Problem Set 6

Due: Wednesday, November 2 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. Prove that two 3×3 matrices over a field F are similar if and only if they have the same characteristic and same minimal polynomials. Give an explicit counterexample to this assertion for 4×4 matrices.

Problem 2. Find all similarity classes of 3×3 matrices A over $\mathbb{F}_2$ satisfying $A^6 = I$.

Problem 3. Determine up to similarity all 2×2 rational matrices A (i.e., $A \in M_2(\mathbb{Q})$) such that $A^4 = I$ and $A^k \neq I$ for $k < 4$. Do the same if the matrix has entries from $\mathbb{C}$.

Problem 4. Determine the Jordan canonical form of the $n \times n$ matrix A with 1's on the diagonal and 2's on the superdiagonal.

$$A := \begin{bmatrix} 1 & 2 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Problem 5. Prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the $n \times n$ matrix A then $\lambda_1^k, \ldots, \lambda_n^k$ are the eigenvalues of A^k for any $k \geq 0$.

Problem 6. Prove that any matrix A is similar to its transpose A^T.

Problem 7. Prove that an $n \times n$ matrix A with entries from $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over *any* field F?

Problem 8. Prove that there are no 3×3 matrices A over $\mathbb{Q}$ with $A^8 = I$ but $A^4 \neq I$.

Problem 9. Show that the following matrices are similar in $M_p(\mathbb{F}_p)$:

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Problem 10. Determine the Jordan canonical form for the $n \times n$ matrix over $\mathbb{F}_p$ whose entries are all equal to 1 (the answer depends on whether or not p divides n).