Problem Set 9
Due: Wednesday, March 13 at the beginning of class

Problem 1. Prove that elements $x$ and $y$ are conjugate in a group $G$ if and only if $\chi(x) = \chi(y)$ for all irreducible characters $\chi$ of $G$.

Problem 2. Let $H$ and $K$ be finite groups and $V$ be a $\mathbb{C}$-vector space. Let $G = H \times K$ and let $\phi : H \to GL(V)$ be an irreducible representation of $H$ with character $\chi$. Then $G \xrightarrow{\pi_H} H \xrightarrow{\phi} GL(V)$ gives an irreducible representation of $G$, where $\pi_H$ is the natural projection; the character $\tilde{\chi}$ of this representation is $\tilde{\chi}((h,k)) = \chi(h)$. Likewise any irreducible character $\psi$ of $K$ gives an irreducible character $\tilde{\psi}$ of $G$ with $\tilde{\psi}((h,k)) = \psi(k)$.

Prove that the product $\tilde{\chi}\tilde{\psi}$ is an irreducible character of $G$.

Problem 3. Show that the character table (over $\mathbb{C}$) is an invertible matrix (you can use the fact that it is a square matrix). Use this to prove the second orthogonality relations: Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of $G$. For any $x, y \in G$,

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Here $C_G(x)$ denotes the centralizer of $x$ in $G$.

Problem 4. The alternating group $A_4$ is a subgroup of $S_4$, hence $\mathbb{C}A_4$ is a subalgebra of $\mathbb{C}S_4$. Therefore any $\mathbb{C}S_4$-module is a $\mathbb{C}A_4$-module by restriction. For each irreducible $\mathbb{C}S_4$-module $V$, determine the decomposition of $V$, regarded as a $\mathbb{C}A_4$-module, into irreducibles. Feel free to use the character table for $A_4$ in 19.1.

Problem 5. Repeat the previous problem with $D_8$ in place of $A_4$, viewing $D_8 \subset S_4$ as arising from thinking of $S_4$ as the permutations of the vertices of a square and $D_8$ as the subgroup of $S_4$ that preserves the edges of the square.

Problem 6. Let $Z_5$ be the cyclic group of order 5. Find five elements $z_1, \ldots, z_5$ of $\mathbb{C}Z_5$ such that

- $z_1 + z_2 + z_3 + z_4 + z_5 = 1$,
- $z_i^2 = z_i$ for all $i$,
- $z_iz_j = z_jz_i = 0$ for $i \neq j$.

Problem 7. Recall that for $FG$-modules $V$ and $W$, $\text{Hom}_F(V,W)$ is an $FG$-module via $(g \cdot \phi)(v) = g\phi(g^{-1}v)$ for every $\phi \in \text{Hom}_F(V,W), g \in G, v \in V$. Show that the character $\chi_{\text{Hom}_F(V,W)}$ is given by

$$\chi_{\text{Hom}_F(V,W)}(g) = \chi_V(g^{-1})\chi_W(g) \quad \text{for every } g \in G.$$ 

Hint: One possible route to proving this is to choose bases for $V$ and $W$ (say $V$ has dimension $n$ and $W$ has dimension $m$); then $\text{Hom}_F(V,W)$ can be identified with $m \times n$-matrices and has a basis $\{E_{ij}\}$, where $E_{ij}$ denotes the matrix with 1 in the $i, j$-th spot and 0 elsewhere. Then the matrix corresponding to the action of $g$ on $\text{Hom}_F(V,W)$ is an $mn \times mn$-matrix that can be computed explicitly in terms of matrices corresponding to the action of $g^{-1}$ on $V$, and the action of $g$ on $W$. 
