Problem Set 9

Due: Wednesday, November 30 at the beginning of class

We will assume rings have identity and ring homomorphisms are unital (send 1 to 1) unless stated otherwise. Turn in Problems 1–10.

Problem 1. Prove that elements x and y are conjugate in a group G if and only if $\chi(x) = \chi(y)$ for all irreducible characters χ of G.

Problem 2. Let H and K be finite groups and V be a $\mathbb{C}$-vector space. Let $G = H \times K$ and let $\phi : H \to GL(V)$ be an irreducible representation of H with character χ. Then $G \xrightarrow{\pi_H} H \xrightarrow{\phi} GL(V)$ gives an irreducible representation of G, where π_H is the natural projection; the character $\widetilde{\chi}$ of this representation is $\widetilde{\chi}((h,k)) = \chi(h)$. Likewise any irreducible character ψ of K gives an irreducible character $\widetilde{\psi}$ of G with $\widetilde{\psi}((h,k)) = \psi(k)$.

Prove that the product $\widetilde{\chi}\widetilde{\psi}$ is an irreducible character of G.

Problem 3. Show that the character table (over $\mathbb{C}$) is an invertible matrix (you can use the fact that it is a square matrix). Use this to prove the *second orthogonality relations*: Let $\chi_1, \dots, \chi_r$ be the irreducible characters of G. For any $x, y \in G$,

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

Here $C_G(x)$ denotes the centralizer of x in G.

Problem 4. The alternating group A_4 is a subgroup of $\mathcal{S}_4$, hence $\mathbb{C}A_4$ is a subalgebra of $\mathbb{C}\mathcal{S}_4$. Therefore any $\mathbb{C}\mathcal{S}_4$-module is a $\mathbb{C}A_4$-module by restriction. For each irreducible $\mathbb{C}\mathcal{S}_4$-module V, determine the decomposition of V, regarded as a $\mathbb{C}A_4$-module, into irreducibles. Feel free to use the character table for A_4 in 19.1.

Problem 5. By the Artin-Wedderburn Theorem and the character table of $\mathcal{S}_3$, we have the following isomorphism of rings:

$$\mathbb{C}\mathcal{S}_3 \cong M_1(\mathbb{C}) \times M_2(\mathbb{C}) \times M_1(\mathbb{C}).$$

Determine explicitly the elements $\left([1], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, [0]\right), \left([0], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [0]\right), \left([0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, [1]\right)$ on the right hand side in terms of the basis $\{\pi : \pi \in \mathcal{S}_3\}$ of $\mathbb{C}\mathcal{S}_3$.

This is equivalent to finding three elements z_1, z_2, z_3 in the center of $\mathbb{C}\mathcal{S}_3$ such that

- $z_1 + z_2 + z_3 = 1$,
- $z_i^2 = z_i$,
- $z_i z_j = z_j z_i = 0$ for $i \neq j$.

Problem 6. Let F be a field and G a finite group. Let V and W be FG-modules.

(a) Consider the F-vector space $V \otimes_F W$. Show that if $g \in G$ acts on $V \otimes_F W$ by

$$g \cdot (v \otimes w) := gv \otimes gw \quad \text{for every } v \in V,\, w \in W,$$

then this gives $V \otimes_F W$ the structure of an FG-module.

(b) Prove that the character of $V \otimes_F W$ is given by $\chi_{V \otimes_F W}(g) = \chi_V(g)\chi_W(g)$ for every $g \in G$.

Problem 7. Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

Problem 8. An element of a tensor product $M \otimes_R N$ is a *simple tensor* if it is of the form $m \otimes n$ for $m \in M$, $n \in N$. Let F be a field and let V be an n-dimensional F-vector space.

(a) The vector space $V \otimes_F V$ can be identified with $M_n(F)$ via $e_i \otimes e_j \mapsto E_{ij}$, where $e_1, \ldots, e_n$ is a basis of V and E_{ij} is the matrix with a 1 in the i, jth spot and 0's elsewhere. Express what it means for an element of $V \otimes_F V$ to be a simple tensor in terms of concepts from linear algebra.

(b) Suppose $n \geq 2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_F V$ is not a simple tensor.

Problem 9. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules.

Problem 10. Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I + J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I + J)$.