Problem Set 4
Due: Thursday, February 7 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise.

Problem 1. For $a, b \in R$, where $R$ is a commutative ring and $a, b$ are nonzero, a least common multiple of $a$ and $b$ is an element $c$ of $R$ such that $a \mid c$, $b \mid c$, and if $(a \mid c'$ and $b \mid c')$ then $c \mid c'$. Prove that any two nonzero elements of a PID have a least common multiple.

Problem 2. Let $R$ be an integral domain. Prove that if the following two conditions hold then $R$ is a PID:
(i) for any $a, b \in R$, there is a $d \in R$ such that $(a, b) = (d)$, and
(ii) if $a_1, a_2, \ldots$ are nonzero elements of $R$ such that $a_{i+1} \mid a_i$ for all $i$, then there is a positive integer $N$ such that $a_n$ is a unit times $a_N$ for all $n \geq N$.

Problem 3. Let $F$ be a field and $R = F[x, y]$. Every ideal of $R$ is finitely generated (we have not proved this, but you can use it for this problem). For a finitely generated ideal $I$, let $s(I)$ be the smallest possible size of a generating set of $I$. Determine $\max\{s(I) \mid I$ is an ideal of $R\}$.
(Define max of a set to be $\infty$ if the set is unbounded from above.)

Problem 4. Let $R$ be a ring. Show that $R$ is a field if and only if every $R$-module has a basis.

Problem 5. Let $M$ be a module over the integral domain $R$.
(a) Let $x \in M$ be a torsion element. Show that $x$ is linearly dependent. Conclude that the rank of $\text{Tor}(M)$ is 0, so that in particular any torsion $R$-module has rank 0.
(b) Show that the rank of $M$ is the same as the rank of the (torsion free) quotient $M/\text{Tor} M$.

Problem 6. Let $M$ be a module over the integral domain $R$. Suppose that $M$ has rank $n$ and that $x_1, \ldots, x_n$ is any maximal set of linearly independent elements of $M$. Let $N = Rx_1 + \cdots + Rx_n$ be the submodule generated by $x_1, \ldots, x_n$. Prove that $N$ is isomorphic to $R^n$ and that the quotient $M/N$ is a torsion $R$-module.

Problem 7. Find an example of a commutative ring $R$ and linearly independent elements $x_1, \ldots, x_n$ of $R^n$ such that these elements do not form a basis of $R^n$.

Problem 8. Let $R$ be a commutative ring and let $b_1, \ldots, b_n$ be a basis of $R^n$. Let $C = [c_{ij}]$ be an $n \times n$ matrix with coefficients in $R$, i.e., $C \in M_n(R)$. Suppose that $\det(C)$ is a unit in $R$.
(a) Show that $C$ is a unit in $M_n(R)$.
(b) For $i = 1, \ldots, n$, let $d_i = \sum_j c_{ij} b_j$. Show that the elements $d_1, \ldots, d_n$ form a basis of $R^n$.

Problem 9. Let $R$ be a commutative ring and let $M$ be the free $R$-module $R^n$. Show that if the elements $x_1, \ldots, x_n \in M$ generate $M$, then they form a basis of $M$. 

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