Homework Set 2

Math 222 — Fall 2021

Due Wednesday, October 6

Problem 2.1. Prove the following identity (for $n \geq 0$):

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Problem 2.2. Determine (with proof) the number of $r$-tuples of integers $(a_1, \ldots, a_r)$ satisfying $a_i \geq i$ for $i = 1, \ldots, r$, and $a_1 + a_2 + \cdots + a_r = n$.

Problem 2.3. We showed in class that the total number of compositions of $n$ is $2^{n-1}$. Find a simple bijective proof of this fact (the set of subsets of $[n-1]$ has size $2^{n-1}$, so this is a natural candidate to show to be in bijection with the set of compositions of $n$, however something similar but slightly different may work better).

Problem 2.4. Prove the trinomial formula:

$$(x + y + z)^n = \sum_{a,b,c \geq 0, a+b+c=n} \binom{n}{a b c} x^a y^b z^c,$$

where $\binom{n}{a b c} = \frac{n!}{a!b!c!}$.

Problem 2.5. Let $A_n$ be the $n \times n$ matrix whose $(i,j)$ entry is $\binom{i}{j}$, with rows and columns numbered starting from 0. So, for example,

$$A_5 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 \\
1 & 4 & 6 & 4 & 1
\end{pmatrix}.$$

Compute $A_2^{-1}$, $A_3^{-1}$ and $A_4^{-1}$. Find and prove a formula for $A_n^{-1}$.

Problem 2.6. Let $R_{d,n}$ be the number of regions cut out by $n$ hyperplanes in general position in $d$-space. In class we found the recurrence $R_{d,n+1} = R_{d,n} + R_{d-1,n}$. Using this, find a formula for $R_{d,n}$ in terms of binomial coefficients.