Name:

Quiz 8
Wednesday, November 30

Directions: Show all your work and carefully justify each claim you make!

Problem 1. Let \( A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \). Find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = PDP^{-1} \). You may use the fact that \( \det(A - \lambda I) = (1 - \lambda)(\lambda + 2)^2 \).

Solution: The eigenvalues of \( A \) are \( \lambda = 1 \) and \( \lambda = -2 \). To find eigenvectors, we compute \( \text{Nul}(A - \lambda I) \):

\[
A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 0 \\ -3 & -6 & -3 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},
\]

thus an eigenvector for \( \lambda = 1 \) is \( \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \).

\[
A - (-2)I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Thus a basis for the eigenspace of \( \lambda = -2 \) is \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} \). Since the sum of the dimensions of the eigenspaces is 3, the size of the matrix, \( A \) is diagonalizable. For \( D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \) and \( P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \), we have \( A = PDP^{-1} \) by the Diagonalization Theorem.
Problem 2. True/False and short answer. No justification is required and no partial credit will be given. Circle your final answers clearly. Each question is worth a half a point.

(a) True/False: An $n \times n$ matrix with $n$ distinct eigenvalues is diagonalizable.

Solution: True. This was a theorem in the last lecture.

(b) Write down a $3 \times 3$ matrix with characteristic polynomial $(3 - \lambda)(4 - \lambda)^2$: 

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

Solution: 

(c) Suppose $A$ is a $4 \times 4$ matrix with real entries. If the eigenvalues of $A$ include $1, 3, 2 + 5i$, then $A$ must have one more eigenvalue, which is equal to 

Solution: $2 - 5i$. For matrices with real entries, non-real eigenvalues come in complex conjugate pairs.

(d) Write down a $2 \times 2$ matrix with real entries but with no real eigenvalues: 

\[
\begin{bmatrix}
0 & -1 \\
1 & 0 \\
\end{bmatrix}
\]

Solution: This matrix has eigenvalues $i$ and $-i$.

(e) Suppose that a $10 \times 10$ matrix $A$ has eigenvalues $1, 4, 6$ (and no other eigenvalues). Suppose that $A$ is not diagonalizable. Then the largest possible dimension of an eigenspace of $A$ is 

Solution: 7. Since $A$ is not diagonalizable, the sum of the dimensions of the eigenspaces must be less than 10. Each eigenspace has dimension at least 1. So the way to maximize the dimension of one of the eigenspaces is to choose the dimensions to be 7,1,1. It is indeed possible to construct a $10 \times 10$ matrix with these given eigenspace dimensions.

(f) Suppose that a $5 \times 5$ matrix $A$ has eigenvalues $1, 4, 6$ (and no other eigenvalues). Suppose that $\text{dim}(\text{Nul}(A - 4I)) = 3$. Then the characteristic polynomial of $A$ is 

Solution: Since the algebraic multiplicity is at least as big as the dimension of the eigenspace, the the algebraic multiplicity of 4 is at least 3. The algebraic multiplicity of 1 and 6 are at least 1, and the sum of the algebraic multiplicities must be 5, so the characteristic polynomial is $(4 - \lambda)^3(1 - \lambda)(6 - \lambda)$. 