Math 201 – Midterm 1
Winter 2015

Time: 90 mins.

1. Answer all questions in the spaces provided.
2. No calculators, notes, books, phones, or other outside assistance allowed.
3. Remember to justify your answers (except for Question 4) by showing all of your work, and by citing Theorems from lecture or from the text where appropriate.

Name: ____________________________

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1. Let \( A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 4 & 12 \end{bmatrix} \) and \( b = \begin{bmatrix} 0 \\ 4 \\ -8 \end{bmatrix} \).

(a) (3 points) Find an echelon form (EF) of the augmented matrix \([A \, b]\).

(b) (2 points) Find the reduced row echelon form (RREF) of the augmented matrix \([A \, b]\).

(c) (2 points) Find the parametric description of the solution set of the system \( A \mathbf{x} = \mathbf{b} \).

(d) (2 points) Do the columns of \( A \) span \( \mathbb{R}^3 \)? Justify your answer.

(e) (1 point) Write down one solution to the linear system

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 4 \\
-4x_1 + 4x_2 + 12x_3 &= -8
\end{align*}
\]
2. Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ -1 & -2 & -3 \end{bmatrix}$.

(a) (4 points) Find the inverse of $A$.

(b) (2 points) Determine the solution set of the system $Ax = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) (2 points) Is there a $b \in \mathbb{R}^3$ such that the system $Ax = b$ has infinitely many solutions?
3. Let 

\[ C = \begin{pmatrix} 
1 & 2 & 1 & 21 \\
2 & 6 & 0 & 20 \\
0 & 0 & 0 & 3 \\
2 & 4 & 3 & 19 
\end{pmatrix}. \]

(a) (4 points) Compute the determinant of \( C \).

(b) (2 points) Compute the determinant of \( C^{-1} \).

(c) (2 points) Compute the determinant of the matrix 

\[ D = \begin{pmatrix} 
1 & 2 & 1 & 21 \\
4 & 12 & 0 & 40 \\
0 & 0 & 0 & 3 \\
2 & 4 & 3 & 19 
\end{pmatrix}. \]
4. For this question, no justification is required (justification is required on all other questions on this midterm). Circle your final answer clearly. For parts (c)–(h), determine whether the statement is true or false.

(a) (1 point) Write down a vector \( \mathbf{v} \in \mathbb{R}^2 \) such that the set \( \{ \mathbf{v}, [1] \} \) is linearly dependent.

(b) (1 point) Let \( A \) be a \( 4 \times 2 \) matrix. Suppose that \( [1] \) is a solution to the system \( A\mathbf{x} = 0 \) and \( [2] \) is a solution to the system \( A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \). Write down a solution to the system \( A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \) that is not equal to \( [2] \).

(c) (1 point) True/False: Let \( A \) be a square matrix. If \( A^2 \) is invertible, then \( A \) is invertible.

(d) (1 point) True/False: Let \( A \) be a square matrix. If \( A \) is invertible, then \( A^2 \) is invertible.

(e) (1 point) True/False: If the columns of a \( 4 \times 7 \) matrix \( A \) span \( \mathbb{R}^4 \), then \( A \) has 7 pivots.

(f) (1 point) True/False: If \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^3 \), then the set \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \} \) is always linearly dependent.

(g) (1 point) True/False: Let \( A \) be a square \( n \times n \) matrix. If the system \( A\mathbf{x} = 0 \) has infinitely many solutions, then the system \( A\mathbf{x} = \mathbf{b} \) has at least one solution for every \( \mathbf{b} \in \mathbb{R}^n \).

(h) (1 point) True/False: Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^4 \) be a linear transformation. The set

\[
\left\{ T \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right\}
\]

is linearly dependent.
5. TRUE/FALSE. Determine whether the following statements are true or false, and give justification.

(a) (3 points) If \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \) is a linearly dependent set then \( \mathbf{v}_3 \) can be written as a linear combination of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).

(b) (3 points) Let \( \mathbf{A} \) be a \( 2 \times 4 \) matrix. If the system \( \mathbf{A} \mathbf{x} = \mathbf{0} \) has infinitely many solutions, then the system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has at least one solution for every \( \mathbf{b} \in \mathbb{R}^2 \).

(c) (3 points) Suppose \( \mathbf{A} \) is a \( 3 \times 4 \) matrix whose columns span \( \mathbb{R}^3 \). There exists a \( 4 \times 3 \) matrix \( \mathbf{B} \) such that \( \mathbf{A} \mathbf{B} = \mathbf{I}_3 \).

(d) (3 points) Let \( \mathbf{A}, \mathbf{B} \) be \( 3 \times 3 \) matrices. Suppose that \( \mathbf{A} \) is invertible and \( \mathbf{A} \) and \( \mathbf{B} \) have different RREFs. Then \( \mathbf{B} \) is not invertible.
6. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that:

$$T \left( \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad T \left( \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) (3 points) Prove that $T \left( \begin{bmatrix} 6 \\ 4 \\ 7 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) (3 points) Is $T$ onto? Justify your answer.