**Problem 8.1.** The **chromatic number** $\chi(G)$ of a graph $G$ is the smallest $k$ such that $G$ is $k$-colorable.

a) Determine the chromatic number of the cube.

b) Determine the chromatic number of the dodecahedron.

c) Determine the chromatic number of the icosahedron.

**Problem 8.2.** Show that the following graphs are planar by drawing them without edge crossings:

![Planar Graphs](image)

**Problem 8.3.** Construct a connected planar graph on 12 vertices such that every vertex has degree 4.

**Problem 8.4.** Prove that every planar graph has a vertex of degree at most 5.

**Problem 8.5.** In this problem, you will prove that the Petersen graph is not planar. Let $P$ denote the Petersen graph. The proof will be a proof by contradiction. We will assume that $P$ can be drawn without edge crossings and then arrive at a contradiction.

a) As we are assuming $P$ is planar, let $f$ be the number of faces in a planar drawing of $P$. Let $n$ be the number of vertices of $P$ and $e$ the number of edges. What are $n$ and $e$? Compute $f$ using Euler’s formula.

b) Let $l_1, l_2, \ldots, l_f$ denote the lengths of the faces of the planar drawing of $P$. Show that $l_1 + l_2 + \cdots + l_f = 2e$.

c) Every cycle of the Petersen graph has length at least 5, hence the integers $l_1, l_2, \ldots, l_f$ are all at least 5. Use this together with parts (a) and (b) to obtain the desired contradiction.

**Problem 8.6.** Prove by induction on the number of faces that a plane graph $G$ is bipartite if and only if every face has even length.
PROBLEM 8.7. Prove that every $n$-vertex plane graph isomorphic to its dual has $2n - 2$ edges. For all $n \geq 4$, construct a simple $n$-vertex plane graph isomorphic to its dual.