Problem 3.1. In class we used inclusion-exclusion with the sets $E_i = \{ \pi \in S_n \mid \pi(i) = i \}$ to compute the number of derangements of $n$. As we did in class for $n = 3$, draw the Venn diagram for $n = 4$ illustrating the sets $E_1, E_2, E_3, E_4$. Since it’s hard to draw Venn diagram 4 sets, one way to draw this is to draw sets $E_1 - E_4, E_2 - E_4, E_3 - E_4$ and as two separate Venn diagrams. For the permutation 1234, how many times is it counted in each sum
\[\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq 4} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_j}|, \text{ for } j = 1, 2, 3, 4?\]

Problem 3.2. For $\pi \in S_n$, let $f(\pi)$ denote the number of fixed points of $\pi$.

(i) Determine the expected number of fixed points in a random permutation, that is, compute
\[\frac{1}{n!} \sum_{\pi \in S_n} f(\pi).\]

(ii) For $0 \leq k \leq n$, give a formula for the number of permutations with exactly $k$ fixed points (and prove that your formula is correct).

Problem 3.3. We will show in class that $F_n$ is the number of ways to tile a $2 \times (n-1)$ chessboard with dominoes. Let $G_n$ be the number of ways to tile a $3 \times (n-1)$ chessboard with $3 \times 1$ or $1 \times 3$ tiles (the same as a domino tiling, but with longer dominoes). Write down a recurrence for $G_n$ and compute $G_1, \ldots, G_{10}$.

Problem 3.4. Prove combinatorially that the number of compositions of $n$ with odd parts is the $n$-th Fibonacci number $F_n$ by finding a simple bijection between such compositions and the compositions of $n - 1$ with parts 1 and 2. (For example, for $n = 5$ this requires finding a bijection between $\{311, 131, 11111, 113, 5\}$ and $\{112, 121, 211, 22, 1111\}$.)

Problem 3.5. Find and prove a formula for the number of 2-dimensional subspaces of the vector space $\mathbb{F}_p^n$, where $p$ is a prime. For example, for $n = 3$ and $p = 2$, there are 7 such subspaces given by the row spaces of the following matrices:
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 0
\end{pmatrix}.
\]

Problem 3.6. Let $\mathbb{F}_p$ be the finite field with $p$ elements for some prime $p$ and let $\mathbb{F}_p[x]$ be the ring of polynomials in the variable $x$ with coefficients in $\mathbb{F}_p$. How many monic polynomials of degree $n$ are there in $\mathbb{F}_p[x]$ that do not take on the value 0 for $x \in \mathbb{F}_p$? (A polynomial is monic if it is of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.)