Problem Set 7
Due: Wednesday, February 28 at the beginning of class

Problem 1. Determine the Jordan canonical form of the $n \times n$ matrix $A$ with 1’s on the diagonal and 2’s on the superdiagonal.

\[
A := \begin{bmatrix}
1 & 2 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 2 & 0 & \cdots & 0 \\
0 & 0 & 1 & 2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 2 \\
0 & 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
\]

Problem 2. We define the quaternion group $Q_8$ by generators and relations:

\[Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle.\]

There is a faithful representation $\phi : Q_8 \to GL_2(\mathbb{C})$ defined by

\[
\phi(i) = \begin{bmatrix}
\sqrt{-1} & 0 \\
0 & -\sqrt{-1}
\end{bmatrix}
\quad \text{and} \quad
\phi(j) = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\]

Write out the matrices $\phi(g)$ for every $g \in Q_8$ for this representation.

Problem 3. Let $R$ be a ring. Let $M, N$ be $R$-modules and $S \subseteq M$ a submodule of $M$. Let $\pi : M \to M/S$ be the natural projection. Let $\Theta : \text{Hom}_R(M/S, N) \to \text{Hom}_R(M, N)$ given by $\phi \mapsto \phi \circ \pi$. Show that $\Theta$ is injective and that the image of $\Theta$ consists of those $\alpha \in \text{Hom}_R(M, N)$ such that $S \subseteq \ker(\alpha)$. This shows that “giving a map from $M/S$ to $N$ is the same as giving a map from $M$ to $N$ that sends $S$ to 0”.

Problem 4. Prove that the degree 1 representations of $G$ are in bijective correspondence with the degree 1 representations of the abelian group $G/G'$ where $G' := \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$ is the commutator subgroup of $G$.

Problem 5. Prove that if $|G| > 1$ then every irreducible $FG$-module has dimension $< |G|$.

Problem 6. Let $\phi : G \to GL_n(\mathbb{C})$ be a representation of the finite group $G$. Show that for every $g \in G$, $\phi(g)$ is diagonalizable and its eigenvalues are roots of unity.

Problem 7. We say that $n \times n$ matrices $A_1, \ldots, A_k$ are simultaneously diagonalizable if there is an invertible matrix $P$ such that $P^{-1}A_ip$ are diagonal matrices for all $i$. Let $\{A_1, \ldots, A_k\} \subseteq GL_n(\mathbb{C})$ be a subgroup of commuting matrices. Show that these matrices are simultaneously diagonalizable using representation theory.

Problem 8. Let $\phi : G \to GL_n(\mathbb{C})$ be an irreducible representation of the finite group $G$. Show that if $g \in Z(G)$, then $\phi(g) = cI_n$ for some $c \in \mathbb{C}$.