**Homework Set 5**

**Math 331 — Fall 2019**

*Due Friday, November 8*

**Problem 5.1.** Let $\sigma \in S_n$ be with cycle notation of the form $(a_1a_2\ldots a_r)$ with $r$ odd. Show that $\sigma^2$ is also a cycle of the same form.

**Problem 5.2.** Let $X \subseteq \{1, 2, \ldots, n\}$. Let $H_X$ be the subset of $S_n$ consisting of bijections $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\pi(x) = x$ for all $x \in X$. Show that $H_X$ is a subgroup of $S_n$.

**Problem 5.3.** Find all subgroups of $S_4$ of size 4. Justify your answer—the main thing to show here is that your list is complete.

**Problem 5.4.** For $n \geq 4$, show that there is no normal subgroup of $S_n$ of size 3.

**Problem 5.5.** Let $H$ be the smallest subgroup of $S_5$ which contains $(12345)$ and $(12)(35)$ (we are using cycle notation). Describe $H$ by listing its elements explicitly in cycle notation. Check that what you have obtained is actually a subgroup by showing that it is closed under multiplication; there is a way to do this without explicitly computing all products of two elements.

**Problem 5.6.** Show that if a finite group $G$ has exactly one subgroup $H$ of a given order, then $H$ is a normal subgroup of $G$.

**Problem 5.7.** Show that if $\phi : \mathbb{Z}_6 \times \mathbb{Z}_3 \to \mathbb{Z}_{14}$ is a nontrivial homomorphism, then $|\ker(\phi)| = 9$.

**Problem 5.8.** Show that if $\phi : G \to H$ is a group homomorphism and $|G|$ and $|H|$ are finite with $\gcd(|G|, |H|) = 1$, then $\phi$ is trivial.

**Problem 5.9.** Suppose that $G$ is finite and $K \trianglelefteq G$ has index $m$. Show that if $\gcd(n, m) = 1$, then $K$ contains every element of $G$ of order $n$. 