1. (a) (5 points) Determine the coefficient of $x_1x_2 \cdots x_n$ in $(x_1 + x_2 + \cdots + x_n)^n$ (this is a polynomial in the $n$ variables $x_1, x_2, \ldots, x_n$).

**Solution:** Expanding out this product, there are $n^n$ terms in bijection with strings of length $n$ in the alphabet $[n]$. The terms which yield $x_1x_2 \cdots x_n$ are the ones which are permutations of $12 \cdots n$, and there are $n!$ such strings.

(b) (5 points) For positive integers $m$ and $n$, determine the number of functions $f : [m] \to \{1, 2\}$ such that $\sum_{i=1}^{m} f(i) = n$.

**Solution:** Given a fixed $n$, we know $n = \sum_{i=1}^{m} f(i) = a + 2b$ where $a = |\{i \in [m] \mid f(i) = 1\}|$, $b = |\{i \in [m] \mid f(i) = 2\}|$.

Since $a + b = m$, we can combine these two facts to see that $b = n - m$, $a = 2m - n$.

Thus, determining a function $f : [m] \to \{1, 2\}$ where $\sum_{i=1}^{m} f(i) = n$ is equivalent to selecting $n - m$ entries of $[m]$ to map to 2, which is counted by $\binom{m}{n-m}$.

(c) (5 points) Prove that the number of honeybees in generation $n$ of a male honeybee is $F_n$.

**Solution:** Let $A_n$ be the number of male honeybees in generation $n$ and $B_n$ be the number of female honeybees in generation $n$ (start with a single male honeybee in generation 1). By construction of the tree, $B_i = A_{i-1} + B_{i-1}$ and $A_i = B_{i-1}$. We prove by induction on $n$ that $B_n = F_{n-1}$ and $A_n = F_{n-2}$ for $n \geq 2$.

The base case is that $B_2 = 1 = F_1$ and $A_2 = 0 = F_0$. We assume the inductive hypothesis, that the statement is true for $n = i$: $B_i = F_{i-1}$ and $A_i = F_{i-2}$. Observe that $B_{i+1} = A_{i+1-1} + B_{i+1-1} = A_i + B_i = F_{i-1} + F_{i-2} = F_i$, and $A_{i+1} = B_{i+1-1} = B_i = F_{i-1}$. Thus both claims hold for $i + 1$, which completes the inductive proof.