Homework Set 8

Math 222 — Fall 2023

Due Wednesday, December 6

Problem 8.1. The chromatic number \(\chi(G)\) of a graph \(G\) is the smallest \(k\) such that \(G\) is \(k\)-colorable.

a) Determine the chromatic number of the cube.

b) Determine the chromatic number of the dodecahedron.

c) Determine the chromatic number of the icosahedron.

Problem 8.2. In this problem, you will prove that the Petersen graph is not planar. Let \(P\) denote the Petersen graph. The proof will be a proof by contradiction. We will assume that \(P\) can be drawn without edge crossings and then arrive at a contradiction.

a) As we are assuming \(P\) is planar, let \(f\) be the number of faces in a planar drawing of \(P\). Let \(n\) be the number of vertices of \(P\) and \(e\) the number of edges. What are \(n\) and \(e\)? Compute \(f\) using Euler’s formula.

b) Let \(l_1, l_2, \ldots, l_f\) denote the lengths of the faces of the planar drawing of \(P\). Show that \(l_1 + l_2 + \cdots + l_f = 2e\).

c) Every cycle of the Petersen graph has length at least 5, hence the integers \(l_1, l_2, \ldots, l_f\) are all at least 5. Use this together with parts (a) and (b) to obtain the desired contradiction.

Problem 8.3. Prove by induction on the number of faces that a plane graph \(G\) is bipartite if and only if every face has even length.

Problem 8.4. Prove that every \(n\)-vertex plane graph isomorphic to its dual has \(2n - 2\) edges. For all \(n \geq 4\), construct a simple \(n\)-vertex plane graph isomorphic to its dual.

Problem 8.5. Let \(S\) be the set \(\{1, 2, \ldots, mn\}\). Partition \(S\) into \(m\) sets \(A_1, \ldots, A_m\) of size \(n\) each. Also partition \(S\) into \(m\) sets \(B_1, \ldots, B_m\) of size \(n\) each. Show that the \(A_i\) can be renumbered so that \(A_i \cap B_i\) is non-empty for every \(i\).

Problem 8.6. For each \(k > 1\), construct a \(k\)-regular graph having no perfect matching.