Problem Set 8
Due: Wednesday, November 16 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. In the problems below, $G$ denotes a finite group. For problems involving decomposing representations into irreducibles, it may be helpful to use the character tables in 19.1. Turn in Problems 1–8.

**Problem 1.** Let $\phi : Q_8 \rightarrow GL_4(\mathbb{C})$ be the representation determined by

$$
i \mapsto \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad j \mapsto \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Determine the decomposition of $\phi$ into irreducible representations.

**Problem 2.** Let $\chi : G \rightarrow \mathbb{C}$ be a character. Show that $\chi(g) = \overline{\chi(g^{-1})}$ for every $g \in G$.

**Problem 3.** Let $\psi : G \rightarrow \mathbb{C}$ be the character of any 2-dimensional representation of a group $G$ and let $x$ be an element of order 2 in $G$. Prove that $\psi(x) = 2, 0, \text{ or } -2$. Generalize this to $n$-dimensional representations.

**Problem 4.** Let $\chi : G \rightarrow \mathbb{C}$ be an irreducible character of $G$. Prove that for every element $z$ in the center of $G$ we have $\chi(z) = \epsilon \chi(1)$, where $\epsilon$ is some root of 1 in $\mathbb{C}$.

**Problem 5.** Let $\phi : G \rightarrow GL(V)$ be a representation and let $\chi : G \rightarrow \mathbb{C}^\times$ be a degree 1 representation. Prove that $\chi \phi : G \rightarrow GL(V)$ defined by $\chi \phi(g) = \chi(g) \phi(g)$ is a representation (note that multiplication of the linear transformation $\phi(g)$ by the complex number $\chi(G)$ is well defined). Show that $\chi \phi$ is irreducible if and only if $\phi$ is irreducible. Show that if $\psi$ is the character afforded by $\phi$ then $\chi \psi$ is the character afforded by $\chi \phi$. Deduce that the product of any irreducible character with a character of degree 1 is also an irreducible character.

**Problem 6.** The action of $S_4$ on $\{1, 2, 3, 4\}$ induces an action of $S_4$ on subsets of $\{1, 2, 3, 4\}$ of size 2. For instance, if $\pi$ is the cycle $(123)$, then

$$
\pi(\{1, 2\}) = \{3, 1\}, \pi(\{1, 3\}) = \{3, 2\}, \pi(\{1, 4\}) = \{3, 4\}, \pi(\{2, 3\}) = \{1, 2\}, \pi(\{2, 4\}) = \{1, 4\}, \pi(\{3, 4\}) = \{2, 4\}.
$$

Let $M_{2,2}$ denote the $\mathbb{C}S_4$-module corresponding to this action. Determine the decomposition of $M_{2,2}$ into irreducibles.

**Problem 7.** Determine the character table of $D_{12}$ (assume the field is $\mathbb{C}$).
Problem 8. Recall that for $FG$-modules $V$ and $W$, $\text{Hom}_F(V, W)$ is an $FG$-module via $(g \cdot \phi)(v) = g\phi(g^{-1}v)$ for every $\phi \in \text{Hom}_F(V, W)$, $g \in G$, $v \in V$. Show that the character $\chi_{\text{Hom}_F(V, W)}$ is given by

$$\chi_{\text{Hom}_F(V, W)}(g) = \chi_V(g^{-1})\chi_W(g)$$

for every $g \in G$.

Hint: One possible route to proving this is to choose bases for $V$ and $W$ (say $V$ has dimension $n$ and $W$ has dimension $m$); then $\text{Hom}_F(V, W)$ can be identified with $m \times n$-matrices and has a basis $\{E_{ij}\}$, where $E_{ij}$ denotes the matrix with 1 in the $i, j$-th spot and 0 elsewhere. Then the matrix corresponding to the action of $g$ on $\text{Hom}_F(V, W)$ is an $mn \times mn$-matrix that can be computed explicitly in terms of matrices corresponding to the action of $g^{-1}$ on $V$, and the action of $g$ on $W$. 