**Homework Set 4**

**Math 221 — Fall 19**

*Do not turn in*

**Problem 4.1.**

a) How many permutations of 7 are there so that the first three numbers are increasing (an example of such a permutation is 2571436)?

b) How many strings of length 5 are there in the alphabet 1, 2, \ldots, 7 such that the first three numbers are increasing (an example of such a string is 25714)?

**Problem 4.2.**

Compute \( |\{A \subseteq \{1, 2, \ldots, 10\} : |A \cup \{1, 17\}| = 5\}| \). In other words, determine the number of subsets \( A \) of \( \{1, 2, \ldots, 10\} \) such that \( A \cup \{1, 17\} \) has size 5.

**Problem 4.3.** A lattice path from a point \((a, b)\) to a point \((c, d)\) is a path from \((a, b)\) to \((c, d)\) using only steps of length 1 going east and steps of length 1 going north. The figure below depicts a lattice path from the point \((0, 0)\) to the point \((5, 6)\).

![Lattice Path Diagram]

a) How many lattice paths are there from \((0, 0)\) to \((5, 6)\)?

b) How many lattice paths are there from \((0, 0)\) to \((5, 6)\) that go through the point \((3, 3)\)?

**Problem 4.4.** We proved the following identity in class (for \(n, m \geq 0\) and \(n + m \geq k\)):

\[
\binom{n}{0} \binom{m}{k} + \binom{n}{1} \binom{m}{k-1} + \cdots + \binom{n}{k-1} \binom{m}{1} + \binom{n}{k} \binom{m}{0} = \binom{n+m}{k}.
\]

Give another proof using the binomial theorem and the fact that \((x + 1)^{n+m} = (x + 1)^n(x + 1)^m\).
Problem 4.5. Suppose we toss a fair coin three times in a row. What is the sample space $S$ that models this situation? Let $E$ be the event that exactly two of the tosses are heads. Write out $E$ explicitly as a subset of $S$. What is $P(E)$, i.e., the probability of getting exactly two heads?

Problem 4.6. Suppose you roll a blue six-sided die and a red six-sided die. Write down the event that a 6 appears on at least 1 of the dice (explicitly as a subset of the sample space). What is the probability of this event?

Problem 4.7. If you roll three six-sided dice, what is the probability that the result sums to 6?

Problem 4.8. Find the number of hands of the type straight flush: five cards of the same suit with consecutive values (we take the convention that A can be high or low but cannot wrap around, so $\{AS, 2S, 3S, 4S, 5S\}$ is a straight flush but $\{QC, KC, AC, 2C, 3C\}$ is not). Find the probability of being dealt such a hand.

Problem 4.9. a) List all the hands that consist of three jacks and two sevens (one such hand is $\{JS, JD, JH, 7C, 7S\}$).

b) How many such hands are there? Show how to count this quickly using binomial coefficients.

c) How many hands are there that consist of three jacks and a pair with value A, 2, 3, or 4 (one such hand is $\{JS, JD, JH, 2C, 2S\}$, another such hand is $\{JC, JD, JH, 4C, 4H\}$)?