Problem 3.1. Prove that for every group $G$ with $|G| = p^n$ for some prime $p$ and $n \geq 1$, $G$ contains an element of order $p$.

Problem 3.2. Suppose that $G$ has subgroups of orders 45 and 75. If $|G| < 400$, determine $|G|$.

Problem 3.3. If $H$ and $K$ are subgroups of a group and $|H|$ is prime, show that $H \leq K$ or $H \cap K = \{e\}$.

Problem 3.4. Determine (with proof) all subgroups of $D_8$ and draw the lattice of subgroups.

Problem 3.5. Let $M_n(\mathbb{R})$ be the additive group of all $n \times n$ matrices with real entries, and let $\mathbb{R}$ be the additive group of real numbers. Determine if the function $\phi : M_n(\mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = \det(A)$ is a group homomorphism.

Problem 3.6. Determine if the function $\phi : GL_n(\mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = tr(A)$ is a group homomorphism, where $tr(A) = \sum_{i=1}^{n} A_{ii}$ is the trace of the matrix $A$. (Here, $\mathbb{R}$ denotes the additive group of real numbers.)

Problem 3.7. Let $H$ be a subgroup of a group $G$. Prove that the partition of $G$ into left cosets of $H$ is the same as the partition into right cosets of $H$ if and only if $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$.

Problem 3.8. Let $H \leq G$ and let $a, b \in G$. Prove or give a counterexample:

(a) If $Ha = Hb$, then $b \in Ha$.

(b) If $aH = bH$, then $a^2H = b^2H$.

Problem 3.9. Complete the classification of groups of order 6 from class: Show that there is no group $G$ of size 6 such that all its non-identity elements have order 2.