Problem Set 2

Due: Wednesday, October 5 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problem A but do not turn it in. Turn in Problems 1–10.

Problem A. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$ (projection onto the 1,1 entry)

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$ (the *trace* of the matrix)

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$ (the *determinant* of the matrix).

Problem 1. For the following two rings, give an example of a prime ideal that is not maximal (and prove that your answer is correct):

(a) $\mathbb{Z}[x]$

(b) $F[x, y]$ for a field F.

Problem 2. Prove that R is a division ring if and only if its only left ideals are (0) and R. (The analogous result holds when "left" is replaced by "right".)

Problem 3. Let R be a commutative ring. Prove that the principal ideal generated by x in the polynomial ring $R[x]$ is a prime ideal if and only if R is an integral domain. Prove that (x) is a maximal ideal if and only if R is a field.

Problem 4. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

Problem 5. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.

Problem 6. Let R be commutative ring. Show that an R-module M is irreducible if and only if M is isomorphic (as an R-module) to R/I where I is a maximal ideal of R.

Problem 7. Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $End_R(M)$ is a division ring (this result is called Schur's Lemma).

Problem 8.

(a) Let $R = M_n(\mathbb{C})$. Let $V = \mathbb{C}^n$ considered as a left R-module in the natural way, i.e., the action of a matrix $A \in M_n(\mathbb{C})$ on a column vector $\mathbf{x}$ of length n is equal to the product $A\mathbf{x}$. Determine the submodules of V.

(b) Now consider $V = \mathbb{C}^n$ as a left $\mathbb{C}\mathcal{S}_n$-module, where the action is given by $\pi e_i = e_{\pi(i)}$ for $\pi \in \mathcal{S}_n$, and where $e_1, e_2, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$. Determine the submodules of V.

Problem 9. Find a ring R and a left R-module M such that M cannot be written as a direct sum of simple modules.

Problem 10. Determine all 2-dimensional $\mathbb{C}$-algebras. This means (1) give a list of nonisomorphic 2-dimensional $\mathbb{C}$-algebras, and (2) show that any 2-dimensional $\mathbb{C}$-algebra is isomorphic to one on the list.