We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problems A–C but do not turn these in. Turn in Problems 1–10.

**Problem A.** Let \( R \) be a ring. Show that \((-1)^2 = 1\) in \( R \).

**Problem B.** Decide which of the following are subrings of the ring of all functions from the closed interval \([0, 1]\) to \( R \):
(a) the set of all functions \( f(x) \) such that \( f(q) = 0 \) for all \( q \in \mathbb{Q} \cap [0, 1] \)
(b) the set of all polynomial functions
(c) the set of all functions which only have a finite number of zeros, together with the zero function
(d) the set of all functions which have an infinite number of zeros
(e) the set of all functions \( f \) such that \( \lim_{x \to 1-} f(x) = 0 \).
(f) the set of all rational linear combinations of the functions \( \sin(nx) \) and \( \cos(mx) \), where \( m, n \in \{0, 1, 2, \ldots\} \).

**Problem C.** Decide which of the following are ideals of the ring \( \mathbb{Z} \times \mathbb{Z} \):
(a) \( \{(a, a) | a \in \mathbb{Z}\} \)
(b) \( \{(2a, 2b) | a, b \in \mathbb{Z}\} \)
(c) \( \{(2a, 0) | a \in \mathbb{Z}\} \)
(d) \( \{(a, -a) | a \in \mathbb{Z}\} \).

**Problem 1.** An element \( x \) in a ring \( R \) is called nilpotent if \( x^m = 0 \) for some \( m \in \mathbb{Z}^+ \). Let \( x \) be a nilpotent element of the commutative ring \( R \).
(a) Prove that \( x \) is either zero or a zero divisor.
(b) Prove that \( rx \) is nilpotent for all \( r \in R \).
(c) Prove that \( 1 + x \) is a unit in \( R \).
(d) Deduce that the sum of a nilpotent element and a unit is a unit.

**Problem 2.** Let \( R \) be a ring with \( 1 \neq 0 \). A nonzero element \( a \) is called a left zero divisor in \( R \) if there is a nonzero element \( x \in R \) such that \( ax = 0 \). Symmetrically, \( b \neq 0 \) is a right zero divisor if there is a nonzero \( y \in R \) such that \( by = 0 \) (so a zero divisor is an element which is neither a left nor a right zero divisor). An element \( u \in R \) has a left inverse in \( R \) if there is some \( s \in R \) such that \( su = 1 \). Symmetrically, \( v \) has a right inverse if \( vt = 1 \) for some \( t \in R \).
(a) Prove that \( u \) is a unit if and only if it has both a right and a left inverse (i.e. \( u \) must have a two-sided inverse).
(b) Prove that if \( u \) has a right inverse then \( u \) is not a right zero divisor.
(c) Prove that if \( u \) has more than one right inverse then \( u \) is a left zero divisor.
(d) Prove that if \( R \) is a finite-dimensional algebra over a field then every element that has a right inverse is a unit (i.e., has a two-sided inverse).

**Problem 3.** Let \( \mathcal{K} = \{k_1, \ldots, k_m\} \) be a conjugacy class in the finite group \( G \).
(a) Prove that the element \( K = k_1 + \cdots + k_m \) is in the center of the group ring \( RG \).
(b) Let $K_1, \ldots, K_r$ be the conjugacy classes of $G$ and for each $K_i$ let $K_i$ be the element of $RG$ that is the sum of the members of $K_i$. Prove that an element $a \in RG$ is in the center of $RG$ if and only if $a = a_1K_1 + a_2K_2 + \cdots + a_rK_r$ for some $a_1, a_2, \ldots, a_r \in R$.

Problem 4. Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.

Problem 5. Decide which of the following are ideals of the ring $\mathbb{Z}[x]$ (and justify your answer):
(a) the set of all polynomials whose constant term is a multiple of 3
(b) the set of all polynomials whose coefficient of $x^2$ is a multiple of 3
(c) the set of all polynomials whose constant term, coefficient of $x$, and coefficient of $x^2$ are zero
(d) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of $x$ appear)
(e) the set of polynomials whose coefficients sum to zero
(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to $x$.

Problem 6. Prove that every (two-sided) ideal of $M_n(R)$ is equal to $M_n(J)$ for some (two-sided) ideal $J$ of $R$.

Problem 7. Let $I$ and $J$ be ideals of $R$.
(a) Prove that $I + J$ is the smallest ideal of $R$ containing both $I$ and $J$.
(b) Prove that $IJ$ is an ideal contained in $I \cap J$.
(c) Give an example where $IJ \neq I \cap J$.
(d) Prove that if $R$ is commutative and if $I + J = R$ then $IJ = I \cap J$.

Problem 8. For the following two rings, give an example of a prime ideal that is not maximal (and prove that your answer is correct):
(a) $\mathbb{Z}[x]
(b) F[x, y]$ for a field $F$.

Problem 9. Let $S_3$ denote the symmetric group on three letters. Determine all two-sided ideals of $\mathbb{C}S_3$.

Problem 10. Let $R$ be the ring of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$ and for each $c \in [0, 1]$ let $M_c = \{ f \in R | f(c) = 0 \}$ (recall that $M_c$ was shown to be a maximal ideal of $R$).
(a) Prove that if $M$ is any maximal ideal of $R$ then there is a real number $c \in [0, 1]$ such that $M = M_c$.
(b) Prove that if $b$ and $c$ are distinct points in $[0, 1]$ then $M_b \neq M_c$.
(c) Prove that $M_c$ is not equal to the principal ideal generated by $x - c$.
(d) Prove that $M_c$ is not a finitely generated ideal.