Problem Set 8
Due: Wednesday, March 6 at the beginning of class

In the problems below, $G$ denotes a finite group. For problems involving decomposing representations into irreducibles, it may be helpful to use the character tables in 19.1.

Problem 1. Let $\phi : Q_8 \to GL_4(\mathbb{C})$ be the representation determined by

$$
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
$$

Determine the decomposition of $\phi$ into irreducible representations.

Problem 2. Let $\chi : G \to \mathbb{C}$ be a character. Show that $\chi(g) = \chi(g^{-1})$ for every $g \in G$.

Problem 3. Let $\chi : G \to \mathbb{C}$ be an irreducible character of $G$. Prove that for every element $z$ in the center of $G$ we have $\chi(z) = \epsilon\chi(1)$, where $\epsilon$ is some root of 1 in $\mathbb{C}$.

Problem 4. Let $\phi : G \to GL(V)$ be a representation and let $\chi : G \to \mathbb{C}^\times$ be a degree 1 representation. Prove that $\chi\phi : G \to GL(V)$ defined by $\chi\phi(g) = \chi(g)\phi(g)$ is a representation (note that multiplication of the linear transformation $\phi(g)$ by the complex number $\chi(G)$ is well defined). Show that $\chi\phi$ is irreducible if and only if $\phi$ is irreducible. Show that if $\psi$ is the character afforded by $\phi$ then $\chi\psi$ is the character afforded by $\chi\phi$. Deduce that the product of any irreducible character with a character of degree 1 is also an irreducible character.

Problem 5. Let $Z_n$ be the cyclic group of order $n$. Viewing the group algebra $\mathbb{Q}Z_4$ as a left module over itself, determine its decomposition into irreducible submodules (this is the same as decomposing the regular representation into irreducible representations). Do the same for $\mathbb{Q}Z_4$.

Problem 6. Let $V = \mathbb{C}\{e_1, e_2, e_3, e_4\}$ be the 4-dimensional $\mathbb{C}S_4$-module and $\phi : S_4 \to GL_4(\mathbb{C})$ be the corresponding representation, where $S_4$ acts on the basis $\{e_i\}$ by permuting indices. Let $\pi : D_8 \to S_4$ be the group homomorphism given by viewing $S_4$ as the permutations of the vertices of a square (this realizes $D_8$ as the subgroup of $S_4$ that preserves the edges of the square). Hence $\phi \circ \pi$ is a representation of $D_8$ and we can consider $V$ as the corresponding $\mathbb{C}D_8$-module. Determine the decomposition of $V$ into irreducible $\mathbb{C}D_8$-modules.

Problem 7. The action of $S_4$ on $\{1, 2, 3, 4\}$ induces an action of $S_4$ on subsets of $\{1, 2, 3, 4\}$ of size 2. For instance, if $\pi$ is the cycle $(132)$, then

$$
\pi(\{1, 2\}) = \{3, 1\}, \pi(\{1, 3\}) = \{3, 2\}, \pi(\{1, 4\}) = \{3, 4\}, \pi(\{2, 3\}) = \{1, 2\}, \pi(\{2, 4\}) = \{1, 4\}, \pi(\{3, 4\}) = \{2, 4\}.
$$

Let $M_{2,2}$ denote the $\mathbb{C}S_4$-module corresponding to this action. Determine the decomposition of $M_{2,2}$ into irreducibles.

Problem 8. Determine the character table of $D_{12}$ (assume the field is $\mathbb{C}$).