.

Let $A \subseteq X$. The image of $A$ under $f$ is the set

$$f(A) = \{ f(x) \mid x \in A \} .$$

Let $B \subseteq Y$. The preimage of $B$ under $f$ is the set

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \} .$$

**Problem 2.1.** Decide whether or not each of the following is a function. Justify your answers.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - 2x + 1$.

(b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ x - 1 & \text{if } x \leq 0. \end{cases}$

(c) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 - 2x + 1 & \text{if } x \geq 0, \\ -x^3 + 1 & \text{if } x \leq 0. \end{cases}$

**Problem 2.2.** Decide whether the following statements are true or false. If true, prove it. If false, provide a counterexample which shows that the statement is false; *i.e.* give an explicit, concrete example of a function $f$ for which the equality fails—don’t forget to provide the domain and codomain in your example!
(a) \( f(f^{-1}(B)) \subseteq B \) for every subset \( B \) of \( Y \).

(b) \( f^{-1}(f(A)) = A \) for every subset \( A \) of \( X \).

(c) \( f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \) for all subsets \( A_1, A_2 \) of \( X \).

(d) \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \) for all subsets \( A_1, A_2 \) of \( X \).