Problem 1. Prove that elements \( x \) and \( y \) are conjugate in a group \( G \) if and only if \( \chi(x) = \chi(y) \) for all irreducible characters \( \chi \) of \( G \).

Problem 2. Let \( H \) and \( K \) be finite groups and \( V \) be a \( \mathbb{C} \)-vector space. Let \( G = H \times K \) and let \( \phi : H \to GL(V) \) be an irreducible representation of \( H \) with character \( \chi \). Then \( G \xrightarrow{\pi_H} H \xrightarrow{\phi} GL(V) \) gives an irreducible representation of \( G \), where \( \pi_H \) is the natural projection; the character \( \tilde{\chi} \) of this representation is \( \tilde{\chi}((h,k)) = \chi(h) \). Likewise any irreducible character \( \psi \) of \( K \) gives an irreducible character \( \tilde{\psi} \) of \( G \) with \( \tilde{\psi}((h,k)) = \psi(k) \).

Prove that the product \( \tilde{\chi} \tilde{\psi} \) is an irreducible character of \( G \).

Problem 3. Show that the character table (over \( \mathbb{C} \)) is an invertible matrix (you can use the fact that it is a square matrix). Use this to prove the second orthogonality relations: Let \( \chi_1, \ldots, \chi_r \) be the irreducible characters of \( G \). For any \( x, y \in G \),

\[
\sum_{i=1}^{r} \chi_i(x)\bar{\chi}_i(y) = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G, \\ 0 & \text{otherwise.} \end{cases}
\]

Here \( C_G(x) \) denotes the centralizer of \( x \) in \( G \).

Problem 4. The alternating group \( A_4 \) is a subgroup of \( S_4 \), hence \( \mathbb{C}A_4 \) is a subalgebra of \( \mathbb{C}S_4 \). Therefore any \( \mathbb{C}S_4 \)-module is a \( \mathbb{C}A_4 \)-module by restriction. For each irreducible \( \mathbb{C}S_4 \)-module \( V \), determine the decomposition of \( V \), regarded as a \( \mathbb{C}A_4 \)-module, into irreducibles. Feel free to use the character table for \( A_4 \) in 19.1.

Problem 5. Repeat the previous problem with \( D_8 \) in place of \( A_4 \), viewing \( D_8 \subset S_4 \) as arising from thinking of \( S_4 \) as the permutations of the vertices of a square and \( D_8 \) as the subgroup of \( S_4 \) that preserves the edges of the square.

Problem 6. By the Artin-Wedderburn Theorem and the character table of \( S_3 \), we have the following isomorphism of rings:

\[
\mathbb{C}S_3 \cong M_1(\mathbb{C}) \times M_2(\mathbb{C}) \times M_1(\mathbb{C}).
\]

Determine explicitly the elements \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \) on the right hand side in terms of the basis \( \{ \pi : \pi \in S_3 \} \) of \( \mathbb{C}S_3 \).

This is equivalent to finding three elements \( z_1, z_2, z_3 \) in the center of \( \mathbb{C}S_3 \) such that

\[
\begin{align*}
\bullet & \quad z_1 + z_2 + z_3 = 1, \\
\bullet & \quad z_i^2 = z_i, \\
\bullet & \quad z_iz_j = z_jz_i = 0 \text{ for } i \neq j.
\end{align*}
\]

Problem 7. Repeat the previous problem with the cyclic group \( Z_5 \): since the irreducible \( \mathbb{C}Z_5 \)-modules are 1-dimensional, we have

\[
\mathbb{C}Z_5 \cong M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}) \times M_1(\mathbb{C}).
\]

Determine the isomorphism explicitly in terms of the natural basis on the right hand side and the basis \( \{ e, g, g^2, g^3, g^4 \} \) of \( \mathbb{C}Z_5 \).
**Problem 8.** Let $F$ be a field and $G$ a finite group. Let $V$ and $W$ be $FG$-modules.

(a) Consider the $F$-vector space $V \otimes_F W$. Show that if $g \in G$ acts on $V \otimes_F W$ by

$$g \cdot (v \otimes w) := gv \otimes gw \quad \text{for every } v \in V, w \in W,$$

then this gives $V \otimes_F W$ the structure of an $FG$-module.

(b) Prove that the character of $V \otimes_F W$ is given by $\chi_{V \otimes_F W}(g) = \chi_V(g)\chi_W(g)$ for every $g \in G$.

**Problem 9.** An element of a tensor product $M \otimes_R N$ is a **simple tensor** if it is of the form $m \otimes n$ for $m \in M$, $n \in N$. Let $F$ be a field and let $V$ be an $n$-dimensional $F$-vector space.

(a) The vector space $V \otimes_F V$ can be identified with $M_n(F)$ via $e_i \otimes e_j \mapsto E_{ij}$, where $e_1, \ldots, e_n$ is a basis of $V$ and $E_{ij}$ is the matrix with a 1 in the $i,j$th spot and 0’s elsewhere. Express what it means for an element of $V \otimes_F V$ to be a simple tensor in terms of concepts from linear algebra.

(b) Suppose $n \geq 2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_F V$ is not a simple tensor.