Project Summary: quantizing Schur functors

Geometric complexity theory (GCT) is an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. A fundamental problem in representation theory, believed to be important for this approach, is the *Kronecker problem*, which asks for a positive combinatorial formula for the multiplicity $g_{\lambda\mu\nu}$ of an irreducible representation M_ν of the symmetric group in the tensor product $M_\lambda \otimes M_\mu$. Blasiak, Mulmuley, and Sohoni have been developing an approach to this problem using quantum groups and canonical bases. The main focus of this proposal is to push this approach further.

Canonical bases were first defined by Kazhdan and Lusztig through their study of singularities of Schubert varieties. Of particular interest for this project is their remarkable ability to connect combinatorics and representation theory. For instance, canonical bases beautifully connect the RSK correspondence with quantum Schur-Weyl duality, give a Littlewood-Richardson rule for all types, and have been used by Blasiak to explain the appearance of the combinatorial operations cyclage and catabolism in the graded characters of Garsia-Procesi modules.

The *nonstandard quantum group* and *nonstandard Hecke algebra* were defined by Mulmuley and Sohoni to study the Kronecker problem. Recently, Mulmuley, Sohoni, and Blasiak have obtained the beginnings of a theory of canonical bases for these nonstandard objects. They construct a canonical basis of a certain representation of the nonstandard quantum group and use this to solve the Kronecker problem in the case of two two-row shapes. The $U_q(\mathfrak{sl}_2)$-graphical calculus is used to organize and count the resulting crystal components, which is believed to be the beginnings of a new kind of graphical calculus in this setting. Blasiak, together with Mulmuley, Sohoni, and others, plans to push this approach further, in the following ways.

- Determine the irreducible (co)modules of the nonstandard quantum group and Hecke algebra.
- Adapt, as extensively as possible, constructions from multilinear algebra like Schur functors and Schur complexes to the nonstandard setting.
- Further develop the theory of canonical bases for nonstandard objects. This will involve creating a new graphical calculus to describe the basis in the case of two two-row shapes.
- Using the new canonical basis theory, solve the Kronecker problem in the case of one two-column shape and the case of one hook shape.
- Building off of work of Berenstein, Zwicknagl, and Mulmuley, further develop a similar quantum groups approach to a special case of the plethysm problem.
- Connect the basis-theoretic approach to the Kronecker problem to the geometry of the $GL_a \times GL_b \times GL_c$-variety $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, with the ultimate goal of better understanding the *tensor decomposition problem*, which asks for a decomposition of a tensor into a sum of as few as possible simple tensors.

Intellectual Merit. This project is an approach to some of the deepest and most important problems in algebraic combinatorics and complexity theory. Its immediate goals are to solve two special cases of the Kronecker problem that would significantly extend the current state of the art, to quantize Schur functors, and develop a canonical basis theory for new quantum objects.

Broader Impacts. This interdisciplinary project will draw on the expertise of researchers across many different fields and cultural backgrounds, and will involve teaching and training of undergraduate and graduate students. It also has the potential to say something important about the tensor decomposition problem, which has applications in medicine, signal processing, chemistry, computer vision, blind source separation, independent component analysis, and fast matrix multiplication.

Project Description: quantizing Schur functors

1.1. **Introduction.** Geometric complexity theory (GCT) is an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. It originated in the 1999 paper [39], in which Mulmuley successfully applied algebraic geometry to prove a weakened version of the $\mathbf{NC} \neq \mathbf{P}$ conjecture. It has since been developed primarily by Mulmuley and Sohoni. In the last several years, GCT has attracted attention from researchers in algebraic geometry, complexity theory, and algebraic combinatorics including Landsberg, Weyman, Kumar, Bürgisser, and myself.

The main focus of GCT so far has been on the permanent vs. determinant problem, which is to show that the permanent of an $n \times n$ variable matrix cannot be expressed as the determinant of an $m \times m$ matrix with constant or single variable entries if $m = O(2^{\text{polylog}\, n})$. A fundamental problem in representation theory arising here is the *Kronecker problem*, which asks for a positive combinatorial formula for the multiplicity $g_{\lambda\mu\nu}$ of an irreducible representation M_ν of the symmetric group $\mathcal{S}_r$ in the tensor product $M_\lambda \otimes M_\mu$.

In [46], Mulmuley and Sohoni define the *nonstandard quantum group* and *nonstandard Hecke algebra* to study the Kronecker problem. Recently, Mulmuley, Sohoni, and I have been advancing this approach by working out the representation theory of and developing a canonical basis theory for these nonstandard objects [4, 5, 8] (see §3). This has resulted in a crystal basis-theoretic solution to the Kronecker problem in the case of two two-row shapes [4]. The main focus of this proposal is to push this approach further. The following specific projects address this, as well as related connections between algebraic combinatorics, complexity theory, and geometry.

(§4) Further uncover the potentially deep connections between complexity theory and positivity in algebraic combinatorics as has been initiated in the GCT paper [42].

(§5) Connect our basis-theoretic approach to the Kronecker problem to the geometry of the $GL_a \times GL_b \times GL_c$-variety $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, with the ultimate goal of better understanding tensor rank.

(§6) Building off of work of Berenstein, Zwicknagl, and Mulmuley, further develop a quantum groups approach to a special case of the plethysm problem.

(§7) Solve a conjecture about canonical basis lengths that arose in [4].

(§8) Further develop the theory of canonical bases for nonstandard objects. In particular, develop a new graphical calculus to describe a canonical basis for the Kronecker problem in the case of two two-row shapes.

(§9) Adapt constructions like Schur functors and Schur complexes to nonstandard objects, and use this to solve the Kronecker problem in the case of one two-column shape and the case of one hook shape.

Since our previous and future research involve common background and notation that is not widely known, we provide this in §2. We then describe our previous research in §3 and future directions in §4–9.

1.2. **Broader Impacts.** This project depends crucially on the interaction of researchers from algebraic combinatorics, complexity theory, and algebraic geometry. One goal of this project is to facilitate communication between researchers in these areas. Toward this end, I have participated in the GCT workshop at Princeton (July 2010) and the ICERM conference on Mathematical Aspects of **P** vs. **NP** and its Variants (August 2011). I have shared the problems and motivation from GCT with my colleagues in algebraic combinatorics at U. Michigan, and I have and will continue to offer my expertise and that of my colleagues to problems arising in GCT.

I am collaborating with researchers across many fields and with a variety of cultural backgrounds. My co-authors Ketan Mulmuley and Milind Sohoni are complexity theorists, both Indian, and with positions at U. Chicago and IIT Bombay, respectively. I have been in constant contact with Josh Grochow (graduate student at U. Chicago) who is quite knowledgeable in complexity theory. I have also been talking to Luke Oeding (postdoctoral fellow at UC Berkeley), J. M. Landsberg, and Jerzy Weyman about the more geometric aspects of these projects. The proposed research in §6 and §5.3 will involve my REU students and will incorporate computer exploration.

Additionally, this project has the potential to say something important about tensor rank and tensor decomposition, which have many practical applications including locating the area causing epileptic seizures in a brain, fluorescence spectroscopy, computer vision, blind source separation, and fast matrix multiplication [32].

2. Background and notation

We work over the ground rings $\mathbb{C}$, $\mathbb{Q}$, $K = \mathbb{Q}(q)$, and $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. Define K_∞ to be the subring of K consisting of rational functions with no pole at $q = \infty$. For a nonnegative integer k, the quantum integer is $[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}} \in \mathbf{A}$. The notation $[k]$ denotes the set $\{1, \ldots, k\}$.

We write $\lambda \vdash_l r$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of size $r = \sum_{i=1}^{l} \lambda_i$. The set of standard Young tableaux is denoted SYT and the subset of SYT of shape λ is denoted $\mathrm{SYT}(\lambda)$. The set of semistandard Young tableaux of shape λ and entries in $[l]$ is denoted $\mathrm{SSYT}_l(\lambda)$. The shape of a tableau T is denoted $\mathrm{sh}(T)$. We let $P(\mathbf{k}), Q(\mathbf{k})$ denote the insertion and recording tableaux produced by the RSK algorithm applied to the word $\mathbf{k}$. Let Z_λ be the superstandard SSYT of shape and content λ and let Z^*_λ be the SYT of shape λ with $1, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row, etc. The notation λ' denotes the conjugate partition of λ and Q^T denotes the transpose of an SYT Q.

We use the term *cells* in the general setting of modules with basis in the sense of [7, §2.3]. This is similar to the notion of Kazhdan-Lusztig cells for Coxeter groups [28] and cellular bases as defined by Graham-Lehrer [20]. If (M, Γ) is an H-module with basis, then we use the phrase *H-cells of* Γ to emphasize that we are considering the action of H on M (and not some other algebra).

2.1. **Canonical bases connect quantum Schur-Weyl duality and RSK.** Let V and V_q be $\mathbb{Q}$- and K-vector spaces, respectively, of dimension d_V. The canonical basis of $V_q^{\otimes r}$ nicely connects quantum Schur-Weyl duality with the RSK correspondence. This is the starting point for our basis-theoretic approach to the Kronecker problem pursued in [4].

Let $U_q(\mathfrak{g}_V)$ be the quantized enveloping algebra of $\mathfrak{g}_V := \mathfrak{gl}(V)$ over $K = \mathbb{Q}(q)$ and $\mathscr{H}_r$ the type A_{r-1} Hecke algebra over $\mathbf{A}$. We let $V_{q,\lambda}$ (resp. $M_{q,\lambda}$) denote the irreducible $U_q(\mathfrak{g}_V)$-module (resp. $K\mathscr{H}_r$-module) corresponding to $\lambda \vdash r$ and let V_λ (resp. M_λ) denote the corresponding $U(\mathfrak{g}_V)$-module (resp. $\mathbb{Q}\mathcal{S}_r$-module). Schur-Weyl duality generalizes nicely to the quantum setting:

Theorem 2.1 (Jimbo [23]). *As a $(U_q(\mathfrak{g}_V), K\mathscr{H}_r)$-bimodule, $V_q^{\otimes r}$ decomposes into irreducibles as*

$$V_q^{\otimes r} \cong \bigoplus_{\lambda \vdash_{d_V} r} V_{q,\lambda} \otimes M_{q,\lambda}. \tag{1}$$

This algebraic decomposition has a combinatorial underpinning, which is the bijection

$$[d_V]^r \cong \bigsqcup_{\lambda \vdash_{d_V} r} \mathrm{SSYT}_{d_V}(\lambda) \times \mathrm{SYT}(\lambda), \ \mathbf{k} \mapsto (P(\mathbf{k}), Q(\mathbf{k})), \tag{2}$$

given by the RSK correspondence.

Now the upper canonical basis $B_V^r := \{c_{\mathbf{k}} : \mathbf{k} \in [d_V]^r\}$ of $V_q^{\otimes r}$ can be defined by $c_{\mathbf{k}} := v_{k_1} \heartsuit \dots \heartsuit v_{k_r}$, $\mathbf{k} = k_1 \cdots k_r \in [d_V]^r$, where $v_1, \dots, v_{d_V}$ is the weight basis of V_q, and $\heartsuit$ is like the $\diamond$ of [38] for tensoring based modules, adapted to upper canonical bases, as explained in [7, 11]. The basis B_V^r has $U_q(\mathfrak{g}_V)$- and $\mathscr{H}_r$-cells corresponding to (1) and labels to (2), which nicely connects the representation theory to the combinatorics (see [7, Corollary 5.7] for a precise statement). This was first shown, in a somewhat different form, by Grojnowski and Lusztig in [21].

2.2. **Graphical calculus for $U_q(\mathfrak{sl}_2)$-modules.** We will make use of the graphical description of the basis B_V^r in the case $d_V = 2$, following [17]. Let $V_q^{i_1}, \dots, V_q^{i_l}$ be cellular subquotients of $(V_q^{\otimes i_1}, B_V^{i_1}), \dots, (V_q^{\otimes i_l}, B_V^{i_l})$ and let $\mathbf{k} = \mathbf{k^1} \cdots \mathbf{k^l}$, $\mathbf{k^j} \in [d_V]^{i_j}$. The image of the canonical basis element $c_{\mathbf{k}} = c_{\mathbf{k^1}} \heartsuit \dots \heartsuit c_{\mathbf{k^l}}$ in $V_q^{i_1} \otimes \cdots \otimes V_q^{i_l}$ is described by the *diagram* of $\mathbf{k}$, which is the picture obtained from $\mathbf{k}$ by pairing 2's and 1's as left and right parentheses and then drawing an arc between matching pairs. We also record in the diagram the lengths $i_1, \dots, i_l$. An arc is *internal* if its ends belong to the same $\mathbf{k}^j$, and is *external* otherwise.

2.3. **Nonstandard Hecke algebras.** The nonstandard Hecke algebra was introduced in [46] to study the Kronecker problem. Its role in this problem will be explained in §3.5 and §8.1, its representation theory discussed in §3.4 and §8.3, and the problem of finding a presentation for it will be addressed in §3.2 and §8.2.

The *nonstandard Hecke algebra* $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by the elements

$$\mathcal{P}_i := C'_{s_i} \otimes C'_{s_i} + C_{s_i} \otimes C_{s_i}, \; i \in [r-1], \tag{3}$$

where $C'_{s_i} = T_{s_i} + q^{-1}$ and $C_{s_i} = T_{s_i} - q$ are the simplest Kazhdan-Lusztig basis elements, which are proportional to the trivial and sign idempotents of the parabolic sub-Hecke algebra $K(\mathscr{H}_r)_{\{s_i\}}$. We think of the inclusion $\check{\Delta} : \check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ as a deformation of the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r} : \mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \otimes w$.

Set $\mathbf{A}' = \mathbf{A}[\frac{1}{[2]_q}]$. A simple but important observation ([5, Proposition 2.3]) is that $\mathbf{A}'\mathscr{H}_2 \cong \mathbf{A}'\check{\mathscr{H}}_2$ is a Hopf algebra with coproduct $\check{\Delta}$ and this is the unique way to make it a Hopf algebra such that it specializes to the Hopf algebra $\mathbb{Z}[\frac{1}{2}]\mathcal{S}_2$ at $q = 1$. For $r > 2$, $\check{\mathscr{H}}_r$ has $\mathbf{A}$-rank much larger than that of $\mathscr{H}_r$ (its rank is 10, 114 for $r = 3, 4$). But, by the above observation, $\check{\mathscr{H}}_r$ is the smallest subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ compatible with the coproduct $\Delta_{\mathbb{Z}\mathcal{S}_r}$ and parabolic sub-Hecke algebras.

We can now define $\Delta^{(k)} : \mathbf{A}'\mathscr{H}_2 \to \mathbf{A}'\mathscr{H}_2^{\otimes k}$ inductively by $\Delta^{(k)} := (\Delta^{(k-1)} \otimes 1) \circ \check{\Delta} = (1 \otimes \Delta^{(k-1)}) \circ \check{\Delta}$ for $k > 2$, and $\Delta^{(2)} := \check{\Delta}$. The *nonstandard Hecke algebra* $\check{\mathscr{H}}_r^{(k)}$ is the subalgebra of $\mathscr{H}_r^{\otimes k}$ generated by the $\mathcal{P}_i^{(k)} := [2]_q^{k-1}\Delta_i^{(k)}(C'_{s_1})$ for all $i \in [r-1]$, where $\Delta_i^{(k)} = \iota_i^{\otimes k} \circ \Delta^{(k)}$ and ι_i is the inclusion $\mathscr{H}_2 \cong (\mathscr{H}_r)_{\{s_i\}} \hookrightarrow \mathscr{H}_r$. Note that $\check{\mathscr{H}}_r^{(2)} = \check{\mathscr{H}}_r$ and $\mathcal{P}_i^{(2)} = \mathcal{P}_i$.

The K-vector space V_q has an integral form $V_{\mathbf{A}} = \mathbf{A}\{v_i : i \in [d_V]\}$ such that $V_{\mathbf{A}}^{\otimes r}$ is a right $\mathscr{H}_r$-module. This defines an $\mathbf{A}$-algebra homomorphism $\mathscr{H}_r \to \mathrm{End}_{\mathbf{A}}(V_{\mathbf{A}}^{\otimes r})$. Define the *Temperley-Lieb quotient* $\mathscr{H}_{r,d_V}$ of $\mathscr{H}_r$ to be the image of this homomorphism. The *nonstandard Temperley-Lieb quotient* $\check{\mathscr{H}}_{r,d}$ of $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_{r,d} \otimes \mathscr{H}_{r,d}$ generated by $\mathcal{P}_i := C'_{s_i} \otimes C'_{s_i} + C_{s_i} \otimes C_{s_i}$, for all $i \in [r-1]$.

3. Results From Prior NSF Support

The papers [6], [5], [7], [8], [4] acknowledge support of my Mathematical Sciences Postdoctoral Research Fellowship (DMS-0903113, $130,000, 7/1/09–8/31/11). This section summarizes the first four of these papers in §3.1, §3.2, §3.3, §3.4, respectively, and the paper [4] in §3.5–3.8.

3.1. **Cyclage, catabolism, and the affine Hecke algebra.** In [6], we identify a subalgebra $\widehat{\mathscr{H}_n}^+$ of the extended affine Hecke algebra $\widehat{\mathscr{H}_n}$ of type A. The subalgebra $\widehat{\mathscr{H}_n}^+$ is a q-analog of the monoid algebra of $\mathcal{S}_n \ltimes \mathbb{Z}^n_{\geq 0}$ and inherits a canonical basis from that of $\widehat{\mathscr{H}_n}$. We show that its left cells are naturally labeled by tableaux filled with positive integer entries having distinct residues mod n, which we term *positive affine tableaux* (PAT).

We then exhibit a cellular subquotient $\mathscr{R}_{1^n}$ of $\widehat{\mathscr{H}_n}^+$ that is a q-analog of the ring of coinvariants $\mathbb{C}[y_1, \ldots, y_n]/(e_1, \ldots, e_n)$ with left cells labeled by PAT that are essentially SYT with cocharge labels. Multiplying canonical basis elements by a certain element $\pi \in \widehat{\mathscr{H}_n}^+$ corresponds to rotations of words, and on cells corresponds to cocyclage. We further show that $\mathscr{R}_{1^n}$ has cellular quotients $\mathscr{R}_\lambda$ that are q-analogs of the Garsia-Procesi modules R_λ with left cells labeled by (a PAT version of) the λ-catabolizable tableaux.

We conjecture that $\widehat{\mathscr{H}_n}^+$ is "tiled" by dual versions of the $\mathscr{R}_\lambda$ and conjecture precisely how this would beautifully connect catabolism to the combinatorics of the cells of $\widehat{\mathscr{H}_n}$ worked out by Shi, Lusztig, and Xi. We also conjecture that the k-atoms of Lascoux, Lapointe, and Morse [35] and the R-catabolizable tableaux of Shimozono and Weyman [56] have cellular counterparts in $\widehat{\mathscr{H}_n}^+$.

3.2. **Nonstandard braid relations and Chebyshev polynomials.** In [5], we study the family of algebras $\breve{\mathscr{H}}_3^{(k)}$, defined in §2.3, in detail. We discover a remarkable connection between the defining relations of these algebras and the Chebyshev polynomials $T_j(x)$. Specifically, we show that $\breve{\mathscr{H}}_3^{(k)}$ is generated by $\mathcal{P}_1^{(k)}, \mathcal{P}_2^{(k)}$ and has a relation, which we call the nonstandard braid relation, that generalizes the braid relation for $k = 1$:

$$\mathcal{P}_1^{(k)}(\mathcal{P}_{21}^{(k)} - a_1^2)(\mathcal{P}_{21}^{(k)} - a_2^2)\ldots(\mathcal{P}_{21}^{(k)} - a_k^2) = \mathcal{P}_2^{(k)}(\mathcal{P}_{12}^{(k)} - a_1^2)(\mathcal{P}_{12}^{(k)} - a_2^2)\ldots(\mathcal{P}_{12}^{(k)} - a_k^2),$$

where $\mathcal{P}_{i_1 i_2}^{(k)} = \mathcal{P}_{i_1}^{(k)}\mathcal{P}_{i_2}^{(k)}$ and $a_j = [2]_q^k T_j(\frac{1}{[2]_q})$. We show that $\breve{\mathscr{H}}_3^{(k)}$ has a cellular basis in the sense of Graham and Lehrer [20], generalizing the Kazhdan-Lusztig basis of $\mathscr{H}_3$. We then generalize these results to nonstandard Hecke algebras of dihedral groups. In this case, the nonstandard braid relation involves a multivariate version of Chebyshev polynomials.

3.3. **Projected canonical bases.** In [7], we study transition matrices between various bases of $\mathscr{H}_r$ and bases of the $K\mathscr{H}_r$-irreducibles $M_{q,\lambda}$. This investigation arose naturally in our approach to the Kronecker problem described in §3.5–3.8. Specifically, we define the *projected upper (resp. lower) Kazhdan-Lusztig basis* of $\mathscr{H}_r$ to be $\{\tilde{C}_w\}_{w\in\mathcal{S}_r}$ (resp. $\{\tilde{C}'_w\}_{w\in\mathcal{S}_r}$), where $\tilde{C}_w := C_w p_\lambda$ (resp. $\tilde{C}'_w := C'_w p_\lambda$) and p_λ is the minimal central idempotent of $K\mathscr{H}_r$ for $\lambda = \mathrm{sh}(P(w)) = \mathrm{sh}(Q(w))$. We study the transition matrices $\tilde{T}, \tilde{T}'$ between the Kazhdan-Lusztig bases of $\mathscr{H}_r$ and their projected counterparts. We also study the transition matrix $S(\lambda)$ between the two bases of $M_{q,\lambda}$ coming from the Kazhdan-Lusztig bases $\{C_w\}$ and $\{C'_w\}$ of $\mathscr{H}_r$. We show that $\tilde{T}, \tilde{T}'$, and $S(\lambda)$ are the identity at $q = 0$ and $q = \infty$, and conjecture that the entries of $S(\lambda)$ satisfy a certain positivity property.

3.4. **Representation theory of the nonstandard Temperley-Lieb algebra.** The nonstandard Temperley-Lieb algebra $\breve{\mathscr{H}}_{r,d}$ has one-dimensional trivial and sign representations $\breve{\epsilon}_+$ and $\breve{\epsilon}_-$, which we identify with the maps $\breve{\mathscr{H}}_{r,d} \to \mathbf{A}$ given by $\breve{\epsilon}_+ : \mathcal{P}_i \mapsto [2]_q^2$, $\breve{\epsilon}_- : \mathcal{P}_i \mapsto 0$. It is easy to show that $\breve{\mathscr{H}}_{r,d} \subseteq S^2\mathscr{H}_{r,d}$, and therefore $\mathrm{Res}_{\breve{\mathscr{H}}_{r,d}} S^2 M_{q,\lambda}$ and $\mathrm{Res}_{\breve{\mathscr{H}}_{r,d}} \Lambda^2 M_{q,\lambda}$ are $\breve{\mathscr{H}}_{r,d}$-modules. Further, $K\breve{\epsilon}_+ \subseteq \mathrm{Res}_{\breve{\mathscr{H}}_{r,d}} S^2 M_{q,\lambda}$ and we define $S'\check{M}_\lambda := \mathrm{Res}_{\breve{\mathscr{H}}_{r,d}} S^2 M_{q,\lambda}/K\breve{\epsilon}_+$. The main result of [8] is

Theorem 3.1. *The algebra $K\breve{\mathscr{H}}_{r,2}$ is split semisimple and the list of distinct irreducibles is*

(1) $\mathrm{Res}_{\breve{\mathscr{H}}_{r,2}} M_{q,\lambda} \otimes M_{q,\mu}$, *for $\lambda, \mu \vdash_2 r$ such that λ is greater than μ in dominance order,*

(2) $S'\check{M}_\lambda$, *for $\lambda \vdash_2 r$, λ not a single row or column shape,*

(3) $\mathrm{Res}_{\check{\mathscr{H}}_{r,2}} \Lambda^2 M_{q,\lambda}$, *for* $\lambda \vdash_2 r$, λ *not a single row or column shape,*

(4) $K\check{\epsilon}_+$.

3.5. **The nonstandard quantum group and Hecke algebra.** In the remainder of this section, we summarize the results of [4]. Let V_q, W_q be K-vector spaces of dimensions d_V, d_W, respectively, considered as the natural representations of $U_q(\mathfrak{g}_V), U_q(\mathfrak{g}_W)$, and set $X_q = V_q \otimes W_q$. Let V, W and X be their $q = 1$ specializations. Schur-Weyl duality for $X^{\otimes r} = V^{\otimes r} \otimes W^{\otimes r}$ as a $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule and a $(U(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathbb{Q}\mathcal{S}_r \otimes \mathbb{Q}\mathcal{S}_r)$-bimodule yields

$$\bigoplus_{\nu} X_\nu \otimes M_\nu \cong X^{\otimes r} = V^{\otimes r} \otimes W^{\otimes r} \cong \bigoplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes M_\lambda \otimes M_\mu.$$

This is a convenient setup for studying the Kronecker problem because the Kronecker coefficient $g_{\lambda\mu\nu}$ can be viewed simultaneously as the multiplicity of M_ν in $M_\lambda \otimes M_\mu$ and the multiplicity of $V_\lambda \otimes W_\mu$ in the restriction $\mathrm{Res}_{U(\mathfrak{g}_V \oplus \mathfrak{g}_W)} X_\nu$ of the $U(\mathfrak{g}_X)$-irreducible X_ν.

We construct two quantum objects, the nonstandard Hecke algebra $\check{\mathscr{H}}_r$ and the nonstandard coordinate algebra $\mathscr{O}(GL_q(\check{X}))$, (we write $\check{X}$ in place of X_q when it is associated to a nonstandard object) that are compatible with the $(U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathscr{H}_r \otimes \mathscr{H}_r)$-bimodule structure on $\check{X}^{\otimes r}$ and quantize the $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule structure on $\check{X}^{\otimes r}$ in a certain sense. We show that $\mathscr{O}(GL_q(\check{X}))$ is a cosemisimple Hopf algebra, that $K\check{\mathscr{H}}_r$ is semisimple, and that a nonstandard analog of quantum Schur-Weyl duality holds.

The Hopf algebra $\mathscr{O}(GL_q(\check{X}))$ is defined by first defining the nonstandard coordinate algebra $\mathscr{O}(M_q(\check{X}))$, which is an FRT-algebra [53] that we will not define in full here. We define a nonstandard version $\check{D} \in \mathscr{O}(M_q(\check{X}))$ of the determinant; this and, more generally, the nonstandard minors are rather interesting objects, discussed further in §7. Then $\mathscr{O}(GL_q(\check{X}))$ is defined by putting a Hopf algebra structure on the bialgebra $\mathscr{O}(M_q(\check{X}))[\frac{1}{\check{D}}]$. This is more delicate than in the standard case, and in particular requires showing that the left and right nonstandard minors agree.

Much of the abstract theory of the standard quantum group can be replicated in the nonstandard case, but explicit computations are much harder. For instance, we can define nonstandard symmetric and exterior algebras $\check{S}(\check{X})$ and $\check{\Lambda}(\check{X})$, which are $\mathscr{O}(M_q(\check{X}))$-comodule algebras and specialize to the symmetric and exterior algebras of X at $q = 1$. However, $\check{S}(\check{X})$ is already isomorphic to the quantum coordinate algebra $\mathscr{O}(M_q(V))$ when $W_q = V_q^*$. The algebra $\mathscr{O}(M_q(\check{X}))$ is yet another level of difficulty beyond this—for example, a natural reduction system for $\mathscr{O}(M_q(\check{X}))$ does not satisfy the diamond property. Nonetheless, at least in the $d_V = d_W = 2$ case, nonstandard Schur-Weyl duality and Theorem 3.1 yield a complete description of the irreducible $\mathscr{O}(M_q(\check{X}))$-comodules.

3.6. **Constructing $\check{X}_\nu$ in the two-row case.** Assume for the remainder of this section that $d_V = d_W = 2$ (the two-row case). We construct an $\mathscr{O}(GL_q(\check{X}))$-comodule (therefore a $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module) $\check{X}_\nu$ that specializes to $\mathrm{Res}_{U(\mathfrak{g}_V \oplus \mathfrak{g}_W)} X_\nu$ at $q = 1$. Our construction of $\check{X}_\nu$ follows the construction of Adsul, Sohoni, and Subrahmanyam [1] of a similar quantum object for the Kronecker problem.

Although the irreducible $\mathscr{O}(GL_q(\check{X}))$-comodules are in general much smaller than those of $\mathscr{O}(GL(X))$, the nonstandard exterior algebra $\check{\Lambda}(\check{X})$ specializes to $\Lambda(X)$ at $q = 1$. Let $\check{\Lambda}^r\check{X}$ denote the degree r part of $\check{\Lambda}(\check{X})$. For a composition α, define

$$\check{Y}_\alpha := \check{\Lambda}^{\alpha_1}\check{X} \otimes \check{\Lambda}^{\alpha_2}\check{X} \otimes \ldots \otimes \check{\Lambda}^{\alpha_l}\check{X}. \quad (4)$$

For each partition ν with $\ell(\nu) \leq \dim(X) = 4$ and $l = \ell(\nu') = 2$, we define a submodule $\check{Y}_{\rhd\nu'}$ of $\check{Y}_{\nu'}$ by explicitly writing down a basis. Then for $\nu' \vdash_l r$, $l > 2$, $\check{X}_\nu$ is defined to be the quotient of $Y_{\nu'}$ by the (generally, not direct) sum over all $i \in [l-1]$ of

$$\check{Y}_{\triangleright^i \nu'} := \check{Y}_{(\nu'_1,\dots,\nu'_{i-1})} \otimes \check{Y}_{\triangleright(\nu'_i,\nu'_{i+1})} \otimes \check{Y}_{(\nu'_{i+2},\dots,\nu'_l)}. \tag{5}$$

3.7. **A global crystal basis for $\check{X}_\nu$.** We now explain the deepest results in crystal basis theory obtained so far in this approach. To define a global crystal basis of $\check{X}_\nu$, we first define a global crystal basis of $\check{\Lambda}^r \check{X}$, whose elements are labeled by what we call *nonstandard columns of height* r (NSC^r). (There are $\binom{4}{r}$ nonstandard columns of height r, several of which appear in Figure 1.) We then define a canonical basis of $\check{Y}_\alpha$ by putting the bases NSC$^{\alpha_i}$ together using Lusztig's construction for tensoring based modules [38, Theorem 27.3.2]. This basis is labeled by *nonstandard tabloids* (NST), which are just sequences of nonstandard columns. We identify a certain subset of (a rescaled version of) NST(ν') such that its image HNSTC(ν) in $\check{X}_\nu$ is a basis (HNSTC stands for *honest nonstandard tabloid class*).

Theorem 3.2 ([4])**.** *The set HNSTC(ν) is a global crystal basis of $\check{X}_\nu$ that solves the two-row Kronecker problem: the number of highest weight elements of HNSTC(ν) of weight (λ,μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$.*

The *symmetric* (resp. *exterior*) *Kronecker coefficient* $g_{+1\lambda\nu}$ (resp. $g_{-1\lambda\nu}$) is the multiplicity of M_ν in $S^2 M_\lambda$ (resp. $\Lambda^2 M_\lambda$). Our approach also yields formulas (Theorem 3.3, below) for these coefficients since any $\mathscr{O}(GL_q(\check{X}))$-comodule is a $U_q^\tau := U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W) \rtimes \mathcal{S}_2$ module.

The deepest part of this theorem from the point of view of crystal basis theory is the rescaling of the NST, which is essentially forced on us (see §8.1 for more about this). Each NST T has a V-column (resp. W-column) reading word and we associate a diagram to T corresponding to the diagrams of these reading words as explained in §2.2, with i_j equal to the height of the j-th column; arcs are decorated by V and W. The *degree* $\deg(T)$ of an NST T is the number of pairs of columns that are paired by two (external) arcs. The rescaled version of T is then $(-\frac{1}{[2]_q})^{\deg(T)}T$.

3.8. **Explicit formulas for two-row Kronecker coefficients.** We go on to describe the crystal components of $(\check{X}_\nu, \text{HNSTC}(\nu))$ in terms of diagrams of NST in a way that is independent of the HNSTC in the component and the rescaled NST representing the HNSTC. This helps us organize and count crystal components. We show that the degree 0 crystal components (degree for NST gives rise to a well-defined notion of degree for crystal components) can be grouped into eight different one-parameter families depending on the heights of the columns that the arcs connect (see Figure 1), and counting crystal components easily reduces to the degree 0 case.

Formulas for two-row Kronecker coefficients are given in [52, 54, 9]. Although these are quite explicit, none is obviously positive. Theorem 3.2 actually produces fairly simple, positive formulas for two-row Kronecker coefficients. For example, define the *symmetric* (resp. *exterior*) *Kronecker generating function*

$$g_{\varepsilon\nu}(x) := \sum_{\lambda \vdash_2 r} g_{\varepsilon\lambda\nu} x^{\lambda_1 - \lambda_2}, \quad \varepsilon = +1 \text{ (resp. } \varepsilon = -1).$$

Here ν is any partition of r with at most 4 parts. For $k \in \mathbb{Z}$, define $[\![k]\!] = x^k + x^{k-2} + \cdots + x^{k'}$, where k' is 0 (resp. 1) if k is even (resp. odd).

Theorem 3.3 ([4])**.** *The symmetric and exterior Kronecker generating functions are given by*

$$g_{\varepsilon\nu}(x) = \begin{cases} [\![n_1]\!][\![n_2]\!][\![n_3]\!] & \text{if} \quad (-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1 - 1]\!][\![n_2 - 1]\!][\![n_3]\!]x & \text{if} \ -(-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1]\!][\![n_2 - 1]\!][\![n_3 - 1]\!]x & \text{if} \ -(-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1 - 1]\!][\![n_2 - 2]\!][\![n_3 - 1]\!]x^2 & \text{if} \quad (-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \end{cases}$$

where n_i is the number of columns of height i in the diagram of ν.

We also easily recover the nice formulas for certain two-row Kronecker coefficients from [10] as well as the exact conditions for two-row Kronecker coefficients to vanish, from [9].

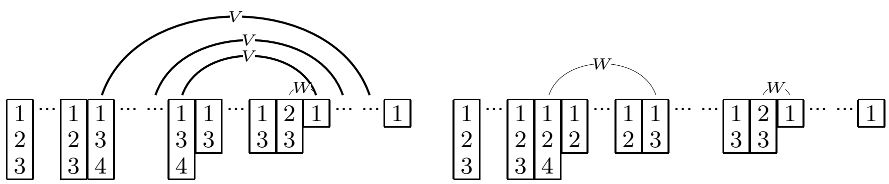


Figure 1: Representatives of two of the eight one-parameter families of degree 0 highest weight HNSTC. Two sets of dots indicates that there is at least one column of that type. The V (resp. W) reading word is obtained by reading columns from bottom to top and then left to right, and then applying the map $1 \mapsto 1$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 2$ (resp. $1 \mapsto 1$, $2 \mapsto 2$, $3 \mapsto 1$, $4 \mapsto 2$).

4. Complexity theory and positivity

Complexity theory provides a new and intriguing motivation for positivity problems in algebraic combinatorics. A positive combinatorial interpretation of some quantity that is already known to be nonnegative is not *just* a more elementary proof of nonnegativity. It is also at the heart of efficient computation, though precisely how is somewhat subtle.

Computing Littlewood-Richardson coefficients is known to be #**P**-complete [12]. The class #**P** is a counting version of **NP**, so being complete for #**P** means that computing these coefficients is expected to be hard. However, deciding nonvanishing of Littlewood-Richardson coefficients can be done in polynomial time [45] by the saturation theorem of Knutson and Tao [30]. This result, together with representation theoretic questions arising in the permanent vs. determinant problem and the Flip Theorem [41], is the basis for a philosophy in GCT called the *flip* [44, 41, 42]. The flip suggests that separating **P** from **NP** will require solving a problem in representation theory in polynomial time, of the same flavor as deciding nonvanishing of Littlewood-Richardson coefficients, but much harder. As part of the flip, Mulmuley hypothesizes

Hypothesis 4.1 (Kronecker Flip Hypothesis (Kronecker FH) [42])**.**

(1) *There is a #**P**-formula for the Kronecker coefficients $g_{\lambda\mu\nu}$.*

(2) *There is a polynomial time algorithm to decide nonvanishing of Kronecker coefficients.*

The hypothesis (1) is a precise complexity-theoretic version of the problem of finding a positive combinatorial formula for Kronecker coefficients. Computing Kronecker coefficients is known to be in Gap**P**, meaning that Kronecker coefficients can be expressed as the difference of two #**P** quantities. The complexity classes Gap**P** and #**P** are both counting analogs of the class **NP**, but the distinction becomes important in light of Kronecker FH (2). It is expected that (1) is easier than (2) and needs to be solved first. Roughly, this is because if a quantity is only known as the difference of two quantities that are hard to compute, then deciding if their difference vanishes is as hard as computing each one. (A precise result along these lines states that Gap**P** = #**P** implies a collapsing of complexity classes that is not expected to occur [49].)

Much of this proposal concerns an approach to Kronecker FH (1). Hypothesis (2) and many of the related hypotheses in [42] are outside the scope of the next few years, however we believe them to be important guides to uncovering a deep connection between complexity theory and algebraic combinatorics. There are some more tractable problems here as well:

Problem 4.2. (1) *Can nonvanishing of the Littlewood-Richardson coefficients $C^{\nu}_{\lambda\,\mu}$ in other types be decided in polynomial time? In [43], it is noted that this would follow from a conjecture* *of De Loera and McAllister stating that the associated stretching quasi-polynomials $C(n) := C^{n\nu}_{n\lambda\, n\mu}$ have nonnegative coefficients.*

(2) *Let G be a connected complex linear algebraic group acting irreducibly on a vector space Y with finitely many orbits. (Such pairs are classified in [24].) Determine the positivity indices of the stretching quasi-polynomials associated to the G-module $\mathbb{C}[Y]$, where the* positivity index *of a quasi-polynomial $f(n)$ is defined to be the smallest integer i such that $f(n+i)$ has nonnegative coefficients [42].*

5. Tensor rank

Given a tensor $x \in U^* \otimes V^* \otimes W^*$, where U, V, W are vector spaces over $\mathbb{C}$, say, the *tensor rank* of x is the minimal k such that $x = \sum_{i=1}^{k} u^i \otimes v^i \otimes w^i$, $u^i \in U^*, v^i \in V^*, w^i \in W^*$. The *border rank* of x is the minimal k such that x is the limit of tensors of tensor rank at most k. Computing tensor and border rank are important problems in statistics and signal processing and have many practical applications (see §1.2). Border rank has also been well studied from the algebro-geometric perspective [34, 33, 32]. From this perspective, the projective space corresponding to the set of tensors with border rank at most k is known as the k-th secant variety $\sigma_k(\mathrm{Seg})$ of the triple Segre variety $\mathrm{Seg} \subseteq \mathbb{P}(U^* \otimes V^* \otimes W^*)$. The goal is to find equations for $\sigma_k(\mathrm{Seg})$.

This is related to the Kronecker problem because the coordinate ring R of the $G := GL(U) \times GL(V) \times GL(W)$-variety $U^* \otimes V^* \otimes W^*$ is (where $X := V \otimes W$)

$$R := \bigoplus_{r \geq 0} S^r(U \otimes V \otimes W) \cong \bigoplus_{\nu} U_\nu \otimes X_\nu \cong \bigoplus_{\lambda,\mu,\nu} U_\nu \otimes (V_\lambda \otimes W_\mu)^{\oplus g_{\lambda\mu\nu}}.$$

Inspired by our successful application of canonical bases to understand the ring $\mathbb{C}[y_1, \ldots, y_n]$ as an $\mathcal{S}_n$-module (see §3.1), we are hopeful that our basis-theoretic approach to the Kronecker problem will help determine G-equivariant equations for the secant varieties $\sigma_k(\mathrm{Seg})$.

5.1. **Covariants in the two-row case.** In the $\dim(V) = \dim(W) = 2$, $\dim(U) = 4$ case, the global crystal basis $\mathrm{HNSTC}(\nu)$ of $\check{X}_\nu$ (see §3.7) can be used to construct a $\mathbb{C}$-basis B for the coordinate ring R (this requires specializing $q = 1$ and choosing bases of the U_ν). Initial computations suggest that B is compatible with the ring structure of R in the sense that each homogeneous prime G-ideal is spanned by a subset of B. We expect this will not be hard to prove. In this case, there are 9 (nonirrelevant) homogeneous prime G-ideals corresponding to the 9 G-orbit closures of $\mathbb{P}(U^* \otimes V^* \otimes W^*)$ (see, e.g., [32, 51]).

Even though the homogeneous prime G-ideals are well understood in this case, we expect that the basis B will lead to a more detailed understanding of R than what was previously known. Specifically, we are close to obtaining a description of the subring of *covariants* of R, i.e. the highest weight spaces of R. Let $\mathbf{T}_1, \ldots, \mathbf{T}_{14}$ be the HNSTC corresponding to the following NST:

$$\boxed{1},\ \begin{array}{|c|c|}\hline 4&1\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\ 2\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\ 3\\\hline\end{array},\ \begin{array}{|c|}\hline 1\\ 2\\ 3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&1\\\cline{2-2} 3&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&1\\ 4&3\\\hline\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\cline{2-2} 2&\multicolumn{1}{|c}{}\\ 4&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\\cline{2-2} 3&\multicolumn{1}{|c}{}\\ 4&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 1&1\\ 2&3\\\cline{2-2} 4&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array},\ \begin{array}{|c|c|}\hline 2&1\\ 3&2\\ 4&3\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&1\\\cline{3-3} 2&3&\multicolumn{1}{|c}{}\\\cline{2-2} 4&\multicolumn{1}{|c}{}&\\\cline{1-1}\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&1\\\cline{3-3} 3&3&\multicolumn{1}{|c}{}\\\cline{2-2} 4&\multicolumn{1}{|c}{}&\\\cline{1-1}\end{array},\ \begin{array}{|c|}\hline 1\\ 2\\ 3\\ 4\\\hline\end{array}.$$

Corollary 5.1. *The subring of covariants of $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4]$ is a quotient of $\mathbb{C}[f_1, \ldots, f_{14}]$, where $f_i = u_i \otimes \mathbf{T}_i|_{q=1}$ and u_i is the highest weight element of U_ν for ν equal to the shape of $\mathbf{T}_i$.*

This is a consequence of the Kronecker graphical calculus (§3.8). Moreover, we expect to be able to calculate the relations among the f_i using this calculus and other results of [4].

5.2. **Beyond the two-row case.** Given the difficulty of extending our basis-theoretic approach to the Kronecker problem [4] beyond the two-row case, we do not expect it to tell us anything new about the geometry of $U^* \otimes V^* \otimes W^*$ anytime soon. However, connecting our approach to geometry may prove fruitful in more immediate ways. For instance, it suggests

Problem 5.2. *Let R_q be the $U_q(\mathfrak{g}_U \oplus \mathfrak{g}_V \oplus \mathfrak{g}_W)$-module $\bigoplus_{\lambda,\mu,\nu}(U_{q,\nu} \otimes V_{q,\lambda} \otimes W_{q,\mu})^{\oplus g_{\lambda\mu\nu}}$. Define a ring structure on R_q so that (for a suitable integral form) $R_q|_{q=1} \cong R$ as rings. Can this be done so that, additionally, there are two-sided ideals $I_{q,k}$ of R_q that specialize to $I(\sigma_k(Seg))$ at $q = 1$?*

This would also give a natural requirement to impose on a K-basis B_q of R_q: each $I_{q,k}$ is spanned by a subset of B_q. The ideas in §9.4 may help solve this problem.

5.3. **Upper bounds on the exponent of matrix multiplication.** My REU student Zeyin Zhang, Josh Grochow (graduate student at U. Chicago), and I also plan to look at the problem of determining border rank from another direction.

The *exponent of matrix multiplication* is the smallest real number ω such that the border rank of the matrix multiplication tensor $M_{n,n,n}$ is $O(n^{\omega+\varepsilon})$ for all $\varepsilon > 0$ [32]. We will investigate obtaining upper bounds on ω using the group-theoretic approach proposed by Cohn and Umans in [15]. In [14], the authors make the following conjecture that would imply $\omega = 2$. A *local strong USP* of width k is a subset $U \subseteq \{1, 2, 3\}^k$ such that for each ordered triple $(u, v, w) \in U^3$, with u, v, and w not all equal, there exists $i \in [k]$ such that (u_i, v_i, w_i) is an element of

$$\{(1, 2, 1), (1, 2, 2), (1, 1, 3), (1, 3, 3), (2, 2, 3), (3, 2, 3)\}.$$

Conjecture 5.3. *The largest constant c such that there exists a local strong USP of size $(c - o(1))^k$ and width k for infinitely many k is $\frac{3}{2^{2/3}}$.*

There are several ideas we plan to test, largely through computer exploration, to construct large local strong USP's. Another idea is to extend the group-theoretic approach to use semisimple algebras other than group algebras. To make this work, some analog of the triple product property (defined in [15]) is needed. The Temperley-Lieb algebra $\mathscr{H}_{r,2}$ (defined in §2.3) has a nice basis that may allow for such an analog, which makes it a particularly good candidate for this approach.

6. Symmetric plethysm

My REU student Howard Xu, John Stembridge, and I will explore an approach to the following special case of the plethysm problem using quantum groups and canonical bases. Such an approach has been pursued by Berenstein, Zwicknagl, Mulmuley, and others [3, 61, 40, 37] (see §6.1, below).

Problem 6.1 (Symmetric plethysm problem)**.** *Find a positive combinatorial formula for the multiplicity $p^{\mu}_{r\lambda}$ of the $GL(V)$-irreducible V_μ in the $GL(V)$ representation $S^r V_\lambda$.*

This is closely related to the problem of computing generators and relations for the subring of covariants of $\mathbb{C}[V_\lambda^*] = \bigoplus_{r \geq 0} S^r V_\lambda$, an intensively studied problem from nineteenth-century invariant theory (see, e.g., [19, 50]). The case $d_V = 2$, $\lambda = (l, 0)$, is the study of covariants of the binary form of degree l. Even this case quite difficult. For example, the size of a minimal generating set for the ring of covariants is 1, 2, 4, 5, 23, 26, 147, 69 for $l = 1, \dots, 8$, and is not known beyond this [48].

The symmetric plethysm problem, which is likely easier than computing a minimal generating set for covariants, is also unsolved, even in the $d_V = 2$ case. In the $d_V = 2$ case, the plethysm coefficients $p^{\mu}_{r\lambda}$ are given by

$$p^{\mu}_{r(l,0)} = P^{\mu_2}_{r,l} - P^{\mu_2 - 1}_{r,l}, \quad \text{where } P^m_{r,l} := \big|\{\nu \vdash m : \nu \text{ fits in an } r \times l \text{ rectangle}\}\big|.$$

This is a particularly good example of how a simple combinatorial formula for a quantity involving signs can be extremely difficult to convert into a positive formula.

6.1. **Quantizing symmetric powers.** There are quantum versions of the symmetric and exterior algebras of the natural representation V_q of $U_q(\mathfrak{g}_V)$. However, there is no quantum analog of the symmetric or exterior algebra known for a general $U_q(\mathfrak{g}_V)$-module X_q. Despite this, Berenstein, Zwicknagl, and Mulmuley [3, 40] suggest a possible quantum groups approach to the symmetric plethysm problem. Berenstein and Zwicknagl [3] define the *braided exterior square* $\Lambda^2_\sigma X_q$ to be the span of the eigenspaces of the braiding endomorphism $\mathcal{R}_{X_q,X_q}$ for eigenvalues of the form $-q^i$. They then define the *braided symmetric algebra* $S_\sigma(X_q) = \bigoplus_{r\geq 0} S_\sigma(X_q)_r$ to be the quotient of the tensor algebra $T(X_q)$ by the ideal generated by $\Lambda^2_\sigma X_q$. The braided symmetric square $S^2_\sigma X_q$ and braided exterior algebra $\Lambda_\sigma(X_q)$ are defined similarly. The difficulty with this approach is that $S_\sigma(X_q)_r$ typically has dimension smaller than that of $S^r X$ (where $X = X_q|_{q=1}$), a fact first noticed by Rossi-Doria, who showed [55] that the symmetric algebra of $V_{(3,0)}$ admits no flat quantum deformation.

Zwicknagl gives a condition [60] for $S_\sigma(X_q)$ to be a flat deformation of $\mathrm{Sym}(X)$ in terms of Poisson structures on $\mathrm{Sym}(X)$. He also determines [61] the decomposition of $S_\sigma(V_{q,\lambda})_r$ into irreducibles in the $d_V = 2$ case. Given these results and connections between braided symmetric algebras and geometric crystals, cluster algebras, and non-commutative geometry [3, 60, 59, 61], it is evident that a rich theory is emerging, but it is not yet clear from this work how it will help solve the plethysm problem. Mulmuley provides a remarkable answer to this in [40]. Unfortunately, this paper remains in preprint form and may not be known to Berenstein, Zwicknagl, and others working on this approach. An important part of this project will be to collaborate with Berenstein and Zwicknagl and combine our different perspectives.

The precise (conjectural) connection between the braided symmetric algebra and the plethysm $S^r X$ for $X = V_{(l,0)}$ comes from another kind of nonstandard quantum group $\mathscr{O}(M'_q(\check{X}))$ and nonstandard Hecke algebra $\check{\mathscr{H}}'_{r,l}$ [40]. Set $X_q := V_{q,(l,0)}$. This is an $\mathscr{O}(M'_q(\check{X}))$-comodule, and $X_q^{\otimes r}$ is a $\mathscr{O}(M'_q(\check{X}))$-comodule and $\check{\mathscr{H}}'_{r,l}$-module. The algebra $\check{\mathscr{H}}'_{r,l}$ has the same flavor as $\check{\mathscr{H}}_r$ but is more difficult. The $l = 3$ case is worked out in detail in [40], where it is shown that $\check{\mathscr{H}}'_{3,3}$ is 21-dimensional and has one one-dimensional irreducible and five two-dimensional irreducibles. To compare, $\check{\mathscr{H}}_3$ is 10-dimensional and has two one-dimensional irreducibles $\check{\epsilon}_\pm$ and two two-dimensional irreducibles. The following conjecture is based on results from [40, 61] and recent computer calculations.

Conjecture 6.2. *For a suitable extension K' of K, $K'\check{\mathscr{H}}'_{3,l}$ is split semisimple and has two (resp. one) one-dimensional irreducible(s) $\check{\epsilon}_+, \check{\epsilon}_-$ (resp. $\check{\epsilon}_+$) if l is even (resp. odd) and all its other irreducibles are two-dimensional. At $q = 1$, (a suitable integral form of) each two-dimensional irreducible specializes to either (a) $M_{(2,1)}$, or (b) a module with composition factors $M_{(3)}$ and $M_{(1,1,1)}$. Further, an analog of Schur-Weyl duality holds for the $(\mathscr{O}(M'_q(\check{X}))^*_3, \check{\mathscr{H}}'_{3,l})$-bimodule $X_q^{\otimes 3}$.*

This would imply that $S_\sigma(X_q)_3 \oplus \bigoplus_i \check{\mathcal{X}}_b^{(i)}$ is a q-analog of $S^3 X$, where the $\check{\mathcal{X}}_b^{(i)}$ are the irreducible $\mathscr{O}(M'_q(\check{X}))$-comodules corresponding to the $\check{\mathscr{H}}'_{3,l}$-irreducibles of type (b). Our computations suggest that the $\check{\mathcal{X}}_b^{(i)}$ are also irreducible as $U_q(\mathfrak{g}_V)$-modules (any $\mathscr{O}(M'_q(\check{X}))$-comodule is a $U_q(\mathfrak{g}_V)$-module). Since $S_\sigma(X_q)_3$ is multiplicity-free [3], this would, quite remarkably, yield a basis (unique up to a diagonal transformation) for the highest weight spaces of $S^3 X = S^3 V_{(l,0)}$. To extend this beyond $r = 3$, we intend to use techniques from [4] to construct a canonical basis for part of $X_q^{\otimes r}$ that avoids a complete understanding of $\mathscr{O}(M'_q(\check{X}))$ and $\check{\mathscr{H}}'_{r,l}$.

6.2. **A crystal basis for $S^2_\sigma V_{q,\lambda}$.** Another important part of this approach will be to construct a global crystal basis for $S^2_\sigma V_{q,\lambda}$ (for arbitrary d_V) and work out the resulting combinatorial formula for the plethysm coefficients $p^\mu_{2\lambda}$. We expect that this can be done using the crystal commutor of Henriques and Kamnitzer [22] and results of Kamnitzer-Tingley [25]. More precisely, this will require [25, Theorem 9.2], which relates Drinfeld's commutor to the crystal commutor with modified signs. Additionally, a formula for the plethysm coefficients $p^{\mu}_{2\lambda}$ is known in terms of domino tableaux [13], and it would be interesting to compare this to the formula resulting from this approach.

7. Canonical basis lengths

The highest weight element of weight (λ, λ') of the global crystal basis NSC^r of $\check{\Lambda}^r \check{X}$ (see §3.7) is

$$\mathrm{NSC}_{Z_\lambda, Z_{\lambda'}} := \sum_{Q \in SYT(\lambda)} (-1)^{\ell(Q^T)} c^V_{RSK^{-1}(Z_\lambda, Q)} \otimes c^W_{RSK^{-1}(Z_{\lambda'}, Q^T)} \in \check{\Lambda}^r \check{X} \subseteq V_q^{\otimes r} \otimes W_q^{\otimes r},$$

where $c^V_{\mathbf{k}} \in B^r_V, c^W_{\mathbf{l}} \in B^r_W$ are as in §2.1, and $\ell(P)$ denotes the distance between P and Z^*_λ in the dual Knuth equivalence graph on $\mathrm{SYT}(\lambda)$ (for a definition of this graph, see [2]).

The *length squared* $\|x\|^2$ of $x \in X_q^{\otimes r}$ (resp. $x \in V_q^{\otimes r}$) is defined to be $\langle x, x\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the symmetric bilinear form in which the monomial basis of $X_q^{\otimes r}$ (resp. $V_q^{\otimes r}$) is orthonormal. A problem arising naturally in the study of the nonstandard minors of $\mathscr{O}(M_q(\check{X}))$ is to compute the lengths of the elements of NSC^r. For $\lambda = (r)$, it follows from well-known facts about Kazhdan-Lusztig polynomials that $\left\|\mathrm{NSC}_{Z_\lambda, Z_{\lambda'}}\right\|^2 = \|c^W_{r\, r-1 \cdots 1}\|^2 = [r]_-!$, where $[k]_-$ is the q-analog $q^{-k+1}[k]_q = 1 + q^{-2} + \cdots + q^{-2k+2}$ of k. We conjecture the following generalization:

Conjecture 7.1 ([4]). *The length squares of the highest weight elements of NSC^r are given by the following q-analogs of $r!$:*

$$\left\|NSC_{Z_\lambda, Z_{\lambda'}}\right\|^2 = |SYT(\lambda)| \prod_{b \in \lambda} [h(b)]_-,$$

where λ is a partition of r that fits in a $d_V \times d_W$ rectangle, the product ranges over the squares b of the diagram of λ, and $h(b)$ denotes the hook length of b.

Our attempts to prove this conjecture led us to the following related problem, which, as far as we know, has not been studied. We can solve it in two substantial special cases.

Problem 7.2. *The length squares $\|c_{\mathbf{k}}\|^2$ of the canonical basis elements $c_{\mathbf{k}} \in B^r_V$ are polynomials in q^{-1} with nonnegative integer coefficients. Find a combinatorial rule for these coefficients.*

8. Canonical bases for $\check{X}^{\otimes r}$ and $\check{\mathscr{H}}_r$

We succeeded in constructing a canonical basis $\mathrm{HNSTC}(\nu)$ of $\check{X}_\nu$ (§3.7) in the two-row case, but this is only the beginning of a theory of canonical bases for nonstandard objects.

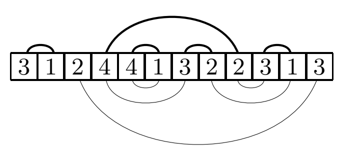
8.1. **A canonical basis of $\check{X}^{\otimes r}$.** In [4] we conjecture that there exists a canonical basis $\check{B}^r$ of $\check{X}^{\otimes r}$ such that it (1) satisfies a certain compatibility with nonstandard Schur-Weyl duality and (2) with Schur-Weyl duality between $GL(X)$ and $\mathcal{S}_r$ at $q = 1$; (3) is a global crystal basis of the $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module $\check{X}^{\otimes r}$; (4) has a $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-cellular subquotient isomorphic to $(\check{X}_\nu, \mathrm{HNSTC}(\nu))$ (and a few other conditions, omitted here). We have succeeded in constructing such a basis for $r \leq 4$, and partially for $r = 5, 6, 7, 8$. This could help develop the theory in several intriguing ways:

- **A generalization of Knuth equivalence**: a Knuth transformation at positions $i, i+1, i+2$ can be thought of as moving between the two Kazhdan-Lusztig basis elements in a two-dimensional cell corresponding to the action of $\langle T_i, T_{i+1}\rangle \cong \mathscr{H}_3$ on the Kazhdan-Lusztig basis of $\mathscr{H}_r$. In the nonstandard setting, we expect that Knuth transformations will come in two flavors corresponding to the two two-dimensional $\check{\mathscr{H}}_3$-irreducibles.

- **New bases for $\mathcal{S}_r$-irreducibles**: specializing the basis $\check{B}^r$ to $q = 1$ would yield interesting bases of the $\mathcal{S}_r$-irreducibles M_ν for $\ell(\nu) \leq 4$. Our computations suggest that for $\ell(\nu) = 3$, these are similar to the Kazhdan-Lusztig basis [28] and web basis [31] (specialized to $q = 1$); this could also yield insight into the difference between these two bases [29].
- **A more complete Kronecker graphical calculus**: as of now, we have used the diagrams of NST to organize the crystal components of HNSTC(ν) (see §3.7). We would like to mimic more of the $U_q(\mathfrak{sl}_2)$ graphical calculus of [17], including a graphical description of each element of $\check{B}^r$, generalizing that for HNSTC(ν), and a graphical description of the action of $\mathcal{P}_i$ on $\check{B}_r$.
- **A tool to solve harder cases of the Kronecker problem**: just as global crystal bases of quantum group representations are built so that each $\mathfrak{sl}_2$-string is isomorphic to a $U_q(\mathfrak{sl}_2)$-irreducible with its global crystal basis, the basis HNSTC(ν) of $\check{X}_\nu$ and its generalization $\check{B}^r$ may become the model for "Kronecker $\mathfrak{sl}_2 \times \mathfrak{sl}_2$-strings" that combine to give a global crystal basis for the general Kronecker problem.

If $\check{B}^r$ is known, then we can define an integral form $\check{Y}^{S\mathbf{A}}_{(1^r)}$ and lattice $\check{\mathscr{L}}_{(1^r)}$ of $\check{X}^{\otimes r} = \check{Y}_{(1^r)}$ to be the $\mathbf{A}$ and K_∞-span of $\check{B}^r$. It follows from requirement (4) and the general theory of crystal bases that once $\check{Y}^{S\mathbf{A}}_{(1^r)}$ and $\check{\mathscr{L}}_{(1^r)}$ are specified, $\check{B}^r$ is determined by the images of the highest weight elements of $\check{B}^r$ in $\check{\mathscr{L}}_{(1^r)}/q^{-1}\check{\mathscr{L}}_{(1^r)}$. One of the subtleties in constructing $\check{B}^r$ (related to the need for rescaling the NST—see §3.7) is that $\check{Y}^{S\mathbf{A}}_{(1^r)}$ and $\check{\mathscr{L}}_{(1^r)}$ are not equal to the obvious integral form and lattice of $X_q^{\otimes r}$.

We want to construct $\check{B}^r$, so by what was just said, we should try to define $\check{Y}^{S\mathbf{A}}_{(1^r)}$ and $\check{\mathscr{L}}_{(1^r)}$ first. We recently discovered what we think to be their correct definitions. These require a new definition of degree for NST$((1^r))$; the definition of degree in §3.7 we now believe to be only an approximation to the correct definition that we are about to give, but these agree in the quotient $\check{X}_{(1^r)}$ of $\check{Y}_{(1^r)}$. Consider the diagram in which the V-arcs and W-arcs of some $T \in$ NST$((1^r))$ are drawn as $\frown$ and $\smile$, respectively, as in the following example (see Figure 1 for conventions):



The *degree* of T, denoted $\deg'(T)$, is the number of closed loops in this diagram. In the example, the degree of the NST is 2. We emphasize here that it is *miraculous that combining V-arcs and W-arcs in this way arose naturally.* Counting loops is a familiar operation from the $U_q(\mathfrak{sl}_2)$ graphical calculus. Indeed, computing the product of two canonical basis elements of the Temperley-Lieb algebra involves computing the number of loops in the concatenation of their diagrams [17, §2.1]. This suggests that counting loops in the concatenation of the V-diagrams of two elements of $\check{B}^r$ is a natural operation, but it does not explain this miracle.

We now define $\check{Y}^{S\mathbf{A}}_{(1^r)}$ and $\check{\mathscr{L}}_{(1^r)}$ to be the $\mathbf{A}$ and K_∞-span of $\big\{(-\frac{1}{[2]_q})^{\deg'(T)}T : T \in \mathrm{NST}((1^r))\big\}$, respectively. Integral forms and lattices $\check{Y}^{S\mathbf{A}}_\alpha$ and $\check{\mathscr{L}}_\alpha$ of $\check{Y}_\alpha$ ($\check{Y}_\alpha$ is as in (4)) for any composition α of r can be defined similarly. The following result makes us confident we are on the right track.

Theorem 8.1. *The integral forms and lattices defined above satisfy (i) $\check{Y}^{S\mathbf{A}}_\alpha \subseteq \check{Y}^{S\mathbf{A}}_{(1^r)}$, (ii) $\check{\mathscr{L}}_\alpha \subseteq \check{\mathscr{L}}_{(1^r)}$, (iii) $\check{\mathscr{L}}_{(1^r)}\mathcal{P}_i \subseteq q^2\check{\mathscr{L}}_{(1^r)}$ for all $i \in [r-1]$, and (iv) $(\check{\mathbf{T}}_{\mathbf{A}}, \overline{\check{\mathscr{L}}_{(1^r)}}, \check{\mathscr{L}}_{(1^r)})$ is a balanced triple (in the sense of Kashiwara [26, 27]), where $\bar{\cdot}$ denotes the bar-involution.*

8.2. **A presentation and canonical basis for $\check{\mathscr{H}}_r$.** A presentation and canonical basis of $\check{\mathscr{H}}_3$ are given in [46, 5]. We have tried hard to extend this beyond $r = 3$, but even the $r = 4$ case is quite difficult. We would like to construct a canonical basis $\check{\mathcal{C}}^{r,d}$ of $K\check{\mathscr{H}}_{r,d}$ ($\check{\mathscr{H}}_{r,d}$ is defined in §3.4) that shares the following properties with the Kazhdan-Lusztig basis of $\mathscr{H}_r$: (1) it is a cellular basis in the sense of Graham-Lehrer [20], and (2) for each $J \subseteq [r-1]$, the *nonstandard right J-descent space*, defined to be $\{h \in \check{\mathscr{H}}_r : h\mathcal{P}_i = 0 \text{ for all } i \in J\}$, is spanned by a subset of $\check{\mathcal{C}}^{r,d}$, and a similar requirement for the left action.

There are two things we now know that might make finding a presentation and canonical basis of $\check{\mathscr{H}}_{r,d}$ more tractable, particularly for $r = 4$. The first is our improved knowledge about the conjectural basis $\check{B}^r$ of $\check{X}^{\otimes r}$, as described in §8.1. This gives another requirement to impose on $\check{\mathcal{C}}^{r,d_V}$: (3) the right cells of $\check{\mathcal{C}}^{r,d_V}$ are isomorphic to the $\check{\mathscr{H}}_{r,d_V}$-cells of $\check{B}^r$.

The second new development is the realization that it is possible to define a nonstandard analog of the 0-Hecke algebra. The *0-nonstandard Hecke algebra* $\check{H}^0_r$ is essentially the $q = 0$ limit of the nonstandard Hecke algebra, though some care is needed to make this precise. It is defined most directly as the subalgebra of $\mathbb{Q}H_0(\mathcal{S}_r) \otimes \mathbb{Q}H_0(\mathcal{S}_r)$ generated by $\pi_i \otimes \pi_i + (1-\pi_i) \otimes (1-\pi_i)$, $i \in [r-1]$, where $H_0(\mathcal{S}_r) = \langle \pi_i : i \in [r-1] \rangle$ is the 0-Hecke monoid of $\mathcal{S}_r$ (see, e.g., [16]).

We have not been able to find a presentation and monomial basis for $\check{\mathscr{H}}_{r,d}$, $r > 3$, analogous to the standard basis $\{T_w : w \in \mathcal{S}_r\}$ of the Hecke algebra. These are important because they might allow $\check{\mathcal{C}}^{r,d}$ to be defined by a globalization procedure similar to that used for the Hecke algebra in [28], or quantum group representations in [26, 27]. In fact, it is conceivable that the combinatorics of decomposing the $q = 1$ specialization of an $\check{\mathscr{H}}_r$-irreducible into $\mathbb{Q}\mathcal{S}_r$-irreducibles (which is close to the Kronecker problem in light of §8.3 and Theorem 3.1) can be read off directly from the combinatorics of a monomial basis of $\check{\mathscr{H}}_r$.

Since the representation theory and combinatorics of the 0-Hecke algebra $\mathbb{Q}H_0(\mathcal{S}_r)$ are well understood [47, 16], and are in some sense easier than that of $\mathscr{H}_r$, we expect that it will be significantly easier to obtain a presentation and monomial basis for $\check{H}^0_r$ than for $\check{\mathscr{H}}_r$. Also, it is not hard (though nontrivial) to show that the $\mathbb{Q}$-dimension of $\check{H}^0_r$ is the same as the $\mathbf{A}$-rank of $\check{\mathscr{H}}_r$ ($\check{\mathscr{H}}_r$ is free as an $\mathbf{A}$-module). This suggests that a presentation and monomial basis of $\check{H}^0_r$ might actually help finding these for $\check{\mathscr{H}}_r$.

8.3. **Representation theory of $\check{\mathscr{H}}_r$.** A complete description of the $K\check{\mathscr{H}}_r$-irreducibles may be within reach using similar techniques to those used to determine the $K\check{\mathscr{H}}_{r,2}$-irreducibles in [8] (Theorem 3.1). Let τ be the flip involution of $\mathscr{H}_r \otimes \mathscr{H}_r$ given by $h_1 \otimes h_2 \mapsto h_2 \otimes h_1$ and let $\theta : \mathscr{H}_r \to \mathscr{H}_r$ be the $\mathbf{A}$-algebra involution defined by $\theta(T_s) = -T_s^{-1}$, $i \in [r-1]$. Twisting an $\mathscr{H}_r$-irreducible by θ corresponds to transposing its shape. The nonstandard Hecke algebra $\check{\mathscr{H}}_r$ is a subalgebra of $(S^2\mathscr{H}_r)^{\theta\otimes\theta}$, the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ fixed by $\theta \otimes \theta$ and τ. Based on computations for $r \leq 6$, it appears that most of the $\check{\mathscr{H}}_r$-irreducibles are restrictions of $(S^2\mathscr{H}_r)^{\theta\otimes\theta}$-irreducibles, except for the trivial $\check{\epsilon}_+$ and sign $\check{\epsilon}_-$ representations.

9. Beyond the two-row case

This work will involve my co-authors Ketan Mulmuley and Milind Sohoni as well as Bharat Adsul and K.V. Subrahmanyam (Chennai Mathematical Institute). To advance our approach to the Kronecker problem described in §3.5–3.8, we now believe it to be important to adapt, as extensively as possible, constructions from multilinear algebra like symmetric and exterior bialgebras, Schur functors, and Schur complexes (as explained, for instance, in [57]) to the nonstandard setting. This is delicate as many definitions that work in the standard setting do not carry over.

9.1. **The Kronecker problem for one two-column shape.** Throughout this subsection let d_V, d_W be arbitrary and $\nu \vdash r$, $\nu' = (a, b)$. We now know how to generalize the construction of $\check{X}_\nu$ from [4] to this case, which gives a basis-theoretic framework for the Kronecker problem in the case that one partition has two columns. However, the correct combinatorics is still missing. Even this special case of the Kronecker problem is already formidable and important because the plethysm coefficients discussed in §6 occur as a special case (precisely, $p^{(a,b)}_{d_V,(d_W,0)} = g_{(d_W{}^{d_V})(d_V{}^{d_W})\nu}$) [58].

To define $\check{X}_\nu$, we first show that the nonstandard exterior algebra $\check{\Lambda}(\check{X})$ is an $\mathscr{O}(GL_q(\check{X}))$-comodule bialgebra. Let $m_{i,j} : \check{\Lambda}^i \check{X} \otimes \check{\Lambda}^j \check{X} \to \check{\Lambda}^{i+j} \check{X}$ and $\Delta_{i,j} : \check{\Lambda}^{i+j} \check{X} \to \check{\Lambda}^i \check{X} \otimes \check{\Lambda}^j \check{X}$ denote the homogeneous components of the multiplication and comultiplication maps m and Δ of $\check{\Lambda}(\check{X})$. Set $\check{Y}_\alpha = \check{\Lambda}^{\alpha_1} \check{X} \otimes \ldots \otimes \check{\Lambda}^{\alpha_l} \check{X}$ as in (4).

Proposition 9.1. *The compositions $\iota^L_{\nu'}$ (top line) and $\iota^R_{\nu'}$ (bottom line)*

$$\check{Y}_{a+1,b-1} \xrightarrow{\Delta_{a,1}\otimes id} \check{Y}_{a,1,b-1} \xrightarrow{id\otimes m_{1,b-1}} \check{Y}_{a,b}$$
$$\check{Y}_{b-1,a+1} \xrightarrow{id\otimes \Delta_{a-b+1,b}} \check{Y}_{b-1,a-b+1,b} \xrightarrow{m_{b-1,a-b+1}\otimes id} \check{Y}_{a,b}$$

are injective $\mathscr{O}(GL_q(\check{X}))$-comodule homomorphisms.

This allows us to define

$$\begin{aligned} \check{Y}^L_{\rhd\nu'} &= \operatorname{im}(\iota^L_{\nu'}), \quad & \check{X}^L_\nu &= \check{Y}_{\nu'}/\check{Y}^L_{\rhd\nu'}, \\ \check{Y}^R_{\rhd\nu'} &= \operatorname{im}(\iota^R_{\nu'}), \quad & \check{X}^R_\nu &= \check{Y}_{\nu'}/\check{Y}^R_{\rhd\nu'}. \end{aligned}$$

This yields two $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-modules $\check{X}^L_\nu, \check{X}^R_\nu$ that specialize to $\mathrm{Res}_{U(\mathfrak{g}_V\oplus\mathfrak{g}_W)} X_\nu$ at $q = 1$. The bases NSC^r and $\mathrm{NST}(\nu')$ of §3.7 can also be defined for general d_V, d_W, so the modules $\check{Y}_{\nu'}$, $\check{Y}_{a+1,b-1}$, $\check{Y}_{b-1,a+1}$ all come with global crystal bases. The key to obtaining a combinatorial formula for Kronecker coefficients is to understand $\iota^L_{\nu'}$ or $\iota^R_{\nu'}$ in terms of these bases (actually, this would yield two different formulas, one corresponding to $\check{X}^L_\nu$ and one to $\check{X}^R_\nu$). We have a pretty good understanding of Δ on the global crystal basis NSC^i of $\check{\Lambda}^i \check{X}$, but m is trickier.

To give an idea of the sort of combinatorics involved, the highest weight elements of weight (λ, μ) of the basis $\mathrm{NST}(\nu')$ of $\check{Y}_{\nu'}$ are (easily shown to be) in bijection with the set

$$\begin{aligned} \mathbf{g}^{\nu'}_{\lambda\mu} := \bigsqcup_{\mu^1\vdash a,\, \mu^2\vdash b} \Big\{ &(Q_V, Q_W) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\mu) : Q_V|_{[a]} = (Z^*_{\mu^1})^T,\ Q_W|_{[a]} = Z^*_{\mu^1}, \\ &\mathrm{jdt}(Q_V|_{[a+1,a+b]} - a) = (Z^*_{\mu^2})^T,\ \mathrm{jdt}(Q_W|_{[a+1,a+b]} - a) = Z^*_{\mu^2} \Big\}, \end{aligned} \tag{6}$$

where $Q|_J$ denotes the (skew) subtableau of Q with entries in J, $\mathrm{jdt}(P)$ denotes the unique straight-shape tableau in the jeu de taquin equivalence class of P, and $P - a$ denotes the result of subtracting the constant a from all entries of a tableau P. Similarly, the highest weight elements of weight (λ, μ) of the basis $\mathrm{NST}((a+1, b-1))$ of $\check{Y}^L_{\rhd\nu'} \cong \check{Y}_{(a+1,b-1)}$ are in bijection with $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu}$. Thus the Kronecker coefficient $g_{\lambda\mu\nu}$ is equal to $\big|\mathbf{g}^{(a,b)}_{\lambda\mu}\big| - \big|\mathbf{g}^{(a+1,b-1)}_{\lambda\mu}\big|$, which is already well known. The difficulty is producing an injection $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu} \hookrightarrow \mathbf{g}^{(a,b)}_{\lambda\mu}$.

Both $\mathbf{g}^{(a,b)}_{\lambda\mu}$ and $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu}$ are subsets of $\mathrm{SYT}(\lambda)\times\mathrm{SYT}(\mu)$, and in lucky cases, $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu} \subseteq \mathbf{g}^{(a,b)}_{\lambda\mu}$. A simple but important observation is that, even when this fails, $\big|\mathbf{g}^{(a,b)}_{\lambda\mu} \setminus \mathbf{g}^{(a+1,b-1)}_{\lambda\mu}\big|$ is an upper bound on $g_{\lambda\mu\nu}$. Our approach suggests this as an approximation and how to correct it.

The way the canonical basis suggests to turn this approximation into an exact formula is subtle. Essentially, the canonical basis provides a guide for turning the linear algebraic problem of computing $\dim_K(\check{Y}_{\nu'}/\check{Y}^L_{\triangleright\nu'})$ into combinatorics. The main subtlety is that the image $\iota^L_{\nu'}(T)$ vanishes at $q=\infty$ for some $T \in \mathrm{NST}((a+1,b-1))$ (precisely, the image belongs to $q^{-1}K_\infty\mathrm{NST}(\nu')$). Such T account for some of the elements of $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu}$ that do not lie in $\mathbf{g}^{(a,b)}_{\lambda\mu}$. Therefore, to obtain an injection $\mathbf{g}^{(a+1,b-1)}_{\lambda\mu} \hookrightarrow \mathbf{g}^{(a,b)}_{\lambda\mu}$, we first have to rescale the NST bases as in §3.7 to reduce the problem to the $q=\infty$ limit, and then understand these $\iota^L_{\nu'}(T)$ combinatorially, which will be difficult.

9.2. **Nonflatness of $\check{X}_\nu$ for ν with more than two columns.** Extending this approach to $\ell(\nu') > 2$ meets with serious difficulties. The K-vector subspaces $\check{Y}^L_{\triangleright\nu'}$ and $\check{Y}^R_{\triangleright\nu'}$ of $\check{Y}_{\nu'}$ are not in general equal, even though they both have integral forms that specialize to the same thing at $q=1$. As a consequence, if we define $\check{X}_\nu$ for $\ell(\nu') > 2$ as in (5), using either $\check{Y}^L_{\triangleright\nu'}$ or $\check{Y}^R_{\triangleright\nu'}$ to define $\check{Y}_{\triangleright^i\nu'}$, then the resulting $\check{X}_\nu$ in general has K-dimension less than the $\mathbb{Q}$-dimension of X_ν. One of the hopes for building the canonical basis of $\check{X}^{\otimes r}$ in the two-row case, as described in §8.1, is that it will suggest a way around this difficulty.

9.3. **The Kronecker problem for one hook shape.** We may be in luck as computer computations indicate that in the case ν is a hook shape, the $\mathscr{O}(GL_q(\check{X}))$-comodule $\check{X}_\nu$ has the correct dimension, i.e., it does not run into the difficulty described above (§9.2). Moreover, this case is likely to be significantly easier than the case ν is a two-column shape because we do understand the maps $\iota^L_{(a,1)}$ and $\iota^L_{(1,1)}$ in terms of the NST basis and combinatorially.

An intriguing combinatorial interpretation of Kronecker coefficients in the case of two hook shapes was given by Lascoux in [36]. Though it is shown in [18] that this does not generalize in an obvious way to other cases, it may nonetheless be fruitful to connect this to our approach.

9.4. **Taking into account more symmetries.** Our approach to the Kronecker problem does not yet take into account all symmetries of the three partitions. For example, $g_{\lambda\mu\nu}$ is the same under any permutation of the three partitions, but our approach only captures the symmetry between λ and μ. Just accounting for the λ, μ symmetry alone has an important application in [4] and makes the computation of the symmetric and exterior Kronecker coefficients (Theorem 3.3) possible.

The nonstandard Hecke algebra offers a way to account for more symmetries, which is to study $\check{\mathscr{H}}^{(3)}_r$ instead of $\check{\mathscr{H}}_r$ (see §2.3). This accounts for the symmetry corresponding to permuting λ, μ, ν since $\check{\mathscr{H}}^{(3)}_r \subseteq S^3\mathscr{H}_r$, as well as symmetries corresponding to transposing shapes. To make use of this, we first need a better understanding of the representation theory of $\check{\mathscr{H}}^{(3)}_r$.

Also, because our approach does not take into account all the symmetries, it gives three ways to look at the same Kronecker coefficients: (1) the case $d_V = 2$, $d_W = 2$, $\ell(\lambda) \leq 2$, $\ell(\mu) \leq 2$, $\ell(\nu) \leq 4$, (2) the case d_V arbitrary, $d_W = 2$, $\ell(\lambda') \leq 4$, $\ell(\mu) \leq 2$, $\ell(\nu') \leq 2$, and (3) the case $d_V = 4$, $d_W = 2$, $\ell(\lambda) \leq 4$, $\ell(\mu) \leq 2$, $\ell(\nu) \leq 2$. Case (1) is worked out in [4], case (2) is within reach using the algebraic framework of §9.1, but the algebraic framework breaks down for (3). Cases (1) and (2) likely yield different combinatorial formulas and it would be interesting to reconcile these, particularly, because this may help us understand why the algebraic framework breaks for (3).

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Biographical Sketch

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a. Professional Preparation

09/00 – 06/04	A.B. in Mathematics, Princeton University, *magna cum laude*. Advisor: Paul Seymour
09/04 – 05/09	Ph.D. in Mathematics, University of California, Berkeley Advisor: Mark Haiman
07/09 – 08/10	NSF Postdoctoral Fellow, University of Chicago Sponsoring Scientist: Ketan Mulmuley
09/10 – 08/11	NSF Postdoctoral Fellow, University of Michigan Sponsoring Scientist: John Stembridge
09/10 – present	Hildebrandt Assistant Professor, University of Michigan Mentor: John Stembridge

b. Appointments

None.

c. Publications

i. The five publications most closely related to the proposed research:

1. Jonah Blasiak. Nonstandard braid relations and Chebyshev polynomials. *ArXiv e-prints*, October 2010, `arXiv:1010.0421`.
2. Jonah Blasiak. Quantum Schur-Weyl duality and projected canonical bases. *ArXiv e-prints*, February 2011, `arXiv:1102.1453`.
3. Jonah Blasiak. Cyclage, catabolism, and the affine hecke algebra. *Advances in Mathematics*, 228(4):2292 – 2351, 2011.
4. Jonah Blasiak. W-graph versions of tensoring with the $\mathcal{S}_n$ defining representation. *Journal of Algebraic Combinatorics*, to appear; `arXiv:0809.4810v1`.
5. J. Blasiak, K. Mulmuley, and M. Sohoni. Geometric complexity theory IV: quantum group for the Kronecker problem. *Preprint*, October 2011. Available at http://www-personal.umich.edu/∼jblasiak.

ii. Five other significant publications:

1. A. Berglund, J. Blasiak, and P. Hersh. Combinatorics of multigraded Poincaré series for monomial rings. *J. Algebra.* 308(1):73–90, 2007.
2. Jonah Blasiak. A special case of Hadwiger's conjecture. *J. Combin. Theory Ser. B.* 97(6):1056–1073, 2007.
3. Jonah Blasiak. The toric ideal of a graphic matroid is generated by quadrics. *Combinatorica.* 28(3):283–297, 2008.
4. Jonah Blasiak. An insertion algorithm for catabolizability. *European J. Combin.*, to appear; `arXiv:0908.1967v1`.

5. Jonah Blasiak and Rick Durrett. Random Oxford graphs. *Stochastic Process. Appl.* 115(8):1257–1278, 2005.

d. Synergistic Activities

1. Mentor for REU students Howard Xu and Zeyin Zhang, August 2011–May 2012 (expected).
2. Reviewer for six journals.
3. Junior staff member at Hampshire College Summer Studies in Mathematics, Summer 2004.

e. Collaborators and Other Affiliations

i. **Collaborators.**

- Anton Geraschenko (Caltech)
- Josh Grochow (U. Chicago)
- Mark Haiman (UC Berkeley)
- Ketan Mulmuley (U. Chicago)
- Milind Sohoni (IIT Bombay)
- John Stembridge (U. Michigan)

ii. **Graduate and Postdoctoral Advisors.**

- Mark Haiman, Graduate Advisor, UC Berkeley.
- Ketan Mulmuley, Postdoctoral Sponsor, U. Chicago.
- John Stembridge, Postdoctoral Sponsor, U. Michigan.

iii. **Graduate Students and Postdoctoral Associates Supervised.**
None.

Data Management Plan

I use the computer algebra system Magma extensively for mathematical exploration. Specifically, I write programs to test and formulate conjectures about objects arising in algebraic combinatorics including tableau, Kazhdan-Lusztig polynomials, and Kronecker coefficients. Since my programs and the data they generate may be valuable to other researchers in these areas, I have made some of these available on my website, and I plan to continue to do so. My computer experiments often result in integer sequences, and I plan to make these easily searchable by entering them into *The On-Line Encyclopedia of Integer Sequences*.

Many of the recent papers I have written or co-authored [6, 7, 4, 8] report on computer generated data. The papers [6, 4, 7] contain substantial pictures and tables that are based on my computer calculations. I plan to continue to aid other researchers by integrating this data into my papers or providing it online.

I am currently teaching my REU students how to use Magma effectively as a tool for mathematical exploration. Several of the incoming postdoctoral students at University of Michigan have expressed interest in learning how to use Magma and in sharing code and data. I will help my colleagues and students become more proficient at mathematical computations and will facilitate making the resulting programs and data available online.