Problem Set 2
Due: Thursday, January 24 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Do Problem A but do not turn it in. Turn in Problems 1–10.

Problem A. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$ (projection onto the 1,1 entry)

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$ (the trace of the matrix)

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$ (the determinant of the matrix).

Problem 1. For the following two rings, give an example of a prime ideal that is not maximal (and prove that your answer is correct):

(a) $\mathbb{Z}[x]$

(b) $F[x, y]$ for a field $F$.

Problem 2. Prove that $R$ is a division ring if and only if its only left ideals are $(0)$ and $R$. (The analogous result holds when “left” is replaced by “right”.)

Problem 3. Let $R$ be a commutative ring. Prove that the principal ideal generated by $x$ in the polynomial ring $R[x]$ is a prime ideal if and only if $R$ is an integral domain. Prove that $(x)$ is a maximal ideal if and only if $R$ is a field.

Problem 4. Prove that $\text{Hom}_\mathbb{Z}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

Problem 5. Let $R$ be a commutative ring. Prove that $\text{Hom}_R(R, M)$ and $M$ are isomorphic as left $R$-modules.

Problem 6. Let $R$ be commutative ring. Show that an $R$-module $M$ is irreducible if and only if $M$ is isomorphic (as an $R$-module) to $R/I$ where $I$ is a maximal ideal of $R$.

Problem 7. Show that if $M_1$ and $M_2$ are irreducible $R$-modules, then any nonzero $R$-module homomorphism from $M_1$ to $M_2$ is an isomorphism. Deduce that if $M$ is irreducible then $\text{End}_R(M)$ is a division ring (this result is called Schur’s Lemma).

Problem 8.

(a) Let $R = M_n(\mathbb{C})$. Let $V = \mathbb{C}^n$ considered as a left $R$-module in the natural way, i.e., the action of a matrix $A \in M_n(\mathbb{C})$ on a column vector $x$ of length $n$ is equal to the product $Ax$. Determine the submodules of $V$.

(b) Now consider $V = \mathbb{C}^n$ as a left $\mathbb{C}S_n$-module, where the action is given by $\pi e_i = e_{\pi(i)}$ for $\pi \in S_n$, and where $e_1, e_2, \ldots, e_n$ denotes the standard basis of $\mathbb{C}^n$. Determine the submodules of $V$.

Problem 9. Find a ring $R$ and a left $R$-module $M$ such that $M$ cannot be written as a direct sum of simple modules.
Problem 10. Determine all 2-dimensional $\mathbb{C}$-algebras. This means (1) give a list of nonisomorphic 2-dimensional $\mathbb{C}$-algebras, and (2) show that any 2-dimensional $\mathbb{C}$-algebra is isomorphic to one on the list.