Homework Set 2

Math 222 — Fall 2017

Due Wednesday, October 11

Problem 2.1. Prove the following identity (for \( n \geq 0 \)):

\[
3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.
\]

Problem 2.2. Determine (with proof) the number of \( r \)-tuples of integers \((a_1, \ldots, a_r)\) satisfying \( a_i \geq i \) for \( i = 1, \ldots, r \), and \( a_1 + a_2 + \cdots + a_r = n \).

Problem 2.3. We showed in class that the total number of compositions of \( n \) is \( 2^{n-1} \). Find a simple bijective proof of this fact (the set of subsets of \([n-1]\) has size \( 2^{n-1} \), so this is a natural candidate to show to be in bijection with the set of compositions of \( n \), however something similar but slightly different may work better).

Problem 2.4. Prove the trinomial formula:

\[
(x + y + z)^n = \sum_{a,b,c \geq 0, a+b+c=n} \binom{n}{a,b,c} x^a y^b z^c,
\]

where \( \binom{n}{a,b,c} = \frac{n!}{a!b!c!} \).

Problem 2.5. Let \( A_n \) be the \( n \times n \) matrix whose \((i,j)\) entry is \( \binom{i}{j} \), with rows and columns numbered starting from 0. So, for example,

\[
A_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}.
\]

Compute \( A_2^{-1}, A_3^{-1} \) and \( A_4^{-1} \). Find and prove a formula for \( A_n^{-1} \).

Problem 2.6. Let \( R_{d,n} \) be the number of regions cut out by \( n \) hyperplanes in general position in \( d \)-space. In class we found the recurrence \( R_{d,n+1} = R_{d,n} + R_{d-1,n} \). Using this, find a formula for \( R_{d,n} \) in terms of binomial coefficients.