HAGLUND'S CONJECTURE ON 3-COLUMN MACDONALD POLYNOMIALS

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Abstract. We prove a positive combinatorial formula for the Schur expansion of LLT polynomials indexed by a 3-tuple of skew shapes. This verifies a conjecture of Haglund [12]. The proof requires expressing a noncommutative Schur function as a positive sum of monomials in Lam's algebra of ribbon Schur operators [15]. Combining this result with the expression of Haglund, Haiman, and Loehr [13] for transformed Macdonald polynomials in terms of LLT polynomials then yields a positive combinatorial rule for transformed Macdonald polynomials indexed by a shape with 3 columns.

1. Introduction

In the late 90's, Lascoux, Leclerc, and Thibon [17] defined a family of symmetric functions depending on a parameter q, in terms of ribbon tableaux and the spin statistic. These symmetric functions, known as *LLT polynomials*, are now fundamental in the study of Macdonald polynomials and diagonal coinvariants, and have intriguing connections to Kazhdan-Lusztig theory, k-Schur functions, and plethysm.

In this paper we work with the version of LLT polynomials from [14], which are indexed by k-tuples of skew shapes. We give a positive combinatorial formula for the Schur expansion of LLT polynomials indexed by a 3-tuple of skew shapes. The Haglund-Haiman-Loehr formula [13] expresses the transformed Macdonald polynomials $\tilde{H}_\mu(\mathbf{x}; q, t)$ as a positive sum of the LLT polynomials indexed by a k-tuple of ribbon shapes, where k is the number of columns of μ. Haglund [12] conjectured a formula for the LLT polynomials that appear in this formula in the case that the partition μ has 3 columns. Our result proves and generalizes this formula.

The *new variant q-Littlewood-Richardson coefficients* $\mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q)$ are the coefficients in the Schur expansion of LLT polynomials, i.e.

$$\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x}; q) = \sum_\lambda \mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q) s_\lambda(\mathbf{x}),$$

where $\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x}; q)$ is the LLT polynomial indexed by a k-tuple $\boldsymbol{\beta}$ of skew shapes; the adjective new variant indicates that these correspond to the version of LLT polynomials from [14], not the version used in [17, 19, 15]. These coefficients are polynomials in q with nonnegative integer coefficients. This was proved in the case that $\boldsymbol{\beta}$ is a tuple of partition shapes [19] by showing that these coefficients are essentially parabolic Kazhdan-Lusztig polynomials.

Key words and phrases. LLT polynomials, q-Littlewood-Richardson coefficients, noncommutative Schur functions, flagged Schur functions, inversion number.

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The general case was proved in [11], also using Kazhdan-Lusztig theory. The paper [3] claims a combinatorial proof of positivity. However, neither of these methods yields an explicit positive combinatorial interpretation of the new variant q-Littlewood-Richardson coefficients for $k > 2$. (The approach of [3], though combinatorial, involves an intricate algorithm to transform a D graph into a dual equivalence graph and has yet to produce explicit formulas for $k > 2$.)

Explicit combinatorial formulas for the new variant q-Littlewood-Richardson coefficients have been found in the following cases. A combinatorial interpretation for the $k = 2$ case was stated by Carré and Leclerc in [7], and its proof was completed by van Leeuwen in [21] (see [13, §9]). Assaf [1] gave another interpretation of these coefficients. Roberts [20] extended the work of Assaf to give an explicit formula for $\mathfrak{c}^{\lambda}_{\boldsymbol{\beta}}(q)$ in the case that the diameter of $\boldsymbol{\beta}$ is ≤ 3, where the diameter of a k-tuple $\boldsymbol{\beta}$ of skew shapes is

$$\max\big\{\big|C \cap \{i, i+1, \dots, i+k\}\big| : i \in \mathbb{Z}\big\}, \tag{1}$$

where C is the set of distinct shifted contents of the cells of $\boldsymbol{\beta}$ (see (2) below). Formulas for the coefficient of $s_\lambda(\mathbf{x})$ in $\tilde{H}_\mu(\mathbf{x}; q, t)$ are known when λ or μ is a hook shape and when μ has two rows or two columns. Fishel [8] gave the first combinatorial interpretation for such coefficients in the case μ has 2 columns using rigged configurations. Zabrocki and Lapointe-Morse also gave formulas for this case [23, 16].

Let $\mathcal{U}$ be the free associative algebra in the noncommuting variables u_i, $i \in \mathbb{Z}$. The *plactic algebra* is the quotient of $\mathcal{U}$ by the Knuth equivalence relations. It has been known since the work of Lascoux and Schützenberger [18] that the plactic algebra contains a subalgebra isomorphic to the ring of symmetric functions, equipped with a basis of noncommutative versions of Schur functions. Fomin and Greene [9] showed that a similar story holds if certain pairs of Knuth relations are replaced by weaker four-term relations. This yields positive formulae for the Schur expansions of a large class of symmetric functions that includes the Stanley symmetric functions and stable Grothendieck polynomials. Lam [15] realized later that some of this machinery can be applied to LLT polynomials. To do this, he defined the *algebra of ribbon Schur operators* $\mathcal{U}/I_{\mathrm{L},k}$ to be the algebra generated by operators u_i which act on partitions by adding k-ribbons. He also explicitly described the relations satisfied by the u_i, which we take here as the definition of this algebra (see §2.4). Lam gave a simple interpretation of LLT polynomials using this algebra (Remark 2.9).

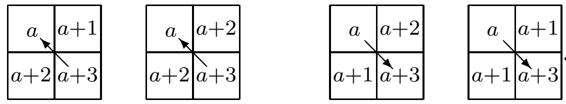
The main difficulty in carrying out the Fomin-Greene approach in this setting is expressing the noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$ (see §4.1) as a positive sum of monomials in $\mathcal{U}/I_{\mathrm{L},k}$. Lam does this for λ of the form $(a, 1^b)$, $(a, 2)$, $(2, 2, 1^a)$ and k arbitrary.

Our main theorem is

Theorem 1.1. *In the algebra $\mathcal{U}/I_{\mathrm{L},3}$, the noncommutative Schur function $\mathfrak{J}_\lambda(\mathbf{u})$ is equal to the following positive sum of monomials*

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{T \in \mathrm{RSST}_\lambda} \mathrm{sqread}(T).$$

Here, RSST_λ is the set of semistandard Young tableaux of shape λ that are row strict and such that entries increase in increments of at least 3 along diagonals. To define $\mathrm{sqread}(T)$, first draw arrows as shown between entries of T for each of its 2×2 subtableaux of the following forms:



Then $\mathrm{sqread}(T)$ is a reading word of T in which the tail of each arrow appears before the head of each arrow. Any reading word satisfying this property can be used in Theorem 1.1 and Corollary 1.2 below—$\mathrm{sqread}(T)$ is just a convenient choice (see Definition 3.4). For example,

$$\mathrm{sqread}\left(\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&4&5&7\\\hline 8\\\cline{1-1}\end{array}\right) = \mathsf{834152476}.$$

We show that by an adaptation of the Fomin-Greene theory of noncommutative Schur functions similar to Lam's [15, Section 6] (see §4.2), the coefficient of a Schur function in an LLT polynomial is equal to $\langle \mathfrak{J}_\lambda(\mathbf{u}), f\rangle$, where f is a certain element of $\mathcal{U}$ that encodes an LLT polynomial and $\langle\cdot,\cdot\rangle$ denotes the symmetric bilinear form on $\mathcal{U}$ for which monomials form an orthonormal basis. Hence Theorem 1.1 yields a positive combinatorial formula for the new variant q-Littlewood-Richardson coefficients $\mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q)$ for 3-tuples $\boldsymbol{\beta}$ of skew shapes. We now state this formula in a special case which implies Haglund's conjecture and therefore, by [13], yields a formula for 3-column Macdonald polynomials (see Corollary 4.3 for the full statement with no restriction on the 3-tuple of skew shapes, and see the discussion after Corollary 4.3 for the precise relation to Haglund's conjecture).

Let $\boldsymbol{\beta} = (\beta^{(0)}, \dots, \beta^{(k-1)})$ be a k-tuple of skew shapes. The *shifted content* of a cell z of $\boldsymbol{\beta}$ is

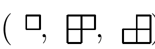
$$\tilde{c}(z) = k \cdot c(z) + i, \tag{2}$$

when $z \in \beta^{(i)}$ and where $c(z)$ is the usual content of z regarded as a cell of $\beta^{(i)}$. For $\boldsymbol{\beta}$ such that each $\beta^{(i)}$ contains no 2×2 square, define $\mathrm{W}'_k(\boldsymbol{\beta})$ to be the set of words v such that

- v is a rearrangement of the shifted contents of $\boldsymbol{\beta}$,
- for each i and each pair z, z' of cells of $\beta^{(i)}$ such that z' lies immediately east or north of z, the letter $\tilde{c}(z)$ occurs before $\tilde{c}(z')$ in v.

Define the following variant of the inversion statistic of [14, 3]:

$$\mathrm{inv}'_k(\mathsf{v}) = |\{(i,j) \mid i < j \text{ and } 0 < \mathsf{v_i} - \mathsf{v_j} < k\}|$$

for any word v in the alphabet of integers. For example, the 3-tuple $\boldsymbol{\beta} = ($  $) = (2, 32, 33)/(1, 11, 21)$ of skew shapes has shifted contents

$$\left(\ \begin{array}{|c|}\hline 3\\\hline\end{array}\,,\quad \begin{array}{|c|c|}\hline 4&7\\\hline 1\\\cline{1-1}\end{array}\,,\quad \begin{array}{cc}\cline{2-2} &\multicolumn{1}{|c|}{8}\\\hline \multicolumn{1}{|c|}{2}&5\\\hline\end{array}\right).$$

The word $\mathsf{8341275}$ belongs to $\mathrm{W}'_3(\boldsymbol{\beta})$ and $\mathrm{inv}'_3(\mathsf{8341275}) = 5$.

Corollary 1.2. *Let $\boldsymbol{\beta} = (\beta^{(0)}, \beta^{(1)}, \beta^{(2)})$ be a 3-tuple of skew shapes such that each $\beta^{(i)}$ contains no 2×2 square. The corresponding new variant q-Littlewood-Richardson coefficients are given by*

$$\mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q) = \sum_{\substack{T \in \mathrm{RSST}_\lambda \\ \mathrm{sqread}(T) \in \mathrm{W}'_3(\boldsymbol{\beta})}} q^{\mathrm{inv}'_3(\mathrm{sqread}(T))}.$$

Note that if the set of shifted contents of $\boldsymbol{\beta}$ is $[n]$, since elements of $\mathrm{W}_3'(\boldsymbol{\beta})$ contain no repeated letter, RSST_λ can be replaced by the set of standard Young tableaux of shape λ in Corollary 1.2.

This paper is part of the series [5, 6], which uses ideas of [15, 3, 2] to generalize the Fomin-Greene theory [9] to quotients of $\mathcal{U}$ with weaker relations than the quotients considered in [9]. In Section 5, we conjecture a strengthening of Theorem 1.1 to a quotient $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$ of $\mathcal{U}$ whose relations combine the Knuth relations and what we call *rotation relations*:

$$u_a u_c u_b = u_b u_a u_c \quad \text{and} \quad u_c u_a u_b = u_b u_c u_a \quad \text{for } a < b < c.$$

Similar algebras are studied more thoroughly in [5, 6]. The idea of combining these two kinds of relations is due to Assaf [3, 2] (see Remark 5.1). Though we have not been able to prove this strengthening, having it as a goal was crucial to our discovery and proof of Theorem 1.1.

This paper is organized as follows: Section 2 introduces Lam's algebra of ribbon Schur operators $\mathcal{U}/I_{\mathrm{L},k}$ and defines LLT polynomials. Section 3 introduces the combinatorics needed for the proof of the main theorem. In Section 4, we state and prove a stronger, more technical version of the main theorem, and we state and prove the full version of Corollary 1.2. In Section 5, we conjecture a strengthening of the main theorem and describe our progress towards proving it.

2. Lam's algebra and LLT polynomials

We introduce Lam's algebra of ribbon Schur operators, reconcile a definition of LLT polynomials from [3] with equivalence classes of words in this algebra, and reformulate this definition in a way that is convenient for our main theorem.

2.1. Diagrams and partitions.

A *diagram* or *shape* is a finite subset of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. A diagram is drawn as a set of square cells in the plane with the English (matrix-style) convention so that row (resp. column) labels start with 1 and increase from north to south (resp. west to east). The *diagonal* or *content* of a cell (i, j) is $j - i$.

A partition λ of n is a weakly decreasing sequence $(\lambda_1, \dots, \lambda_l)$ of nonnegative integers that sum to n. The *shape* of λ is the subset $\{(r, c) \mid r \in [l],\ c \in [\lambda_r]\}$ of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Write $\mu \subseteq \lambda$ if the shape of μ is contained in the shape of λ. If $\mu \subseteq \lambda$, then λ/μ denotes the *skew shape* obtained by removing the cells of μ from the shape of λ. The *conjugate partition* λ' of λ is the partition whose shape is the transpose of the shape of λ.

We will make use of the following partial orders on $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$:

$$(r, c) \leq_{\searrow} (r', c') \text{ whenever } r \leq r' \text{ and } c \leq c',$$
$$(r, c) \leq_{\nearrow} (r', c') \text{ whenever } r \geq r' \text{ and } c \leq c'.$$

It will occasionally be useful to think of diagrams as posets for the order $<_{\searrow}$ or $<_{\nearrow}$.

2.2. **The algebra $\mathcal{U}$.** We will mostly work over the ring $\mathbf{A} = \mathbb{Q}[\hat{q}, \hat{q}^{-1}]$ of Laurent polynomials in the indeterminate $\hat{q}$. Let $\mathcal{U}$ be the free associative $\mathbf{A}$-algebra in the noncommuting variables u_i, $i \in \mathbb{Z}$. We think of the monomials of $\mathcal{U}$ as words in the alphabet of integers and frequently write n for the variable u_n.

2.3. **The nil-Temperley-Lieb algebra.** The nil-Temperley-Lieb algebra [4, 9] is the $\mathbf{A}$-algebra generated by s_i, $i \in \mathbb{Z}$, and relations

$$\begin{aligned} s_i^2 &= 0, \\ s_{i+1}s_is_{i+1} &= 0, \\ s_is_{i+1}s_i &= 0, \\ s_is_j &= s_js_i \qquad \text{for } |i-j| > 1. \end{aligned}$$

The reduced words of 321-avoiding permutations of $[n]$ form a basis for the subalgebra generated by $s_1, s_2, \dots, s_{n-1}$. Here and throughout the paper, $[n] := \{1, 2, \dots, n\}$.

Let $\mathcal{P}$ denote the set of partitions and write $\mathbf{A}\mathcal{P}$ for the free $\mathbf{A}$-module with basis indexed by $\mathcal{P}$. By [4] (see also [9]), there is a faithful action of the nil-Temperley-Lieb algebra on $\mathbf{A}\mathcal{P}$, defined by

$$\nu \circ s_i = \begin{cases} \mu & \text{if } \mu/\nu \text{ is a cell of content } i, \\ 0 & \text{otherwise.} \end{cases}$$

A *skew shape with contents* is an equivalence class of skew shapes, where two skew shapes are equivalent if there is a content and $<_{\searrow}$-order preserving bijection between their diagrams. By [4, §2], the map sending an element $v = s_{i_1} \cdots s_{i_t}$ of the nil-Temperley-Lieb algebra to those skew shapes μ/ν such that $\nu \circ v = \mu$ defines a bijection from the monomial basis of the nil-Temperley-Lieb algebra to skew shapes with contents.

2.4. **Lam's algebra of ribbon Schur operators.** Lam defines [15] an algebra of ribbon Schur operators, which gives an elegant algebraic framework for LLT polynomials. It sets the stage for applying the theory of noncommutative symmetric functions from [9] to the problem of computing Schur expansions of LLT polynomials. We have found it most convenient to work with the following variant[1] of Lam's algebra. Set $q = \hat{q}^{-2}$. Let $\mathcal{U}/I_{\mathrm{L},k}$ be the quotient of $\mathcal{U}$ by the following relations (let $I_{\mathrm{L},k}$ denote the corresponding two-sided ideal of $\mathcal{U}$):

$$u_i^2 = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{3}$$

$$u_{i+k}u_iu_{i+k} = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{4}$$

$$u_iu_{i+k}u_i = 0 \qquad \text{for } i \in \mathbb{Z}, \tag{5}$$

$$u_iu_j = u_ju_i \qquad \text{for } |i-j| > k, \tag{6}$$

$$u_iu_j = q^{-1}u_ju_i \qquad \text{for } 0 < j-i < k. \tag{7}$$

We refer to (6) as the *far commutation* relations.

[1] There is an algebra antiautomorphism from $\mathbb{Q}(\hat{q}) \otimes_{\mathbf{A}} \mathcal{U}/I_{\mathrm{L},k}$ to the algebra in [15], defined by sending $u_i \mapsto u_i$ and setting $q = \hat{q}^{-2}$, where $\hat{q}$ denotes the q from [15].

A k-ribbon is a connected skew shape of size k containing no 2×2 square. The *content* of a ribbon is the maximum of the contents of its cells. The *spin* of a ribbon R, denoted $\mathrm{spin}(R)$, is the number of rows in the ribbon, minus 1. By [15], the following defines a faithful right action of $\mathcal{U}/I_{\mathrm{L},k}$ on $\mathbf{A}\mathcal{P}$:

$$\nu \cdot u_i = \begin{cases} \hat{q}^{\mathrm{spin}(\mu/\nu)}\mu & \text{if } \mu/\nu \text{ is a } k\text{-ribbon of content } i, \\ 0 & \text{otherwise.} \end{cases}$$

For the variant of LLT polynomials we prefer to work with in this paper (see §2.6), it is better to work with an action on k-tuples of partitions. We now describe this action and its relation to the one just defined. Unfortunately, we only know how to relate these actions at $\hat{q} = 1$.

We briefly recall the definitions of k-cores and k-quotients (see [14] for a more detailed discussion). The *k-core* of a partition μ, denoted $\mathrm{core}_k(\mu)$, is the unique partition obtained from μ by removing k-ribbons until it is no longer possible to do so. Let μ be a partition with k-core ν. Then for each $i = 0, 1, \dots, k-1$, there is exactly one way to add a k-ribbon of content $c_i \equiv i \bmod k$ to ν. The *k-quotient of* μ, denoted $\mathrm{quot}_k(\mu)$, is the unique k-tuple $\boldsymbol{\gamma} = (\gamma^{(0)}, \dots, \gamma^{(k-1)})$ of partitions such that the multiset of integers $k \cdot c(z) + c_i$ for $z \in \gamma^{(i)}$ and $i = 0, 1, \dots, k-1$ is equal to the multiset of contents of the ribbons in any k-ribbon tiling of μ/ν.

Let $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$ denote the algebra $\mathcal{U}/I_{\mathrm{L},k}$ specialized to $q = 1$. It is equal to the k-fold tensor product over $\mathbb{Q}$ of the nil-Temperley-Lieb algebra. For each $\mathbf{d} = (d_0, \dots, d_{k-1}) \in \mathbb{Z}^n$, define an action of $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$ on k-tuples of partitions by

$$\boldsymbol{\delta} \circ_{\mathbf{d}} u_i = (\delta^{(0)}, \dots, \delta^{(\hat{i}-1)}, \delta^{(\hat{i})} \circ s_{(i-\hat{i})/k - d_{\hat{i}}}, \delta^{(\hat{i}+1)}, \dots),$$

where $\hat{i}$ denotes the element of $\{0, 1, \dots, k-1\}$ congruent to $i \bmod k$. This action is equivalent to the action of $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$ on partitions with k-core ν when $d_i = (c_i - i)/k$ and the c_i are determined by ν as above; the precise relation is $\nu \cdot \mathsf{v} = \hat{q}^a \mu$ for some $a \in \mathbb{Z}$ if and only if $\mathrm{quot}_k(\nu) \circ_{\mathbf{d}} \mathsf{v} = \mathrm{quot}_k(\mu)$, for any word $\mathsf{v} \in \mathcal{U}$.

2.5. **Statistics on words and equivalence classes.** Following [3], we introduce statistics Des and inv on words and the set of k-ribbon words, which will prepare us to define LLT polynomials. We then relate these to the algebra $\mathcal{U}/I_{\mathrm{L},k}$.

Given a word w of positive integers (not thought of as an element of $\mathcal{U}$) and a weakly increasing sequence of integers c of the same length, define the following statistics on pairs (w, c)

$$\mathrm{Des}_k(w, c) = \{(i, j) \mid i < j,\ w_i > w_j, \text{ and } c_j - c_i = k\},$$
$$\mathrm{inv}_k(w, c) = |\{(i, j) \mid i < j,\ w_i > w_j, \text{ and } 0 < c_j - c_i < k\}|.$$

These are called the *k-descent set* and *the k-inversion number of the pair* (w, c). Define the corresponding statistics on words $\mathsf{v} \in \mathcal{U}$:

$$\mathrm{Des}'_k(\mathsf{v}) = \{(\mathsf{v_i}, \mathsf{v_j}) \mid i < j \text{ and } \mathsf{v_i} - \mathsf{v_j} = k\} \text{ (a multiset)},$$
$$\mathrm{inv}'_k(\mathsf{v}) = |\{(i, j) \mid i < j \text{ and } 0 < \mathsf{v_i} - \mathsf{v_j} < k\}|.$$

See Example 2.5. For a pair (w, c) as above with w a permutation, define $\mathsf{v} = (w, c)^{-1}$ by $\mathsf{v_i} = c_j$ where j is such that $w_j = i$. Given c as above and a k-descent set D, let $(c, D)^{-1}$ denote the multiset $\{(c_j, c_i) \mid (i, j) \in D\}$. One checks that

$$\operatorname{Des}'_k((w,c)^{-1}) = (c, \operatorname{Des}_k(w,c))^{-1} \quad \text{and} \quad \operatorname{inv}'_k((w,c)^{-1}) = \operatorname{inv}_k(w,c). \tag{8}$$

Definition 2.1 (Assaf [3])**.** A *k-ribbon word* is a pair (w, c) consisting of a word w and a weakly increasing sequence of integers c of the same length such that if $c_i = c_{i+1}$, then there exist integers h and j such that $(h, i), (i+1, j) \in \operatorname{Des}_k(w, c)$ and $(i, j), (h, i+1) \notin \operatorname{Des}_k(w, c)$. In other words, $c_h = c_i - k$ and $w_i < w_h \leq w_{i+1}$ while $c_j = c_i + k$ and $w_i \leq w_j < w_{i+1}$.

We now relate k-ribbon words to the algebra $\mathcal{U}/I_{\mathrm{L},k}$.

Say that two words v, v' of $\mathcal{U}$ are *k-equivalent at $q = 1$* if $\mathsf{v} = \hat{q}^a \mathsf{v}'$ in the algebra $\mathcal{U}/I_{\mathrm{L},k}$ for some a. Define *$q = 1$ k-equivalence classes* in the obvious way; we omit the k from these definitions when it is clear. Note that v and v' are equivalent at $q = 1$ if and only if they are equal in the algebra $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$: given that $\mathcal{U}/I_{\mathrm{L},k}$ is defined by binomial and monomial relations, the only way this can fail is if $\mathsf{v} \neq 0$ in $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$ but $f\mathsf{v} = 0$ in $\mathcal{U}/I_{\mathrm{L},k}$ for some nonzero $f \in \mathbf{A}$ (which must satisfy $f|_{\hat{q}=1} = 0$). However, if this occurs then $\boldsymbol{\delta} \circ_{\mathbf{0}} \mathsf{v} = \boldsymbol{\gamma}$ for some k-tuples of partitions $\boldsymbol{\delta}, \boldsymbol{\gamma}$. Therefore, letting ν, μ be shapes with empty k-cores such that $\operatorname{quot}_k(\nu) = \boldsymbol{\delta}$, $\operatorname{quot}_k(\mu) = \boldsymbol{\gamma}$, we have $\nu \cdot f\mathsf{v} = f\hat{q}^a\mu$ for some $a \in \mathbb{Z}$, hence $f\mathsf{v} \neq 0$ in $\mathcal{U}/I_{\mathrm{L},k}$. It follows that $\mathcal{U}/I_{\mathrm{L},k}$ is a free $\mathbf{A}$-module and the nonzero $q = 1$ equivalence classes form an $\mathbf{A}$-basis for $\mathcal{U}/I_{\mathrm{L},k}$.

Proposition-Definition 2.2. *The following are equivalent for a word $\mathsf{v} \in \mathcal{U}$:*

(i) $\mathsf{v} \neq 0$ *in* $\mathcal{U}/I_{\mathrm{L},k}$,
(ii) $\mathsf{v} \neq 0$ *in* $\mathcal{U}/I_{\mathrm{L},k}|_{\hat{q}=1}$,
(iii) *for every pair $i < j$ such that $\mathsf{v_i} = \mathsf{v_j}$, there exists s, t such that $i < s < t < j$ and $\{\mathsf{v_s}, \mathsf{v_t}\} = \{\mathsf{v_i} - \mathsf{k}, \mathsf{v_i} + \mathsf{k}\}$,*
(iv) $\mathsf{v} = (w, c)^{-1}$ *for some k-ribbon word (w, c) with w a permutation.*

If v satisfies these properties, then we say it is a nonzero k-word*.*

Proof. By the discussion above, (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from the bijection between the monomial basis of the nil-Temperley-Lieb algebra and skew shapes with contents (see §2.3). The equivalence of (iii) and (iv) is a straightforward unraveling of definitions. □

We record the following immediate consequence of the equivalence of (i) and (iii) for later use.

Corollary 2.3. *Let $\mathsf{v} = \mathsf{v_1} \cdots \mathsf{v_t} \in \mathcal{U}$ be a word such that $\mathsf{v_1} = \mathsf{v_t}$. If either v does not contain $\mathsf{v_1} + \mathsf{k}$ or v does not contain $\mathsf{v_1} - \mathsf{k}$, then $\mathsf{v} = 0$ in $\mathcal{U}/I_{\mathrm{L},k}$.*

A *k-tuple of skew shapes with contents* is a k-tuple $\boldsymbol{\beta} = (\beta^{(0)}, \ldots, \beta^{(k-1)})$ such that each $\beta^{(i)}$ is a skew shape with contents. The *shifted content* of a cell z of $\boldsymbol{\beta}$ is

$$\tilde{c}(z) = k \cdot c(z) + i,$$

when $z \in \beta^{(i)}$ and where $c(z)$ is the usual content of z regarded as a cell of $\beta^{(i)}$. The *content vector* of $\boldsymbol{\beta}$ is the weakly increasing sequence of integers consisting of the shifted contents of all the cells of $\boldsymbol{\beta}$ (with repetition).

Definition 2.4. Define the following sets of words:

- $\mathrm{WRib}_k(c, D)$ = the set of k-ribbon words (w, c) with k-descent set D.
- $\mathrm{W}'_k(c, D')$ = the set of nonzero k-words $\mathsf{v} \in \mathcal{U}$ such that sorting v in weakly increasing order yields c and $\mathrm{Des}'_k(\mathsf{v}) = D'$.
- $\mathrm{W}'_k(\boldsymbol{\beta})$ = the set of words $\mathsf{v} \in \mathcal{U}$ such that $\boldsymbol{\delta} \circ_{\mathbf{0}} \mathsf{v} = \boldsymbol{\gamma}$, where $\boldsymbol{\beta} = \boldsymbol{\gamma}/\boldsymbol{\delta}$ is any k-tuple of skew shapes with contents.

For a word v, the *standardization* of v, denoted v^{st}, is the permutation obtained from v by first relabeling, from left to right, the occurrences of the smallest letter in v by $1, \dots, t$, then relabeling the occurrences of the next smallest letter of v by $t+1, \dots, t+t'$, etc.

Example 2.5. Let $\boldsymbol{\beta}$ be the 3-tuple $\left(\,\begin{array}{|c|}\hline \ \\ \hline\end{array},\ \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{\ } \\ \hline \multicolumn{1}{|c|}{\ } & \ \\ \hline\end{array}\right)$ of skew shapes with contents. Its shifted contents are

$$\left(\ \begin{array}{|c|}\hline 3 \\ \hline\end{array},\ \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 1 & 4 \\ \hline\end{array},\ \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{8} \\ \hline \multicolumn{1}{|c|}{2} & 5 \\ \hline\end{array}\right).$$

The content vector c of $\boldsymbol{\beta}$, a word $\mathsf{v} \in \mathrm{W}'_3(\boldsymbol{\beta})$, and $w = (\mathsf{v}^{\mathrm{st}})^{-1}$ (note $\mathsf{v} = (w, c)^{-1}$):

$$\begin{aligned} w &= 46715832 \\ c &= 12344578 \\ \mathsf{v} &= 48714235. \end{aligned}$$

The statistics defined above, on the pair (w, c) and on v:

$$\begin{aligned} \mathrm{Des}_3(w, c) &= \{(1, 4), (5, 7), (6, 8)\} \\ \mathrm{Des}'_3(\mathsf{v}) &= \{(4, 1), (7, 4), (8, 5)\} \\ \mathrm{inv}_3(w, c) &= \mathrm{inv}'_3(\mathsf{v}) = 6. \end{aligned}$$

We also have $(w, c) \in \mathrm{WRib}_3(c, D)$ for $D = \mathrm{Des}_3(w, c)$, and $\mathrm{W}'_3(\boldsymbol{\beta}) = \mathrm{W}'_3(c, D')$ for $D' = \mathrm{Des}'_3(\mathsf{v})$.

Proposition 2.6. *Let $\boldsymbol{\beta}$ be a k-tuple of skew shapes with contents and let c be its content vector. Let $\tilde{c}^{\mathrm{st}}$ be the function on the cells of $\boldsymbol{\beta}$ that assigns to the cells of shifted content i the letters of c^{st} that relabel the i's in c, increasing in the $\searrow$ direction.*

(i) *The rule $\boldsymbol{\beta} \mapsto \mathrm{W}'_k(\boldsymbol{\beta})$ defines a bijection between k-tuples of skew shapes with contents and nonzero $q = 1$ equivalence classes.*

(ii) *Suppose D' is determined from $\boldsymbol{\beta}$ as follows: the number of pairs of cells z, z' in $\boldsymbol{\beta}$ such that $\tilde{c}(z) = \tilde{c}(z') + k$ and $z <_{\searrow} z'$ is the multiplicity of $(\tilde{c}(z), \tilde{c}(z'))$ in D'. Then $\mathrm{W}'_k(\boldsymbol{\beta}) = \mathrm{W}'_k(c, D')$.*

(iii) *Suppose D is such that $\mathrm{WRib}_k(c, D)$ is nonempty and let $D' = (c, D)^{-1}$. Then we have the following bijection*

$$\{(w, c) \in \mathrm{WRib}_k(c, D) \mid w \text{ a permutation}\} \to \mathrm{W}'_k(c, D'),\ \ (w, c) \mapsto (w, c)^{-1},$$

with inverse given by $((\mathsf{v}^{\mathrm{st}})^{-1}, c) \leftarrow\!\shortmid \mathsf{v}$.

(iv) *If $\boldsymbol{\beta}$ and D' are related as in (ii), and $D' = (c, D)^{-1}$ as in (iii), then D is determined from $\boldsymbol{\beta}$ as follows: D is the set of pairs $(\tilde{c}^{\mathrm{st}}(z'), \tilde{c}^{\mathrm{st}}(z))$ with z, z' cells of $\boldsymbol{\beta}$ such that $\tilde{c}(z) = \tilde{c}(z') + k$ and $z <_{\searrow} z'$.*

If $\boldsymbol{\beta}$ is as in Example 2.5, then $c^{\mathrm{st}} = 12345678$ and $\tilde{c}^{\mathrm{st}}$ is given by

$$\left(\begin{array}{c} \boxed{3} \\ \end{array}, \begin{array}{|c|c|}\hline 4 & 7 \\ \hline 1 & 5 \\ \hline \end{array}, \begin{array}{cc} & \boxed{8} \\ \boxed{2} & \boxed{6} \end{array} \right).$$

Proof. By the discussion before Proposition-Definition 2.2, (i) reduces to the case $k = 1$. The case $k = 1$ follows from [4]. For (ii), the relations of $\mathcal{U}/I_{\mathrm{L},k}$ preserve the statistic $\mathrm{Des}'_k(\mathsf{v})$ on nonzero k-words, hence $\mathrm{W}'_k(c, D')$ is a union of $q = 1$ equivalence classes. That $\mathrm{W}'_k(c, D')$ is the single equivalence class $\mathrm{W}'_k(\boldsymbol{\beta})$ follows from the observation that a skew shape with contents is determined by its multiset of contents and, for each of its pairs of diagonals G, G' with the content of G 1 more than that of G', the number of pairs of cells (z, z'), $z \in G$, $z' \in G'$ such that $z <_{\searrow} z'$. Statements (iii) and (iv) are straightforward from definitions, Proposition-Definition 2.2, and (8). □

2.6. **LLT polynomials.** LLT polynomials are certain q-analogs of products of skew Schur functions, first defined by Lascoux, Leclerc, and Thibon in [17]. There are two versions of LLT polynomials (which we distinguish following the notation of [11]): the combinatorial LLT polynomials of [17] defined using spin, and the new variant combinatorial LLT polynomials of [14] defined using inversion numbers (we called these LLT polynomials in the introduction). Although Lam's algebra is well suited to studying the former, we prefer to work with the latter because they can be expressed entirely in terms of words, and because inversion numbers are easier to calculate than spin.

Let $\mathsf{v} = \mathsf{v}_1 \cdots \mathsf{v}_t$ be a word. We write $\mathrm{Des}(\mathsf{v}) := \{i \in [t-1] \mid \mathsf{v}_i > \mathsf{v}_{i+1}\}$ for the *descent set* of v. Let

$$Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x}) = \sum_{\substack{1 \le i_1 \le \cdots \le i_t \\ j \in \mathrm{Des}(\mathsf{v}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_t}$$

be Gessel's *fundamental quasisymmetric function* [10] in the commuting variables $x_1, x_2, \ldots$.

The result [3, Corollary 4.3], which we take here as a definition, expresses the new variant combinatorial LLT polynomials of [14] in terms of k-ribbon words.

Definition 2.7. Let $\boldsymbol{\beta}$ be a k-tuple of skew shapes with contents, and let c, D be the corresponding content vector and k-descent set from Proposition 2.6 (iv). The *new variant combinatorial LLT polynomials* are the generating functions

$$\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x}; q) = \sum_{\substack{(w,c) \in \mathrm{WRib}_k(c,D) \\ w \text{ a permutation}}} q^{\mathrm{inv}_k(w,c)} Q_{\mathrm{Des}(w^{-1})}(\mathbf{x}).$$

For this paper, we have found the following expressions for LLT polynomials to be the most useful. The second involves only words and the statistics inv'_k and Des'_k. Also, as will be seen in §4.2, these expressions allow for the application of the machinery of [9].

Proposition 2.8. *Let $\boldsymbol{\beta}$ be a k-tuple of skew shapes with contents and let c, D' be determined from $\boldsymbol{\beta}$ as in Proposition 2.6 (ii). Then*

$$\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x};q) = \sum_{\mathsf{v}\in \mathrm{W}'_k(\boldsymbol{\beta})} q^{\mathrm{inv}'_k(\mathsf{v})} Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x}) = \sum_{\mathsf{v}\in \mathrm{W}'_k(c,D')} q^{\mathrm{inv}'_k(\mathsf{v})} Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x}).$$

Proof. This follows from Definition 2.7, Proposition 2.6, (8), and the fact $\mathrm{Des}(\mathsf{v}^{\mathrm{st}}) = \mathrm{Des}(\mathsf{v})$. □

Remark 2.9. The *combinatorial LLT polynomials* $G^{(k)}_{\mu/\nu}(\mathbf{x};\hat{q})$ are the $\hat{q}$-generating functions over k-ribbon tableaux weighted by spin ([19, 15, 11] use spin, whereas [17] uses cospin). We refer to [11] for their precise definition. Lam shows [15] that

$$G^{(k)}_{\mu/\nu}(\mathbf{x};\hat{q}) = \langle \nu \cdot \Omega(\mathbf{x},\mathbf{u}), \mu\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the symmetric bilinear form on $\mathbf{A}\mathcal{P}$ for which $\mathcal{P}$ is an orthonormal basis, and

$$\Omega(\mathbf{x},\mathbf{u}) = \prod_{j=1}^{\infty}\prod_{i=-\infty}^{\infty}(1-x_j u_i)^{-1} = \sum_{\text{words } \mathsf{v}\in\mathcal{U}} Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x})\mathsf{v}$$

is a noncommutative Cauchy product in which the x_j commute with the u_i.

The two types of LLT polynomials are related as follows. Suppose μ/ν can be tiled by k-ribbons. Let $\boldsymbol{\beta}$ be the k-tuple of skew shapes with contents obtained from $\mathrm{quot}_k(\mu)/\mathrm{quot}_k(\nu) = (\gamma^{(0)}/\delta^{(0)},\ldots,\gamma^{(k-1)}/\delta^{(k-1)})$ by translating each $\gamma^{(i)}/\delta^{(i)}$ east by $(c_i - i)/k$, where the c_i are determined by the k-core of μ ($=\mathrm{core}_k(\nu)$) as in §2.4. Then by [14] (see e.g. [11, Proposition 6.17]), there is an integer e such that (recall $q = \hat{q}^{-2}$)

$$\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x};q) = \hat{q}^e G^{(k)}_{\mu/\nu}(\mathbf{x};\hat{q}).$$

3. Reading words for $\mathfrak{J}_\lambda(\mathbf{u})$ when $k=3$

Here we introduce new kinds of tableaux and reading words that arose naturally in our efforts to write $\mathfrak{J}_\lambda(\mathbf{u})$ as a positive sum of monomials in $\mathcal{U}/I_{\mathrm{L},3}$. The main new feature of these objects is that they involve posets obtained from posets of diagrams by adding a small number of covering relations. We are hopeful that this will become part of a more general theory that extends tools from tableaux combinatorics to posets more general than partition diagrams.

In Sections 3 and 4, we write $f \equiv g$ to mean that f and g are equal in $\mathcal{U}/I_{\mathrm{L},k}$, when the value of k is clear from context (typically it is 3 or arbitrary).

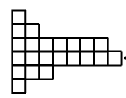
3.1. **Tableaux.** Let θ be a diagram (see §2.1). A *tableau of shape* θ is the diagram θ together with an integer in each of its cells. The *size* of a tableau T, denoted $|T|$, is the number of cells of T, and $\mathrm{sh}(T)$ denotes the shape of T. For a tableau T and a set of cells S such that $S \subseteq \mathrm{sh}(T)$, T_S denotes the subtableau of T obtained by restricting T to the diagram S. If z is a cell of T, then T_{z} denotes the entry of T in z. When it is clear, we will occasionally identify a tableau entry with the cell containing it.

A *standard Young tableau* (SYT) is a tableau T of partition shape filled with the entries $1, 2, \dots, |T|$ such that entries increase from north to south in each column and from west to east in each row. The set of standard Young tableaux of shape λ is denoted SYT_λ.

3.2. Restricted shapes and restricted square strict tableaux.

Definition 3.1. A *restricted shape* is a lower order ideal of a partition diagram for the order $<_{\nearrow}$. We will typically specify a restricted shape as follows: for any weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$, let α' denote the diagram $\{(r, c) \mid c \in [l],\ r \in [\alpha_c]\}$. Now let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition and $\alpha = (\alpha_1, \dots, \alpha_l)$ a weak composition such that $0 \le \alpha_1 \le \cdots \le \alpha_{j'}$, $\alpha_1 < \lambda_1,\ \alpha_2 < \lambda_2, \dots, \alpha_{j'} < \lambda_{j'}$, and $\alpha_{j'+1} = \lambda_{j'+1}, \dots, \alpha_l = \lambda_l$ for some $j' \in \{0, 1, \dots, l\}$. Then the set difference of λ' by α', denoted $\lambda' \setminus \alpha'$, is a restricted shape and any restricted shape can be written in this way.

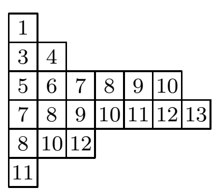
Note that, just as for skew shapes, different pairs λ, α may define the same restricted shape $\lambda' \setminus \alpha'$. An example of a restricted shape is

$$(65544444221)' \setminus (01222223221)' \ =$$ .

Definition 3.2. A *restricted tableau* is a tableau whose shape is a restricted shape. A *restricted square strict tableau* (RSST) is a restricted tableau such that entries

- strictly increase from north to south in each column,
- strictly increase from west to east in each row,
- satisfy $R_z + 3 \le R_{z'}$ whenever $z <_{\searrow} z'$ and z, z' do not lie in the same row or column.

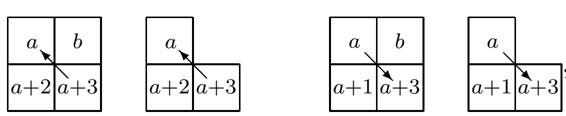
For example,



is an RSST of shape $(6554444)' \setminus (0122223)'$.

3.3. Square respecting reading words.

Definition 3.3. An *arrow square* S of an RSST R is a subtableau of R such that $\mathrm{sh}(S)$ is the intersection of $\mathrm{sh}(R)$ with a 2×2 square, and S is of the form



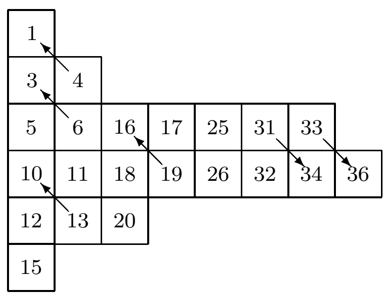
with $b \in \{a+1, a+2\}$. The first two of these are called $\nwarrow$ *arrow squares* and the last two are $\searrow$ *arrow squares*. Also, *an arrow* of R is a directed edge between the two cells of an arrow square, as indicated in the picture; we think of the arrows of R as the edges of a directed graph with vertex set the cells of R.

Definition 3.4. A *reading word* of a tableau R is a word w consisting of the entries of R such that for any two cells z and z' of R such that $z <_{\nearrow} z'$, $\mathsf{R_z}$ appears to the left of $\mathsf{R_{z'}}$ in w.

A *square respecting reading word* w of an RSST R is a reading word of R such that for each arrow of R, the tail of the arrow appears to the left of the head of the arrow in w.

There is no reason to prefer one square respecting reading word over another (see Theorem 3.11), but it is useful to have notation for one such word. So we define, for any RSST R, the word sqread(R) as follows: let $D^1, D^2, \dots, D^t$ be the diagonals of R, starting from the southwest. Let $\mathsf{w^i}$ be the result of reading, in the $\nwarrow$ direction, the entries of D^i that are $\nwarrow$ arrow tails followed by the remaining entries of D^i, read in $\searrow$ direction. Set sqread$(R) = \mathsf{w^1 w^2 \cdots w^t}$. It is easy to check that sqread(R) is a square respecting reading word of R.

Example 3.5. Here is an RSST drawn with its arrows:

$R =$ 

Of the following three reading words of R, the first two are square respecting, but the last is not.

$$\mathsf{15\ 12\ 13\ 10\ 5\ 11\ 20\ 6\ 3\ 18\ 19\ 4\ 1\ 16\ 17\ 26\ 25\ 32\ 31\ 34\ 33\ 36} = \text{sqread}(R)$$
$$\mathsf{15\ 12\ 13\ 10\ 5\ 11\ 20\ 6\ 3\ 4\ 1\ 18\ 19\ 16\ 17\ 26\ 25\ 32\ 31\ 34\ 33\ 36} \quad \text{(square respecting)}$$
$$\mathsf{15\ 12\ 10\ 13\ 11\ 20\ 18\ 19\ 26\ 5\ 6\ 3\ 4\ 1\ 16\ 17\ 25\ 32\ 34\ 36\ 31\ 33} \quad \text{(not square respecting)}$$

A $\nearrow$*-maximal cell* of a diagram is a cell that is maximal for the order $<_{\nearrow}$. A *nontail removable cell* of an RSST R is a $\nearrow$-maximal cell of R that is not the tail of an arrow of R. The last letter of a square respecting reading word of R must lie in a nontail removable cell of R.

Remark 3.6. If a tableau has no repeated letter, then its entries are naturally in bijection with the letters in any of its reading words. We can force this to be true for tableaux with repeated entries by adding subscripts to repeated entries and matching subscripts to the letters of its reading words. We will abuse notation and do this without saying so explicitly. So for example, if w is a reading word of R and z a cell of R, then by *the letter* $\mathsf{R_z}$ *of* w we really mean the unique subscripted letter of w that is equal to the subscripted version of $\mathsf{R_z}$. Also, if vw is a reading word of R, then *the canonical subtableau* of R with reading word v is the subtableau of R whose subscripted entries are the subscripted versions of the letters of v.

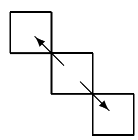
Remark 3.7. All of the results of §3.3 and §3.4 about restricted shapes and RSST can be generalized to the following setting: a $\llcorner$*-diagram* is a diagram θ such that for any two cells

$z = (r, c)$, $z' = (r', c')$ of θ such that $z <_{\searrow} z'$, the cell (r', c) is also in θ. A $\llcorner$*-tableau* is a tableau whose shape is a $\llcorner$-diagram, and a *square strict* $\llcorner$*-tableau* is a $\llcorner$-tableau satisfying the three conditions from Definition 3.2. An *arrow square* of a $\llcorner$-tableau R is a subtableau of R whose shape is an interval $[z, z']$ for the poset $\mathrm{sh}(R)$ with the order $<_{\searrow}$ such that z and z' do not lie in the same row or column and $R_{z'} - R_z = 3$.

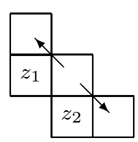
We have little use for this generality in this paper, so we do not discuss it further except to note one useful application. Unlike restricted diagrams, the set of $\llcorner$-diagrams is closed under the operation of reflecting across a line in the direction $\nearrow$. Also, the set of square strict $\llcorner$-tableaux with entries in $[n]$ is closed under the operation of reflecting across a line in the direction $\nearrow$ and sending the entry a to $n+1-a$. Arrows of a square strict $\llcorner$-tableau are reflected along with the cells.

3.4. **Combinatorics of restricted square strict tableaux.** We assemble some basic results about RSST, square respecting reading words, and the images of these words in $\mathcal{U}/I_{\mathrm{L},3}$. These are needed for the proof of the main theorem.

Proposition 3.8. *The following configuration of arrows cannot occur in an RSST:*



Proof. Let z_1, z_2 be the two cells of R as shown.



If this configuration occurs, then $R_{z_1} = R_{z_2} - 2$, contradicting the definition of an RSST. □

Lemma 3.9. *Let R be an RSST and z a cell of R such that the only cells $>_{\nearrow} z$ lie in the same column as z, and z is not the tail of a $\searrow$ arrow. Then R has a nontail removable cell that is weakly north of z.*

Similarly, if z is a cell of R such that the only cells $>_{\nearrow} z$ lie in the same row as z, and z is not the tail of a $\nwarrow$ arrow, then R has a nontail removable cell that is weakly east of z.

As an example of the second statement, if R is as in Example 3.5 and z is the cell containing 31, then there is exactly one nontail removable cell weakly east of z, the cell containing 36.

Proof. We prove only the first statement. The second then follows from Remark 3.7. Let D be the northeasternmost diagonal of R that has nonempty intersection with the cells of R weakly north of z. Let G be the directed graph with vertex set D and edges given by the arrows of R. Let D^N be the cells of D that are weakly north of z. Then the underlying undirected graph of G is contained in a path and there is no edge from D^N to vertices not in D^N. Hence D^N contains a vertex that is not an arrow tail and hence is a nontail removable cell of R. ☐

Lemma 3.10. *Let R be an RSST and z a cell of R such that the only cells $>_{\nearrow} z$ lie in the same column as z, and z is not the tail of a $\searrow$ arrow. Then there is a square respecting reading word w of R such that the letters appearing to the right of $\mathsf{R_z}$ in w are $< R_z$.*

Similarly, if z is a cell of R such that the only cells $>_{\nearrow} z$ lie in the same row as z, and z is not the tail of a $\nwarrow$ arrow, then there is a square respecting reading word v of R such that the letters appearing to the right of $\mathsf{R_z}$ in v are $> R_z$.

Proof. This follows by induction on $|R|$ using Lemma 3.9. ☐

The next result gives a natural way to associate an element of $\mathcal{U}/I_{\mathrm{L},3}$ to any RSST.

Theorem 3.11. *Any two square respecting reading words of an RSST R are equal in $\mathcal{U}/I_{\mathrm{L},3}$.*

Proof. Let $\mathscr{R}_R$ denote the graph with vertex set the square respecting reading words of R and an edge for each far commutation relation. We prove that $\mathscr{R}_R$ is connected by induction on $|R|$. Let $z_1, \dots, z_t$ denote the nontail removable cells of R ($t \geq 1$ by Lemma 3.9). By induction, each $\mathscr{R}_{R-z_i}$ is connected. Hence the induced subgraph of $\mathscr{R}_R$ with vertex set consisting of those words that end in $\mathsf{R_{z_i}}$, call it $\mathscr{R}_{R-z_i}\mathsf{R_{z_i}}$, is connected. Let H be the graph (with t vertices) obtained from $\mathscr{R}_R$ by contracting the subgraphs $\mathscr{R}_{R-z_i}\mathsf{R_{z_i}}$. We must show that H is connected.

It follows from the definition of an RSST that if z, z' are $\nearrow$-maximal cells of R, then $|R_z - R_{z'}| \leq 3$ if and only if there is an arrow between z and z'. Hence the $\mathsf{R_{z_i}}$ pairwise commute. Since $\mathscr{R}_R$ contains a word ending in $\mathsf{R_{z_i}R_{z_j}}$ for every $i \neq j$, H is a complete graph. ☐

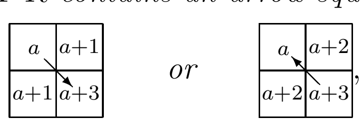
Lemma 3.10 and Theorem 3.11 have the following useful consequence.

Corollary 3.12. *If R is an RSST and z is a $\nearrow$-maximal cell of R, then $\mathsf{vR_z} = 0$ in $\mathcal{U}/I_{\mathrm{L},3}$ for every square respecting reading word v of R.*

Proof. By Theorem 3.11, it suffices to exhibit a single square respecting reading word v of R such that $\mathsf{vR_z} \equiv 0$. By Proposition 3.8 one of the two statements of Lemma 3.10 applies, thus R has a square respecting reading word ending in $\mathsf{R_z}$ followed by letters all $> R_z$ or all $< R_z$. Hence Corollary 2.3 with repeated letter $\mathsf{R_z}$ yields $\mathsf{vR_z} \equiv 0$. ☐

Say that an RSST R is *nonzero* if any (equivalently, every) square respecting reading word of R is nonzero in $\mathcal{U}/I_{\mathrm{L},3}$ (see Proposition-Definition 2.2). Although we will not need the next proposition in the proof of the main theorem, it is useful because it constrains the form of nonzero RSST.

Proposition 3.13. *If an RSST R contains an arrow square of the form*



(9)

then every square respecting reading word of R is 0 in $\mathcal{U}/I_{\mathrm{L},3}$.

Proof. By Remark 3.7, we may assume without loss of generality that R contains an arrow square S of the form on the left of (9), with cells labeled as follows

$$S = \begin{array}{|c|c|}\hline s_1 & s_3 \\ \hline s_2 & s_4 \\ \hline \end{array} \subseteq R.$$

It suffices to exhibit one square respecting reading word of R that is 0 in $\mathcal{U}/I_{\mathrm{L},3}$.

By induction on $|R|$, we may assume s_3 is the only nontail removable cell of R and S is the only arrow square of R of either of the forms in (9). Since s_3 cannot be the head of a $\searrow$ arrow of R, it follows from Lemma 3.9 that if R has a cell north of s_1 and in the same column as s_1, then there is a nontail removable cell to the north of s_1; we are assuming there is no such cell, so any cell $>_{\nearrow} s_1$ can only be in the same row as s_1. By Proposition 3.8, s_1 is not the tail of a $\nwarrow$ arrow. Hence by Lemma 3.10 (with $z = s_1$), there is a square respecting reading word $\mathsf{v}\mathsf{R}_{\mathsf{s}_1}\mathsf{w}$ of R such that the letters of w are $> R_{s_1} = a$; also we must have $\mathsf{R}_{\mathsf{s}_3} = \mathsf{a}+1$ and $\mathsf{R}_{\mathsf{s}_4}$ contained in w.

Now observe that s_2 is a $\nearrow$-maximal cell of R', where R' is the canonical sub-RSST of R with reading word v. There is no $\nwarrow$ arrow of R with tail s_2 because this would force an arrow square in R of the form on the right of (9), which we are assuming does not exist. This implies by Lemma 3.10 that we can assume v ends in $\mathsf{R}_{\mathsf{s}_2}\mathsf{v}'$ where v' has letters $> R_{s_2} = a+1$. Hence $\mathsf{R}_{\mathsf{s}_2}\mathsf{v}'\mathsf{R}_{\mathsf{s}_1}\mathsf{w} \equiv 0$ by Corollary 2.3 with repeated letter $\mathsf{a}+1$, implying that the square respecting reading word $\mathsf{v}\mathsf{R}_{\mathsf{s}_1}\mathsf{w}$ of R is 0 in $\mathcal{U}/I_{\mathrm{L},3}$. ☐

4. Positive monomial expansion of noncommutative Schur functions

After some preliminary definitions, we recall the main theorem and use it to give an explicit positive combinatorial formula for new variant q-Littlewood-Richardson coefficients indexed by a 3-tuple of skew shapes. In §4.3, we prove a stronger, more technical version of the main theorem.

4.1. **Noncommutative flagged Schur functions.**

Here we introduce the noncommutative Schur functions $\mathfrak{J}_\lambda(\mathbf{u})$ from [9, 15] and their flagged generalizations.

The *noncommutative elementary symmetric functions* are given by

$$e_d(S) = \sum_{\substack{i_1 > i_2 > \cdots > i_d \\ i_1,\ldots,i_d \in S}} u_{i_1} u_{i_2} \cdots u_{i_d},$$

for any subset S of $\mathbb{Z}$ and positive integer d; set $e_0(S) = 1$ and $e_d(S) = 0$ for $d < 0$. By [15], $e_i(S)e_j(S) = e_j(S)e_i(S)$ in $\mathcal{U}/I_{\mathrm{L},k}$ for all i and j.

Given a weak composition $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}^l_{\geq 0}$ and sets $S_1, S_2, \ldots, S_l \subseteq \mathbb{Z}$, define the *noncommutative column-flagged Schur function* by

$$J_\alpha(S_1, S_2, \ldots, S_l) := \sum_{\pi \in \mathcal{S}_l} \mathrm{sgn}(\pi)\, e_{\alpha_1+\pi(1)-1}(S_1) e_{\alpha_2+\pi(2)-2}(S_2) \cdots e_{\alpha_l+\pi(l)-l}(S_l). \qquad (10)$$

For an l-tuple $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l_{\geq 0}$, we use the following shorthands:

$$J_\alpha(\mathbf{n}) = J_\alpha(n_1, \dots, n_l) = J_\alpha([n_1], \dots, [n_l]).$$

Let λ be a partition and λ' its conjugate partition. We define the *noncommutative Schur function* $\mathfrak{J}_\lambda(\mathbf{u})$ by

$$\mathfrak{J}_\lambda(\mathbf{u}) = J_{\lambda'}(\mathbb{Z}, \dots, \mathbb{Z}).$$

These noncommutative (column-flagged) Schur functions are related to the noncommutative Schur functions $\mathfrak{J}_\lambda(u_1, \dots, u_n)$ of [9, Equation 3.3] by $\mathfrak{J}_\lambda(u_1, \dots, u_n) = J_{\lambda'}(n, n, \dots, n)$. When the u_i commute, $\mathfrak{J}_\lambda(u_1, \dots, u_n)$ becomes the ordinary Schur function s_λ in n variables, and $J_\lambda(\mathbf{n})$ becomes the column-flagged Schur function $S^*_\lambda(\mathbf{1}, \mathbf{n})$ studied in [22], where $\mathbf{1}$ denotes the all-ones vector of length l.

The *augmented noncommutative column-flagged Schur function* labeled by a weak composition $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{Z}^l_{\geq 0}$, an l-tuple $\mathbf{n} = (n_1, \dots, n_l) \in \mathbb{Z}^l_{\geq 0}$, and words $\mathsf{w}^1, \dots, \mathsf{w}^{l-1} \in \mathcal{U}$ is defined by

$$\begin{aligned} J_\alpha(\mathbf{n} : \underset{1}{\mathsf{w}^1}, \underset{2}{\mathsf{w}^2}, \dots, \underset{l-1}{\mathsf{w}^{l-1}}) &= J_\alpha(n_1, n_2, \dots, n_l : \underset{1}{\mathsf{w}^1}, \underset{2}{\mathsf{w}^2}, \dots, \underset{l-1}{\mathsf{w}^{l-1}}) \\ &:= \sum_{\pi \in \mathcal{S}_l} \operatorname{sgn}(\pi)\, e_{\alpha_1+\pi(1)-1}([n_1])\, \mathsf{w}^1\, e_{\alpha_2+\pi(2)-2}([n_2])\, \mathsf{w}^2 \cdots \mathsf{w}^{l-1}\, e_{\alpha_l+\pi(l)-l}([n_l]). \end{aligned}$$

We omit $\underset{i}{\mathsf{w}^i}$ from the notation if the word w^i is empty.

The (augmented) noncommutative column-flagged Schur functions will be considered here as elements of $\mathcal{U}/I_{\mathrm{L},k}$, unless stated otherwise.

Note that because the noncommutative elementary symmetric functions commute in $\mathcal{U}/I_{\mathrm{L},k}$,

$$J_\alpha(\mathbf{n}) \equiv -J_{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1}-1,\alpha_j+1,\dots,\alpha_l}(\mathbf{n}) \quad \text{whenever } n_j = n_{j+1}. \tag{11}$$

In particular,

$$J_\alpha(\mathbf{n}) \equiv 0 \quad \text{whenever } \alpha_j = \alpha_{j+1} - 1 \text{ and } n_j = n_{j+1}. \tag{$\diamond$}$$

More generally, (11) and ($\diamond$) hold for the augmented case provided w^j is empty.

We will make frequent use of the following fact (interpret $[0] = \{\}$):

$$e_d([m]) = \mathsf{m}e_{d-1}([m-1]) + e_d([m-1]) \qquad \text{if } m > 0 \text{ and } d \text{ is any integer.} \tag{12}$$

Note that

$$e_d([0]) = \begin{cases} 1 & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will often apply this to $J_\alpha(\mathbf{n})$ and its variants by expanding $e_{\alpha_j+\pi(j)-j}(S_j)$ in (10) using (12) (so that (12) is applied once to each of the $l!$ terms in the sum in (10)). We refer to this as a *j-expansion of* $J_\alpha(\mathbf{n})$ or simply a *j-expansion*.

4.2. **The main theorem.** Let RSST_λ denote the set of restricted square strict tableaux of shape λ. For a diagram θ contained in columns $1, \dots, l$ and nonnegative integers $n_1, \dots, n_l$, let $\mathrm{Tab}_\theta(n_1, n_2, \dots, n_l)$ denote the set of tableaux of shape θ such that the entries in column c lie in $[n_c]$. Recall from Definition 3.4 that $\mathrm{sqread}(T)$ is a specially chosen reading word of T. We have the following generalization of Theorem 1.1 from the introduction to the noncommutative column-flagged Schur functions:

Theorem 4.1. *If λ is a partition with $l = \lambda_1$ columns and $0 \le n_1 \le n_2 \le \cdots \le n_l$, then in the algebra $\mathcal{U}/I_{\mathrm{L},3}$,*

$$J_{\lambda'}(n_1, n_2, \dots, n_l) = \sum_{T \in \mathrm{RSST}_\lambda,\ T \in \mathrm{Tab}_\lambda(n_1, n_2, \dots, n_l)} \mathrm{sqread}(T).$$

In the next subsection we will prove a stronger, more technical version of this theorem. Recall that Theorem 1.1 states that

$$\mathfrak{J}_\lambda(\mathbf{u}) = \sum_{T \in \mathrm{RSST}_\lambda} \mathrm{sqread}(T) \qquad \text{in } \mathcal{U}/I_{\mathrm{L},3}.$$

This follows from Theorem 4.1 since $\mathfrak{J}_\lambda(\mathbf{u}) = J_{\lambda'}(\mathbb{Z}, \dots, \mathbb{Z})$ can be written as a sum of words in $\mathcal{U}$, grouped according to the multiset of letters appearing in each word, and each group can be computed using Theorem 4.1.

We now use Theorem 1.1 to deduce an explicit combinatorial formula for the coefficients of the Schur expansion of new variant combinatorial LLT polynomials indexed by a 3-tuple of skew shapes. This is a straightforward application of the Fomin-Greene machinery [9] (see also [15, 6]).

Define the $\mathbf{A}$-linear map

$$\Delta : \mathcal{U} \to \mathbf{A}[x_1, x_2, \dots] \ \text{ by } \ \mathsf{v} \mapsto Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x}).$$

Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on $\mathcal{U}$ in which the monomials form an orthonormal basis. Note that any element of $\mathcal{U}/I_{\mathrm{L},k}$ has a well-defined pairing with any element of $(I_{\mathrm{L},k})^\perp$. We need the following variant of Theorem 1.2 of [9] and results in Section 6 of [15]; see [6] for a detailed proof.

Theorem 4.2. *For any $f \in (I_{\mathrm{L},k})^\perp$,*

$$\Delta(f) = \Big\langle \sum_{\text{words } \mathsf{v} \in \mathcal{U}} Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x})\mathsf{v}, f \Big\rangle = \sum_\lambda s_\lambda(\mathbf{x}) \langle \mathfrak{J}_\lambda(\mathbf{u}), f \rangle.$$

Recall that the *new variant q-Littlewood-Richardson coefficients* $\mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q)$ are the coefficients in the Schur expansion of the new variant combinatorial LLT polynomials, i.e.

$$\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x}; q) = \sum_\lambda \mathfrak{c}^\lambda_{\boldsymbol{\beta}}(q) s_\lambda(\mathbf{x}).$$

The new variant q-Littlewood-Richardson coefficients are known to be polynomials in q with nonnegative integer coefficients [11], but positive combinatorial formulae for these coefficients have only been given in the cases that $k \le 2$ and the diameter of $\boldsymbol{\beta}$ is ≤ 3 (see (1)). Below is the first positive combinatorial interpretation in the $k = 3$ case.

In order to give an expression for the $\mathfrak{c}^{\lambda}_{\boldsymbol{\beta}}(q)$ that does not depend on arbitrary choices of square respecting reading words, we adapt the statistics Des'_3 and inv'_3 to RSST:

$$\begin{aligned}\mathrm{Des}'_3(T) = \{(T_z, T_{z'}) \mid z, z' \in \mathrm{sh}(T),\ T_z - T_{z'} = 3,\\ \text{and } (z <_{\nearrow} z' \text{ or there is a } \nwarrow \text{ arrow from } z \text{ to } z' \text{ in } T)\},\\ \mathrm{inv}'_3(T) = |\{(z, z') \mid z, z' \in \mathrm{sh}(T),\ 0 < T_z - T_{z'} < 3,\ \text{and } z <_{\nearrow} z'\}|,\end{aligned}$$

where T is any RSST and the first expression is a multiset. We refer to these as the *3-descent multiset* and the *3-inversion number* of T. One checks easily that

$$\mathrm{Des}'_3(T) = \mathrm{Des}'_3(\mathsf{v}) \text{ and } \mathrm{inv}'_k(T) = \mathrm{inv}'_k(\mathsf{v}) \tag{13}$$

for any square respecting reading word v of T. Recall that an RSST T is *nonzero* if any (equivalently, every) square respecting reading word of T is nonzero in $\mathcal{U}/I_{\mathrm{L},3}$.

Recall from Definition 2.4 that $\mathrm{W}'_k(\boldsymbol{\beta}) = \{\text{words } \mathsf{v} \in \mathcal{U} \mid \boldsymbol{\delta} \circ_{\mathbf{0}} \mathsf{v} = \boldsymbol{\gamma}\}$, where $\boldsymbol{\beta} = \boldsymbol{\gamma}/\boldsymbol{\delta}$ and $\circ_{\mathbf{0}}$ is an action of $\mathcal{U}$ on k-tuples of partitions.

Corollary 4.3. *Let $\boldsymbol{\beta}$ be a 3-tuple of skew shapes with contents, and let c, D' be the corresponding content vector and multiset from Proposition 2.6 (ii). Then the new variant q-Littlewood-Richardson coefficients are given by*

$$\mathfrak{c}^{\lambda}_{\boldsymbol{\beta}}(q) = \sum_{\substack{\mathsf{v} \in \mathrm{W}'_3(\boldsymbol{\beta}) \\ \mathsf{v} \in \{\mathrm{sqread}(T) \mid T \in \mathrm{RSST}_\lambda\}}} q^{\mathrm{inv}'_3(\mathsf{v})} = \sum_{\substack{T \in \mathrm{RSST}_\lambda,\ T \text{ nonzero} \\ \mathrm{Des}'_3(T) = D',\, c = \text{ sorted entries of } T}} q^{\mathrm{inv}'_3(T)}.$$

The advantage of the second expression is that it makes it clear that the answer depends only on a set of RSST and not on choices for their square respecting reading words. In the special case that c contains no repeated letter, this second expression is equal to

$$\sum_{S \in \mathrm{SYT}_\lambda,\, \mathrm{Des}''_3(c,S) = D'} q^{\mathrm{inv}''_3(c,S)},$$

where Des''_3 and inv''_3 are defined so that $\mathrm{Des}''_3(c, T^{\mathrm{st}}) = \mathrm{Des}'_3(T)$ and $\mathrm{inv}''_3(c, T^{\mathrm{st}}) = \mathrm{inv}'_3(T)$ whenever c is equal to the sorted entries of T; here, T^{st} denotes the SYT obtained from T by replacing its smallest entry with 1, its next smallest entry with 2, etc. This reformulation is essentially Haglund's conjecture [12, Conjecture 3]. (Haglund's conjecture is for those new variant combinatorial LLT polynomials which appear in the expression for transformed Macdonald polynomials as a positive sum of LLT polynomials [13]. In particular, all such LLT polynomials correspond to a tuple of skew shapes that have a content vector with no repeated letter.)

Proof of Corollary 4.3. If $\mathsf{v} = \mathsf{v}_1 \cdots \mathsf{v}_t \in \mathcal{U}$ is a word with $0 < \mathsf{v}_{i+1} - \mathsf{v}_i < 3$ and $\mathsf{w} = \mathsf{v}_1 \cdots \mathsf{v}_{i-1}\mathsf{v}_{i+1}\mathsf{v}_i\mathsf{v}_{i+2} \cdots \mathsf{v}_t$, then $\mathsf{v} \equiv q^{-1}\mathsf{w}$ by (7) and $q^{-\mathrm{inv}'_3(\mathsf{v})}\mathsf{v} - q^{-\mathrm{inv}'_3(\mathsf{w})}\mathsf{w} \equiv 0$. It follows from this and Proposition 2.6 (i) that

$$\sum_{\mathsf{v} \in \mathrm{W}'_3(\boldsymbol{\beta})} q^{\mathrm{inv}'_3(\mathsf{v})}\mathsf{v} \in (I_{\mathrm{L},3})^{\perp}.$$

By definition of Δ and Proposition 2.8,

$$\Delta\Big(\sum_{\mathsf{v}\in \mathrm{W}_3'(\boldsymbol{\beta})} q^{\mathrm{inv}_3'(\mathsf{v})}\mathsf{v}\Big) = \sum_{\mathsf{v}\in \mathrm{W}_3'(\boldsymbol{\beta})} q^{\mathrm{inv}_3'(\mathsf{v})} Q_{\mathrm{Des}(\mathsf{v})}(\mathbf{x}) = \mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x};q).$$

By Theorem 4.2, the coefficient of $s_\lambda(\mathbf{x})$ in $\mathcal{G}_{\boldsymbol{\beta}}(\mathbf{x};q)$ is

$$\mathfrak{c}^{\lambda}_{\boldsymbol{\beta}}(q) = \Big\langle \mathfrak{J}_\lambda(\mathbf{u}), \sum_{\mathsf{v}\in \mathrm{W}_3'(\boldsymbol{\beta})} q^{\mathrm{inv}_3'(\mathsf{v})}\mathsf{v}\Big\rangle = \sum_{\substack{\mathsf{v}\in \mathrm{W}_3'(\boldsymbol{\beta})\\ \mathsf{v}\in\{\mathrm{sqread}(T)\,|\,T\in \mathrm{RSST}_\lambda\}}} q^{\mathrm{inv}_3'(\mathsf{v})},$$

where the second equality is by Theorem 1.1. The second expression for $\mathfrak{c}^{\lambda}_{\boldsymbol{\beta}}(q)$ follows from this one by Proposition 2.6 (ii) and (13). □

Example 4.4. Let $\boldsymbol{\beta} = (2/1, 33/11, 33/21)$, $\mathsf{v} = 48714235$, and $c = 12344578$ be as in Example 2.5. Let $D' = \mathrm{Des}_3'(\mathsf{v}) = \{(4,1),(7,4),(8,5)\}$. The coefficient $\mathfrak{c}^{(4,3,1)}_{\boldsymbol{\beta}}(q)$ is computed by finding all the nonzero RSST T of shape $(4,3,1)$ with 3-descent multiset D'. These are shown below, along with their square respecting reading words and 3-inversion numbers.

T	1 3 4 5 / 2 7 8 / 4	1 2 4 5 / 3 7 8 / 4	1 2 4 5 / 3 4 7 / 8
$\mathrm{sqread}(T)$	42173845	43172845	83412745
$\mathrm{inv}_3'(T)$	4	5	5

Hence

$$\mathfrak{c}^{(4,3,1)}_{\boldsymbol{\beta}}(q) = q^4 + 2q^5.$$

4.3. **Proof of Theorem 4.1.** After three preliminary results, we state and prove a more technical version of Theorem 4.1, which involves computing $J_{\lambda'}(\mathbf{n})$ inductively by peeling off diagonals from the shape λ. Recall that $\equiv$ denotes equality in $\mathcal{U}/I_{\mathrm{L},k}$.

The following lemma is key to the proof of Theorem 4.1.

Lemma 4.5. *If $m < x$, then in $\mathcal{U}/I_{\mathrm{L},k}$,*

$$J_{(a,a)}(m, m : \overset{\mathsf{x}}{_1}) = \mathsf{x} J_{(a,a)}(m,m).$$

More generally, if $m < x$ and α is a weak composition satisfying $\alpha_j = \alpha_{j+1}$ and $\mathbf{n} = (n_1,\dots,n_l)$ with $n_j = n_{j+1}$, then in $\mathcal{U}/I_{\mathrm{L},k}$,

$$J_\alpha(\mathbf{n} : \overset{\mathsf{x}}{_j}) = J_\alpha(\mathbf{n} : \overset{\mathsf{x}}{_{j-1}}).$$

Proof. Since the proofs of both statements are essentially the same, we prove only the first to avoid extra notation. We compute

$$\begin{aligned}
0 &\equiv J_{(a,a+1)}(\{x\} \cup [m], \{x\} \cup [m]) \\
&= \mathsf{x}J_{(a-1,a+1)}([m], \{x\} \cup [m]) + J_{(a,a+1)}([m], \{x\} \cup [m]) \\
&= \mathsf{x}J_{(a-1,a)}(m, m : {}^{\mathsf{x}}_{1}) + \mathsf{x}J_{(a-1,a+1)}(m, m) + J_{(a,a)}(m, m : {}^{\mathsf{x}}_{1}) + J_{(a,a+1)}(m, m) \\
&\equiv \mathsf{x}J_{(a-1,a+1)}(m, m) + J_{(a,a)}(m, m : {}^{\mathsf{x}}_{1}) \\
&\equiv -\mathsf{x}J_{(a,a)}(m, m) + J_{(a,a)}(m, m : {}^{\mathsf{x}}_{1}).
\end{aligned}$$

The first congruence is by (⋄). The equalities are a 1-expansion and a 2-expansion (see §4.1 for notation). The second congruence is by Corollary 2.3 with repeated letter x and (⋄). The last congruence is by (11). □

The next lemma is a slight improvement on Lemma 4.5 needed to handle the case of repeated letters.

Lemma 4.6. *Let k be a positive integer. Suppose*

(a) *α is a weak composition with l parts and $\alpha_j = \alpha_{j+1}$,*
(b) *$\mathbf{n} = (n_1, \dots, n_l)$ satisfies $0 \le n_1 \le n_2 \le \cdots \le n_l$, and $n_{j-1} < n_{j+1} - 3$, and $n_{j+1} = n_j + 1$,*
(c) *x is a letter such that $x > n_{j+1}$ and $x \neq n_{j+1} + k$,*
(d) *$\mathsf{v} \in \mathcal{U}$ is a word such that $\mathsf{v}\mathsf{n}_{\mathsf{j+1}} \equiv 0$.*

Then in the algebra $\mathcal{U}/I_{\mathrm{L},k}$,

$$\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j}) = \mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}). \tag{14}$$

Proof. Set $\alpha_- = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \dots)$ and $\mathbf{n}_+ = (n_1, \dots, n_{j-1}, n_{j+1}, n_{j+1}, n_{j+2}, \dots)$. By Lemma 4.5,

$$\mathsf{v}J_\alpha(\mathbf{n}_+ : {}^{\mathsf{x}}_{j}) \equiv \mathsf{v}J_\alpha(\mathbf{n}_+ : {}^{\mathsf{x}}_{j-1}).$$

Apply a j-expansion to both sides to obtain

$$\mathsf{v}J_{\alpha_-}(\mathbf{n} : {}^{\mathsf{n_{j+1}}}_{j-1}, {}^{\mathsf{x}}_{j}) + \mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j}) \tag{15}$$

$$\equiv \mathsf{v}J_{\alpha_-}(\mathbf{n} : {}^{\mathsf{x}\mathsf{n_{j+1}}}_{j-1}) + \mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}). \tag{16}$$

It suffices to show that the first term of (15) and the first term of (16) are $\equiv 0$. We only show the latter, as the former is similar and easier. There holds

$$\mathsf{v}J_{\alpha_-}(\mathbf{n} : {}^{\mathsf{x}\mathsf{n_{j+1}}}_{j-1}) \equiv q^a\mathsf{v}J_{\alpha_-}(\mathbf{n} : {}^{\mathsf{n_{j+1}}\mathsf{x}}_{j-1}) \equiv q^a\mathsf{v}\mathsf{n_{j+1}}J_{\alpha_-}(\mathbf{n} : {}^{\mathsf{x}}_{j-1}) \equiv 0,$$

where $a \in \{0, 1\}$. The first congruence is by (c) and the relations (7) and the far commutation relations (6), the second congruence is by (b) and the far commutation relations, and the last congruence is by (d). □

Corollary 4.7. *Maintain the notation of Lemma 4.6 and assume (a)–(d) of the lemma hold. Assume in addition*

(e) *$\mathsf{w} \in \mathcal{U}$ is a word such that $\mathsf{w} = \mathsf{x}\mathsf{w}'$ and the letters of w' are $> n_{j+1} + k$.*

Then in the algebra $\mathcal{U}/I_{\mathrm{L},k}$,

$$\mathsf{v}J_\alpha(\mathbf{n}:\ {}^{\mathsf{w}}_{j}) = \mathsf{v}J_\alpha(\mathbf{n}:{}^{\mathsf{x}}_{j-1},\ {}^{\mathsf{w}'}_{j}).$$

Proof. We compute in $\mathcal{U}/I_{\mathrm{L},k}$,

$$\mathsf{v}J_\alpha(\mathbf{n}:\ {}^{\mathsf{w}}_{j}) = \mathsf{v}J_\alpha(\mathbf{n}:\ {}^{\mathsf{x}}_{j}, {}^{\mathsf{w}'}_{j+1}) = \mathsf{v}J_\alpha(\mathbf{n}:{}^{\mathsf{x}}_{j-1}, {}^{\mathsf{w}'}_{j+1}) = \mathsf{v}J_\alpha(\mathbf{n}:{}^{\mathsf{x}}_{j-1},\ {}^{\mathsf{w}'}_{j}).$$

The first and third equalities are by the far commutation relations and the second equality is by Lemma 4.6 (the proof of the lemma still works with the word w' present). □

For a weak composition or partition $\alpha = (\alpha_1, \dots, \alpha_l)$, define $\alpha_{l+1} = 0$.

Theorem 4.8. *Let* $\lambda' \setminus \alpha'$ *be a restricted shape and let* l *be the number of parts of* λ. *Set* $j = \min(\{i \mid \alpha_i > 0,\ \alpha_i \geq \alpha_{i+1}\} \cup \{l+1\})$ *and* $j' = \max(\{i \mid \alpha_i < \lambda_i\} \cup \{0\})$ *(see Figure 1 and the discussion following Remark 4.9). Suppose*

(i) α *is of the form*

$$(0, \dots, 0, 1, 2, \dots, a, a, a+1, a+2, \dots, \alpha_{j'}, \lambda_{j'+1}, \lambda_{j'+2}, \dots), \qquad or$$
$$(0, \dots, 0, 1, 2, \dots, \alpha_{j'}, \lambda_{j'+1}, \lambda_{j'+2}, \dots),$$

where $\alpha_{j'} \geq \alpha_{j'+1} - 1 = \lambda_{j'+1} - 1$, *the initial run of 0's may be empty, and on the top line (resp. bottom line)* j *is* $< j'$ *and is the position of the first* a *(resp.* j *is* j' *or* $j'+1$*).*

(ii) R *is an RSST of shape* $\lambda' \setminus \alpha'$ *with entries* $r_1 = R_{\alpha_1+1,1}, \dots, r_{j'} = R_{\alpha_{j'}+1,j'}$ *along its northern border,*

(iii) vw *is a square respecting reading word of* R *such that* w *is a subsequence of* $\mathsf{r}_{j+1} \cdots \mathsf{r}_{j'}$ *which contains* r_{j+1} *if* $r_{j+1} > r_j + 1$,

(iv) $\mathbf{n} = (n_1, \dots, n_l)$ *is a tuple which satisfies* $0 \leq n_1 \leq n_2 \leq \cdots \leq n_l$, *and*

(v) $n_c = r_c - 1$ *for* $c \in [j']$ *with the exception that* n_j *may be* $< r_j - 1$ *if* w *is empty or* $w_1 \neq r_j + 1$.

Then in the algebra $\mathcal{U}/I_{\mathrm{L},3}$,

$$\mathsf{v}J_\alpha(\mathbf{n}:\ {}^{\mathsf{w}}_{j}) = \sum_{\substack{T \in \mathrm{RSST}_{\lambda'},\ T_{\lambda' \setminus \alpha'} = R \\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})}} \mathrm{sqread}(T). \tag{17}$$

To parse this statement, it is instructive to first understand the case when $j \geq j'$ (which implies w is empty) and $\alpha_2 > 0$. In this case the theorem becomes

> Suppose $\lambda' \setminus \alpha'$ is a restricted shape with $\alpha_1 = \alpha_2 - 1 = \cdots = \alpha_j - j + 1$ and $\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_l > \alpha_{l+1} = 0$. Let R be an RSST of shape $\lambda' \setminus \alpha'$ with entries $r_1, \dots, r_{j'}$ along its northern border. Suppose $\mathbf{n} = (n_1, \dots, n_l)$ satisfies $0 \leq n_1 \leq n_2 \leq \cdots \leq n_l$, $n_c = r_c - 1$ for $c \in [j-1]$, and $n_j < r_j$. Then in $\mathcal{U}/I_{\mathrm{L},3}$,
>
> $$\mathrm{sqread}(R) J_\alpha(\mathbf{n}) = \sum_{\substack{T \in \mathrm{RSST}_{\lambda'},\ T_{\lambda' \setminus \alpha'} = R \\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})}} \mathrm{sqread}(T).$$

Remark 4.9.

(a) Theorem 4.1 is the special case of Theorem 4.8 when $\alpha = \lambda$ (the λ and λ' of Theorem 4.1 must be interchanged to match the notation here).

(b) A reading word vw as in (iii) always exists—for instance take $\mathsf{vw} = \text{sqread}(R)$ with w the subsequence of $\mathsf{r_{j+1}} \cdots \mathsf{r_{j'}}$ consisting of those entries which are not the tail of a $\nwarrow$ arrow of R.

(c) For any vw satisfying the the assumptions of the theorem, w does not contain the tail of a $\nwarrow$ arrow of R.

(d) It follows from the theorem that $\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = 0$ if R cannot be completed to an RSST of shape λ', however we purposely do not make this assumption so that it does not have to be verified at the inductive step.

The assumptions on j, j', and α look complicated, but their purpose is simply to peal off the entries of a tableau of shape λ' one diagonal at a time, starting with the southwesternmost diagonal, and reading each diagonal in the $\nwarrow$ direction (see Figure 1). The proof goes by induction, peeling off one letter at a time from $J_\alpha(\mathbf{n})$, in the order just specified, and incorporating them into R. The index j indicates the column of the next letter to be removed.

The reader is encouraged to follow along the proof below with Example 4.10.

Proof of Theorem 4.8. Write J for $J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j})$. The proof is by induction on $|\alpha|$ and the n_i. If $|\alpha| = 0$, $J_\alpha(\mathbf{n})$ is (a noncommutative version of) the determinant of an upper unitriangular matrix, hence $J = \mathsf{w}$. The theorem then states that $\mathsf{vw} \equiv \text{sqread}(R)$, which holds by Theorem 3.11.

Next consider the case $n_1 = 0$ and $\alpha_1 > 0$. This implies that $J_\alpha(\mathbf{n})$ is (a noncommutative version of) the determinant of a matrix whose first row is all 0's, hence $J = 0$. The right side of (17) is also 0 because $\text{Tab}_{\alpha'}(\mathbf{n})$ is empty for $n_1 = 0$ and $\alpha_1 > 0$.

If $n_i = n_{i+1}$ for any $i \neq j$ such that $\alpha_i = \alpha_{i+1} - 1$, then $J = 0$ by ($\diamond$). To see that the right side of (17) is also 0 in this case, observe that if T is an RSST from this sum, then

$$n_i + 1 = r_i < T_{\alpha_{i+1}, i+1} \leq n_{i+1} = n_i,$$

hence the sum is empty.

By what has been said so far, we may assume $|\alpha| > 0$, $\alpha_j > 0$, and $0 \leq n_{j-1} < n_j$ if $j > 1$ and $0 < n_j$ if $j = 1$. We proceed to the main body of the proof. There are two cases depending on whether (w is empty or $w_1 > r_j + 1$) or $w_1 = r_j + 1$. Set $\mathbf{n}_- = (n_1, n_2, \ldots, n_{j-1}, n_j - 1, n_{j+1}, \ldots, n_l)$ and $\alpha_- = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots)$.

Case w is empty or $w_1 > r_j + 1$.

There are three subcases depending on whether $j = 1$, ($j > 1$ and $\alpha_j = 1$), or ($j > 1$ and $\alpha_j > 1$). We argue only the last, as the first two are similar and easier. A j-expansion yields

$$\mathsf{v}J = \mathsf{v}J_{\alpha_-}(\mathbf{n}_- : {}^{\mathsf{n_j}}_{j-1}, {}^{\mathsf{w}}_{j}) + \mathsf{v}J_{\alpha}(\mathbf{n}_- : {}^{\mathsf{w}}_{j}) \tag{18}$$

$$\equiv \mathsf{v}J_{\alpha_-}(\mathbf{n}_- : {}^{\mathsf{n_j w}}_{j-1}) + \mathsf{v}J_{\alpha}(\mathbf{n}_- : {}^{\mathsf{w}}_{j}) \tag{19}$$

$$\equiv \sum_{\substack{T\in \mathrm{RSST}_{\lambda'},\ T_{\lambda'\setminus\alpha'_-}=R\sqcup \boxed{\mathsf{n_j}}_{\alpha_j,j} \\ T_{\alpha'_-}\in \mathrm{Tab}_{\alpha'_-}(\mathbf{n}_-)}} \mathrm{sqread}(T) + \sum_{\substack{T\in \mathrm{RSST}_{\lambda'},\ T_{\lambda'\setminus\alpha'}=R \\ T'_{\alpha}\in \mathrm{Tab}_{\alpha'}(\mathbf{n}_-)}} \mathrm{sqread}(T), \tag{20}$$

where $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ denotes the result of adding the cell (α_j, j) to R and filling it with $\mathsf{n_j}$. The first congruence is by the far commutation relations since the letters of w are $> n_j+2$. The second term of (19) is congruent (mod $I_{\mathrm{L},3}$) to the second sum of (20) by induction. The conditions of the theorem are satisfied here: (i)–(iii) are clear, (iv) holds since $n_{j-1} < n_j$, and (v) holds as w is empty or $w_1 \neq r_j + 1$ so it is okay that $(n_-)_j < r_j - 1$.

We next claim that either the first term of (19) satisfies conditions (i)–(v) of the theorem (with α_- in place of α, $\mathbf{n}_-$ in place of $\mathbf{n}$, $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ in place of R, $\mathsf{n_j w}$ in place of w, $j-1$ in place of j), or (ii) fails and the first term of (19) is 0. This will show that the first term of (19) is congruent to the first sum of (20) (by induction in the former case, and because both quantities are $\equiv 0$ in the latter case). We now assume that (ii) holds, i.e. $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ is an RSST, and we verify the remaining conditions. Conditions (i) and (v) are clear, and (iv) follows from $n_{j-1} < n_j$. Next we check (iii), which requires showing that $\mathsf{n_j w}$ is the end of a square respecting reading word of $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ satisfying an extra condition. There are two ways this can fail: either (I) $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ has a $\searrow$ arrow from n_j to r_{j+1} and $w_1 \neq r_{j+1}$, or (II) $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ has a $\nwarrow$ arrow from r_{j+1} to n_j and $r_{j+1} = w_1$; (I) implies $r_{j+1} > r_j + 1$ so by assumption (iii) of the theorem $w_1 = r_{j+1}$, thus (I) cannot occur; (II) implies $w_1 = r_j + 1$, contradicting our assumption $w_1 > r_j + 1$, thus (II) cannot occur.

It remains to show that if (ii) does not hold, i.e. $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ is not an RSST, then the first term of (19) is $\equiv 0$. There are two ways $R\sqcup\boxed{\mathsf{n_j}}_{\alpha_j,j}$ can fail to be an RSST: (A) $n_j \leq r_{j-1}$, or (B) $n_j = r_{j+1} - 2$ (and $j < j'$).

In the case (A) holds, we have $n_j \leq r_{j-1} = n_{j-1} + 1$; in fact, since $n_{j-1} < n_j$, there holds $n_{j-1} + 1 = r_{j-1} = n_j$. By (18) and (19), it suffices to show $\mathsf{v}J \equiv 0$ since $\mathsf{v}J_{\alpha}(\mathbf{n}_- : {}^{\mathsf{w}}_{j}) \equiv 0$ by ($\diamond$) as $(n_-)_{j-1} = (n_-)_j$ and $\alpha_{j-1} = \alpha_j - 1$. Let $\hat{R}$ be the canonical sub-RSST of R with reading word v. Corollary 3.12 with RSST $\hat{R}$ and z the cell containing r_{j-1} implies $\mathsf{v r_{j-1}} \equiv 0$, hence

$$0 \equiv \mathsf{v r_{j-1}} J_{\alpha-\epsilon_{j-1}}(\mathbf{n} : {}^{\mathsf{w}}_{j}) \equiv \mathsf{v}J_{\alpha-\epsilon_{j-1}}(\mathbf{n} : {}^{\mathsf{r_{j-1}}}_{j-2}, {}^{\mathsf{w}}_{j}), \tag{21}$$

where the last congruence is by the far commutation relations since $n_{j-2} = r_{j-2} - 1 \leq r_{j-1} - 4$; here and throughout the proof, ϵ_i denotes the vector with a 1 in the i-th position and 0's elsewhere.

Next, we compute

$$0 \equiv \mathsf{v}J_{\alpha}(n_1, \dots, n_{j-2}, n_j, n_j, n_{j+1}, \dots : {}^{\mathsf{w}}_{j}) = \mathsf{v}J_{\alpha-\epsilon_{j-1}}(\mathbf{n} : {}^{\mathsf{r_{j-1}}}_{j-2}, {}^{\mathsf{w}}_{j}) + \mathsf{v}J \equiv \mathsf{v}J.$$

The first congruence uses ($\diamond$), the equality is a $j-1$-expansion, and the last congruence is by (21).

In the case that (B) holds, we must have $n_j + 1 = r_j = r_{j+1} - 1$. This, together with $r_{j+2} \geq r_{j+1} + 3$, implies

$$\mathsf{r_{j+1}} \cdots \mathsf{r_{j'}} \text{ does not contain } r_j + 3 \text{ or } r_j - 3, \tag{22}$$

which will be important later. Apply a $j+1$-expansion twice to the first term of (19) and for convenience set $m = n_j$ (note that $n_{j+1} = r_{j+1} - 1 = r_j = n_j + 1$):

$$\begin{aligned}
&\mathsf{v}J_{\alpha_-}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots : \overset{\mathsf{n_jw}}{\scriptstyle j-1}) \\
&= \mathsf{v}J_{\beta}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1} - 2, \dots : \overset{\mathsf{n_jw}}{\scriptstyle j-1}, \ \overset{\mathsf{n_{j+1}(n_{j+1}-1)}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\gamma}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1} - 2, \dots : \overset{\mathsf{n_jw}}{\scriptstyle j-1}, \ \overset{\mathsf{n_{j+1}}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\gamma}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1} - 2, \dots : \overset{\mathsf{n_jw}}{\scriptstyle j-1}, \ \overset{\mathsf{n_{j+1}-1}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\alpha_-}(n_1, \dots, n_{j-1}, n_j - 1, n_{j+1} - 2, \dots : \overset{\mathsf{n_jw}}{\scriptstyle j-1}) \\
&= \mathsf{v}J_{\beta}(n_1, \dots, n_{j-1}, m - 1, m - 1, \dots : \overset{\mathsf{mw}}{\scriptstyle j-1}, \ \overset{\mathsf{r_jm}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\gamma}(n_1, \dots, n_{j-1}, m - 1, m - 1, \dots : \overset{\mathsf{mw}}{\scriptstyle j-1}, \ \overset{\mathsf{r_j}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\gamma}(n_1, \dots, n_{j-1}, m - 1, m - 1, \dots : \overset{\mathsf{mw}}{\scriptstyle j-1}, \ \overset{\mathsf{m}}{\scriptstyle j}) \\
&+ \mathsf{v}J_{\alpha_-}(n_1, \dots, n_{j-1}, m - 1, m - 1, \dots : \overset{\mathsf{mw}}{\scriptstyle j-1}),
\end{aligned}$$

where $\beta = \alpha_- - 2\epsilon_{j+1}$ and $\gamma = \alpha_- - \epsilon_{j+1}$. The first and third terms are $\equiv 0$ by Corollary 2.3 with repeated letter m. The fourth term is $\equiv 0$ by ($\diamond$). The second term is equal to $\mathsf{v}J_{\gamma}(n_1, \dots, n_{j-1}, m - 1, m - 1, \dots : \overset{\mathsf{mwr_j}}{\scriptstyle j-1})$ by Lemma 4.5. Since $n_{j-1} = r_{j-1} - 1 \leq r_j - 4$ and (22) holds, this is equal in $\mathcal{U}/I_{\mathrm{L},3}$ to an expression beginning with $\mathsf{vr_j}$ times some power of q.

We claim that $\mathsf{vr_j} \equiv 0$, as follows: by Theorem 3.11, $\mathsf{v} \equiv \mathsf{v'w'}$, where $\mathsf{v'w'}$ is a square respecting reading word of R such that $\mathsf{w'w}$ is a permutation of $r_{j+1} \cdots r_{j'}$. Let R' be the canonical sub-RSST of R with reading word $\mathsf{v'}$ i.e. the RSST obtained from R by deleting its northeasternmost diagonal. Now apply Corollary 3.12 with RSST R' and z the cell containing $\mathsf{r_j}$ to obtain $\mathsf{v'r_j} \equiv 0$, hence

$$\mathsf{vr_j} \equiv \mathsf{v'w'r_j} \equiv q^a \mathsf{v'r_jw'} \equiv 0,$$

where second congruence is by (22) (a is some integer). This completes the proof that the first term of (19) is $\equiv 0$ when (B) holds. This, in turn, completes the proof that the first term of (19) is congruent to the first sum of (20).

Now (20) is equal to the right side of (17) because (20) is simply the result of partitioning the set $\{T \in \mathrm{RSST}_{\lambda'} \mid T_{\lambda' \setminus \alpha'} = R,\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})\}$ into two, depending on whether or not $T_{\alpha_j, j}$ is equal to or less than $\mathsf{n_j}$. This completes the case w is empty or $w_1 > r_j + 1$.

Case $w_1 = r_j + 1$**.**

We have $w_1 = r_{j+1}$ and $n_j = r_j - 1 = w_1 - 2 = n_{j+1} - 1$. Note that (22) holds here as well. We now verify the hypotheses (a)–(e) of Corollary 4.7. Conditions (a) and (b) are clear from what has just been said and from $n_{j-1} < r_{j-1} \leq r_j - 3 = n_{j+1} - 3$. Conditions (c) and (e) follow from (22). Condition (d) holds because the same argument given two paragraphs above also applies here to give $\mathsf{vn_{j+1}} = \mathsf{vr_j} \equiv 0$.

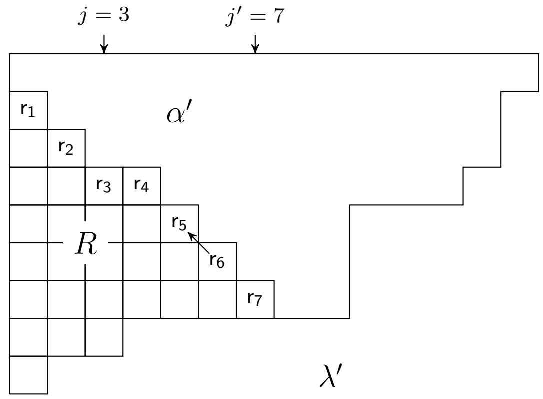


Figure 1: The setup of the proof of Theorem 4.8 for $\lambda = (9,8,8,7,7,7,7,7,7,4,4,4,3,1)$, $\alpha = (1,2,3,3,4,5,6,7,7,4,4,4,3,1)$. A possibility for the arrows of R is shown. A possibility for w is $\mathsf{r}_4\mathsf{r}_5\mathsf{r}_7$.

Set $t = |\mathsf{w}|$. Corollary 4.7 yields

$$\begin{aligned}\mathsf{v}J_\alpha(\mathbf{n} :\ {}^{\mathsf{w}}_{j}) &\equiv \mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}_1}_{j-1},\ {}^{\mathsf{w}_2\cdots\mathsf{w}_t}_{j}) \\ &\equiv \mathsf{v}\mathsf{w}_1 J_\alpha(\mathbf{n} :\ {}^{\mathsf{w}_2\cdots\mathsf{w}_t}_{j}). \end{aligned} \tag{23}$$

The last congruence is by the far commutation relations if $j > 1$ since $n_{j-1} < r_{j-1} \leq r_j - 3 = w_1 - 4$ (if $j = 1$ there is nothing to prove).

Finally, observe that the case (w is empty or $w_1 > r_j + 1$) applies to (23) (with $\mathsf{v}\mathsf{w}_1$ in place of v, $\mathsf{w}_2 \cdots \mathsf{w}_t$ in place of w). Since the right side of (17) depends only on R and not directly on v and w, (23) is congruent to this right side, and this gives the desired statement in the present case $w_1 = r_j + 1$. □

Example 4.10. Figure 2 illustrates the inductive computation of $J_{(3,3)}(6,6)$ from the proof of Theorem 4.8.

Write $A, B, \ldots, G$ for $10, 11, \ldots, 16$. Let $\lambda = (4,4,4,4)$. Let $R = \begin{array}{|c|c|c|}\hline 3 \\ \hline 5 & 6 \\ \hline 7 & 8 & F \\ \hline\end{array}$; then $\mathrm{sqread}(R) = \mathsf{78563F}$. We illustrate several steps of the inductive computation of $\mathsf{78563F}J_{(1,2,3,4)}(2,5,E,G)$ from the proof of Theorem 4.8. After each step in which we add an entry to R, we record the new values of R, j, and j'. We first apply the proof of the theorem to $\mathsf{78563F}J_{(1,2,3,4)}(2,5,E,G)$ ($\mathsf{v} = \mathsf{78563F}$, $\mathsf{w} = \emptyset$, R as above, $j = 4$, $j' = 3$) and expand as in (18):

$$\begin{aligned} &\mathsf{78563F}J_{(1,2,3,4)}(2,5,E,G) \\ &= \mathsf{78563F}\Big(J_{(1,2,3,3)}(2,5,E,F :\ {}^{\mathsf{G}}_{3}) + J_{(1,2,3,4)}(2,5,E,F)\Big) \end{aligned} \tag{24}$$

Next, apply the theorem to the first term of (24) ($\mathsf{v} = \mathsf{78563F}$, $\mathsf{w} = \mathsf{G}$, $R = \begin{array}{|c|c|c|c|}\hline 3 \\ \hline 5 & 6 \\ \hline 7 & 8 & F & G \\ \hline\end{array}$, $j = 3$, $j' = 4$). This is handled by the case $w_1 = r_j + 1$, hence the computation proceeds by

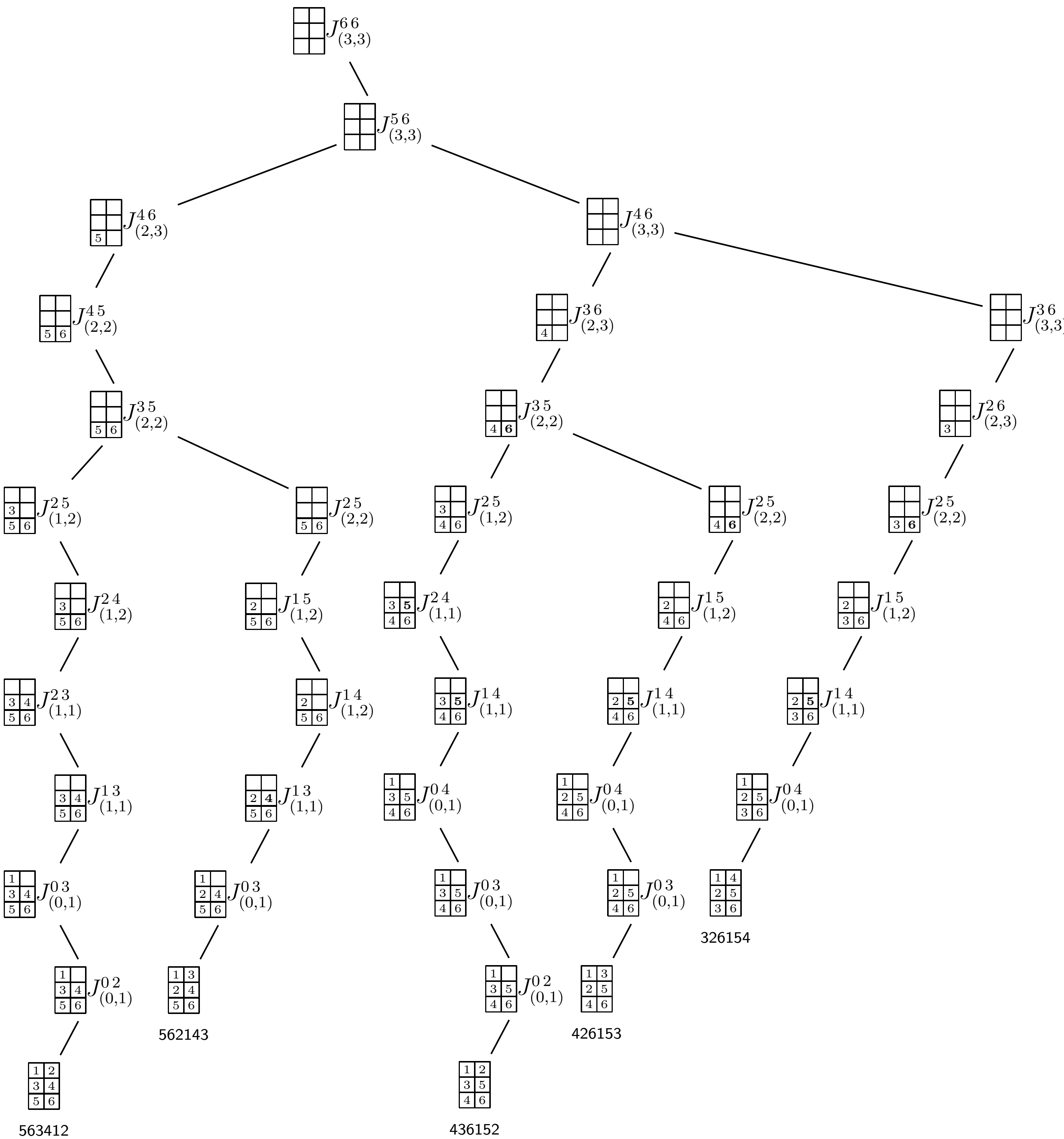


Figure 2: The inductive computation of $J_{(3,3)}(6,6)$ from the proof of Theorem 4.8, depicted with the following conventions: instead of writing $\mathsf{v}J_\alpha(\mathbf{n} : \genfrac{}{}{0pt}{}{\mathsf{w}}{j})$, we write $RJ_\alpha^{\mathbf{n}}$ where R is an RSST with square respecting reading word vw; the entries of R corresponding to w are in bold. A left (resp. right) branch of the tree corresponds to the first (resp. second) term of (18). All branches that eventually lead to 0 have been pruned. The words below the leaves indicate the final square respecting reading words produced by this computation.

applying (23) and then a 3-expansion:

$$\begin{aligned}
&\mathsf{78563F}\,J_{(1,2,3,3)}(2,5,E,F:\ {}^{\mathsf{G}}_{3}) \\
\equiv\ &\mathsf{78563FG}\,J_{(1,2,3,3)}(2,5,E,F) \\
=\ &\mathsf{78563FG}\Big(J_{(1,2,2,3)}(2,5,D,F:\ {}^{\mathsf{E}}_{2}) + J_{(1,2,3,3)}(2,5,D,F)\Big).
\end{aligned}$$

The first term is $\equiv 0$ by the proof of Theorem 4.8 for the case that (B) holds. We expand the second term as in (18) ($j = 3$):

$$\begin{aligned}
&\mathsf{78563FG}\,J_{(1,2,3,3)}(2,5,D,F) \\
=\ &\mathsf{78563FG}\Big(J_{(1,2,2,3)}(2,5,C,F:\ {}^{\mathsf{D}}_{2}) + J_{(1,2,3,3)}(2,5,C,F)\Big).
\end{aligned}$$

Next, we proceed with the inductive computation of the first term above ($R = \begin{array}{|c|c|c|c|}\hline 3 \\ \cline{1-1} 5 & 6 & D \\ \cline{1-3} 7 & 8 & F & G \\ \hline\end{array}$, $j = 2$, $j' = 4$)

$$\begin{aligned}
&\mathsf{78563FG}\,J_{(1,2,2,3)}(2,5,C,F:\ {}^{\mathsf{D}}_{2}) \\
=\ &\mathsf{78563FG}\Big(J_{(1,1,2,3)}(2,4,C,F:\ {}^{\mathsf{5}}_{1},\ {}^{\mathsf{D}}_{2}) + J_{(1,2,2,3)}(2,4,C,F:\ {}^{\mathsf{D}}_{2})\Big).
\end{aligned}$$

By Proposition 3.13 and Theorem 4.8, this first term is 0. We continue the computation with the second term:

$$\begin{aligned}
&\mathsf{78563FG}\,J_{(1,2,2,3)}(2,4,C,F:\ {}^{\mathsf{D}}_{2}) \\
=\ &\mathsf{78563FG}\Big(J_{(1,1,2,3)}(2,3,C,F:\ {}^{\mathsf{4}}_{1},\ {}^{\mathsf{D}}_{2}) + J_{(1,2,2,3)}(2,3,C,F:\ {}^{\mathsf{D}}_{2})\Big) \\
\equiv\ &\mathsf{78563FG}\Big(J_{(1,1,2,3)}(2,3,C,F:\ {}^{\mathsf{4D}}_{1}) + J_{(1,2,2,3)}(2,3,C,F:\ {}^{\mathsf{D}}_{2})\Big).
\end{aligned}$$

The input data to the theorem for this first term is $\mathsf{v} = \mathsf{78563FG}$, $\mathsf{w} = \mathsf{4D}$, $R = \begin{array}{|c|c|c|c|}\hline 3 & 4 \\ \cline{1-2} 5 & 6 & D \\ \cline{1-3} 7 & 8 & F & G \\ \hline\end{array}$, $j = 1$, $j' = 4$, $\alpha = (1,1,2,3)$, $\mathbf{n} = (2,3,C,F)$.

5. Strengthenings of the main theorem

Theorem 4.8 does not hold in $\mathcal{U}/I_{\mathrm{L},k}$ for $k > 3$, nor does it seem likely that $J_\lambda(\mathbf{n})$ can be computed positively in $\mathcal{U}/I_{\mathrm{L},k}$ for $k > 3$ using a sequence of j-expansions and manipulations like that of Lemma 4.5. The papers [5, 6] discuss the difficulties that must be overcome to apply the Fomin-Greene approach to this more general setting. Here we discuss two extensions of the main theorem to quotients of $\mathcal{U}$ that are similar to $\mathcal{U}/I_{\mathrm{L},3}$ but have weaker relations. These relations were inspired by the work of Assaf [3, 2].

5.1. **An easy strengthening.** Define $\mathcal{U}/I_{\mathrm{L},\leq k}$ to be the quotient of $\mathcal{U}$ by the following relations:

$$\begin{aligned} \mathsf{v} &= 0 && \text{if the word } \mathsf{v} \in \mathcal{U} \text{ lies in } I_{\mathrm{L},k}, \\ u_i u_j &= u_j u_i && \text{for } |i-j| > k, \\ (u_a u_c - u_c u_a) u_b &= u_b (u_a u_c - u_c u_a) && \text{for } a < b < c \text{ and } c - a \leq k. \end{aligned}$$

All the results in Sections 3 and 4 hold in this variant $\mathcal{U}/I_{\mathrm{L},\leq 3}$ of Lam's algebra $\mathcal{U}/I_{\mathrm{L},3}$. All the proofs in Sections 3 and 4 work for this variant with no essential change, except that we require [6] to prove that the elementary symmetric functions commute in $\mathcal{U}/I_{\mathrm{L},\leq 3}$. Since $\mathcal{U}/I_{\mathrm{L},3}$ is a quotient of $\mathcal{U}/I_{\mathrm{L},\leq 3}$, this is a slight strengthening of the main theorem.

5.2. **A conjectured strengthening.** Most of our work on this project was towards proving Theorem 4.8 in the algebra $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$, where $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq k}$ is the quotient of $\mathcal{U}$ by the following relations

$$\mathsf{v} = 0 \qquad \text{if the word } \mathsf{v} \in \mathcal{U} \text{ has a repeated letter,} \tag{25}$$

$$u_a u_c u_b = u_c u_a u_b \qquad \text{for } a < b < c \text{ and } c - a > k, \tag{26}$$

$$u_b u_a u_c = u_b u_c u_a \qquad \text{for } a < b < c \text{ and } c - a > k, \tag{27}$$

$$(u_a u_c - u_c u_a) u_b = u_b (u_a u_c - u_c u_a) \qquad \text{for } a < b < c \text{ and } c - a \leq k. \tag{28}$$

The quotient of $\mathcal{U}$ by (26)–(28) has the plactic algebra and $\mathcal{U}/I_{\mathrm{L},k}$ as quotients. Also, the noncommutative elementary symmetric functions commute in this quotient of $\mathcal{U}$ by (26)–(28) and hence in $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq k}$ (see [6]). Note that for any $a < b < c$, the relation in (28) is implied by either of the following pairs of relations

$$u_a u_c u_b = u_c u_a u_b \quad \text{and} \quad u_c u_a u_b = u_a u_c u_b \quad \text{(Knuth relations)} \tag{29}$$

$$u_a u_c u_b = u_b u_a u_c \quad \text{and} \quad u_c u_a u_b = u_b u_c u_a \quad \text{(rotation relations).} \tag{30}$$

A *bijectivization* of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq k}$ is a quotient of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq k}$ obtained by adding, independently for every $a < b < c$ such that $c - a \leq k$, either (29) or (30) to its list of relations.

Remark 5.1. Let $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{A},k}$ be the bijectivization of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq k}$ which uses only (30). It is related to the graphs $\mathcal{G}^{(k)}_{c,D}$ defined in [3, §4.2] as follows (see [5, §5.3] and [6] for further discussion): the bijection $\{(w,c) \in \mathrm{WRib}_k(c,D) \mid w \text{ a permutation}\} \to \mathrm{W}'_k(c,D')$ of Proposition 2.6 (iii) is a bijection between the vertex set of $\mathcal{G}^{(k)}_{c,D}$ and $\mathrm{W}'_k(c,D') \subseteq \mathcal{U}$; if each word of $\mathrm{W}'_k(c,D')$ contains no repeated letter, then this bijection takes connected components of $\mathcal{G}^{(k)}_{c,D}$ to nonzero equivalence classes of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{A},k}$; moreover, all the nonzero equivalence classes of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{A},k}$ are obtained in this way. Here, a *nonzero equivalence class* of an algebra $\mathcal{U}/I$ is a maximal set C of words of $\mathcal{U}$ such that $\mathsf{v} = \mathsf{w} \neq 0$ in $\mathcal{U}/I$ for all $\mathsf{v}, \mathsf{w} \in C$.

We conjecture that Theorem 4.8 holds exactly as stated with $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$ in place of $\mathcal{U}/I_{\mathrm{L},3}$. We believe that the same inductive structure of the proof works in this setting, except some of the steps are much more difficult to justify. In fact, we know how to prove the slightly weaker statement—that Theorem 4.8 holds in any bijectivization of $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$—assuming the following

Conjecture 5.2. *Suppose $m \leq n_1 \leq \cdots \leq n_t$, $n_i < y_i$ for $i \in [t]$, $m < x$, $y_1 - x \geq 3$, and $y_{i+1} - y_i \geq 3$ for $i \in [t-1]$. Set $\alpha = (a, a, a+1, a+2, \dots, a+t)$. Then in $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$,*

$$J_\alpha(m, m, n_1, \dots, n_t :\ {}^{\mathsf{x}\mathsf{y}_1\cdots\mathsf{y}_t}_{\ 1}) = \mathsf{x} J_\alpha(m, m, n_1, \dots, n_t :\ {}^{\mathsf{y}_1\cdots\mathsf{y}_t}_{\ 1}).$$

This is a replacement of Corollary 4.7 and is needed because, without the far commutation relations, this corollary can no longer be deduced from Lemma 4.5. When $t = 0$ this becomes Lemma 4.5 for $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$, and the given proof of this lemma carries over with no essential change. We can also prove the $t = 1$ case, which takes about a page of delicate algebraic manipulations. Conjecture 5.2 does not seem to be easy to prove even in the quotient of the plactic algebra by the additional relation (25).

Another ingredient required to extend the proof of Theorem 4.8 to this setting is the following variant of Theorem 3.11. The proof is substantial and requires identifying a nonobvious binomial relation in $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$.

Theorem 5.3. *Any two square respecting reading words of an RSST are equal in $\mathcal{U}/I^{\mathrm{st}}_{\mathrm{KR},\leq 3}$.*

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