Homework Set 6

MATH 331 — FALL 2019

Due Friday, November 15

PROBLEM 6.1. List all possible cycle types for $S_5$. For each cycle type, write down one permutation of that cycle type and indicate how many elements there are of this cycle type. No proof is required.

PROBLEM 6.2. Show that in $S_n$, the transposition $(12)$ is not equal to a product of 3-cycles.

PROBLEM 6.3. A group $G$ is generated by $a_1, a_2, \ldots, a_k \in G$ if every element of $G$ can be written as a product $a_{i_1}^{n_1} a_{i_2}^{n_2} \cdots a_{i_\ell}^{n_\ell}$ for some $i_1, i_2, \ldots, i_\ell \in \{1, 2, \ldots, k\}$ and $n_1, n_2, \ldots, n_\ell \in \mathbb{Z}$. Show that $S_n$ is generated by $(12), (23), \ldots, ((n-1)n)$; these elements are called the simple reflections.

PROBLEM 6.4. In the previous problem set, we showed that
\[
\{e, (12345), (13524), (14253), (15432), (12)(35), (23)(41), (34)(52), (45)(13), (51)(24)\}
\]
is a subgroup of $S_5$. This is called the dihedral group of order 10, denoted $D_{10}$. Determine (with proof) the conjugacy classes of $D_{10}$.

PROBLEM 6.5. Determine (with proof) the conjugacy classes of $A_4$. Warning: these are not necessarily the same as the intersection of a conjugacy class of $S_4$ with $A_4$.

PROBLEM 6.6. An automorphism of a group $G$ is called an inner automorphism if it is of the form $\psi_g : G \to G$ for some $g \in G$, where $\psi_g$ denotes the conjugation by $g$ automorphism. Find an automorphism of a group $G$ which is not an inner automorphism. Justify your answer.

PROBLEM 6.7. Let $G$ be a group acting on a set $X$. Let $Y \subseteq X$. Let $G_Y = \{g \in G \mid g \cdot y = y \text{ for all } y \in Y\}$. Show $G_Y$ is a subgroup of $G$.

PROBLEM 6.8. Consider the natural action of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$, where the action of $A \in GL_2(\mathbb{R})$ on $v \in \mathbb{R}^2$ is $Av$. Let $H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{R} \right\}$. This is a subgroup of $GL_2(\mathbb{R})$ and by restriction it acts on $\mathbb{R}^2$ (you do not need to prove this). Determine the orbits for the action of $H$ on $\mathbb{R}^2$. 