Problem Set 3

Due: Wednesday, October 12 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Read 10.3 Problem 27. Turn in Problems 1–8.

Problem 1. An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R, prove that $M = eM \oplus (1 - e)M$.

Problem 2. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).
(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule.
(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

Problem 3. Let $\phi : M \to N$ be an R-module homomorphism. Prove that $\phi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$.

Problem 4. Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and x, considered as a submodule of R. Show that $\{2, x\}$ is not a basis of M. Show that the rank of M is 1 but that M is not free of rank 1.

Problem 5. Determine all nilpotent elements of $M_2(\mathbb{C})$.

Problem 6. Let F be a field. Give a simple description of the set of zero divisors of $M_n(F)$ in terms of concepts from linear algebra.

Problem 7. Show that if $R = \mathbb{Z}$, $I = \mathbb{Z}_{>0}$, and $M_i = \mathbb{Z}/i\mathbb{Z}$ for each $i \in I$, then $\bigoplus_{i \in I} M_i$ is not isomorphic to $\prod_{i \in I} M_i$.

Problem 8. Let R be a commutative ring. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e., two free R-modules of finite rank are isomorphic if and only if they have the same rank.