Problem Set 5

Due: Wednesday, October 26 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–8.

Problem 1. Let $R = \mathbb{Z}$. Use the row and column operations discussed in the proof in class to bring the matrix $A = \begin{bmatrix} 2 & 4 & 4 \\ -4 & 8 & 8 \\ 3 & 2 & 4 \end{bmatrix}$ into the form $\begin{bmatrix} a_1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & \cdots & 0 \end{bmatrix}$ such that $a_1 \mid a_2 \mid \cdots \mid a_m$ and $m \leq 3$.

Problem 2. Repeat the previous problem with $R = \mathbb{C}[x]$ and the matrix

$$A = \begin{bmatrix} x & x+1 & x^2+2 \\ x & x^2+2x+1 & 2 \\ x+x^2 & x^3+2x^2+2x+1 & x^2+2x+2 \end{bmatrix}.$$

Then let M be the submodule of R^3 generated by the rows of A. Give an explicit description of R^3/M as a direct sum of modules of the form R or $R/(a)$.

Problem 3. Let $A_1, \ldots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, \ldots, n$. Prove that

$$(A_1 \oplus \cdots \oplus A_n)/(B_1 \oplus \cdots \oplus B_n) \cong (A_1/B_1) \oplus \cdots \oplus (A_n/B_n).$$

Problem 4. Find an integral domain R and an R-module M such that M is torsion-free and M is not a free module.

Problem 5. Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is torsion-free but not free. Why does this not contradict the Structure Theorem proven in class?

Problem 6. Let M be a finitely generated module over a PID R. Show that any submodule of M is finitely generated. (Do not use the Structure Theorem since we needed this to prove the Structure Theorem.)

Problem 7. Find the rational canonical forms of the following matrices over $\mathbb{Q}$:

$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} c & 0 & -1 \\ 0 & c & 1 \\ -1 & 1 & c \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 8. Determine representatives for the conjugacy classes for $GL_3(\mathbb{F}_2)$.