QUANTUM SCHUR-WEYL DUALITY AND PROJECTED CANONICAL BASES

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ABSTRACT. Let $\mathscr{H}_r$ be the generic type A Hecke algebra defined over $\mathbb{Z}[u, u^{-1}]$. The Kazhdan-Lusztig bases $\{C_w\}_{w\in\mathcal{S}_r}$ and $\{C'_w\}_{w\in\mathcal{S}_r}$ of $\mathscr{H}_r$ give rise to two different bases of the Specht module M_λ, $\lambda \vdash r$, of $\mathscr{H}_r$. These bases are not equivalent and we show that the transition matrix $S(\lambda)$ between the two is the identity at $u = 0$ and $u = \infty$. To prove this, we first prove a similar property for the transition matrices $\tilde{T}, \tilde{T}'$ between the Kazhdan-Lusztig bases and their projected counterparts $\{\tilde{C}_w\}_{w\in\mathcal{S}_r}$, $\{\tilde{C}'_w\}_{w\in\mathcal{S}_r}$, where $\tilde{C}_w := C_w p_\lambda$, $\tilde{C}'_w := C'_w p_\lambda$ and p_λ is the minimal central idempotent corresponding to the two-sided cell containing w. We prove this property of $\tilde{T}, \tilde{T}'$ using quantum Schur-Weyl duality and results about the upper and lower canonical basis of $V^{\otimes r}$ (V the natural representation of $U_q(\mathfrak{gl}_n)$) from [14, 11, 7]. We also conjecture that the entries of $S(\lambda)$ have a certain positivity property.

1. Introduction

Let $\{C_w : w \in \mathcal{S}_r\}$ and $\{C'_w : w \in \mathcal{S}_r\}$ be the Kazhdan-Lusztig bases of the type A Hecke algebra $\mathscr{H}_r$, which we refer to as the upper and lower canonical basis of $\mathscr{H}_r$, respectively. After working with these bases for a while, we have convinced ourselves that it is not particularly useful to look at both at once—one can work with one or the other and it is easy to go back and forth between the two (precisely, there is an automorphism θ of $\mathscr{H}_r$ such that $\theta(C'_w) = (-1)^{\ell(w)} C_w$). However, our recent work on the nonstandard Hecke algebra [4, 6] has forced us to look at both these bases simultaneously. Before explaining how this comes about, let us describe our results and conjectures.

Let $K = \mathbb{Q}(u)$, where u is the Hecke algebra parameter, and let M_λ be the $K\mathscr{H}_r$-irreducible of shape $\lambda \vdash r$. The upper and lower canonical basis of $\mathscr{H}_r$ give rise to bases $\{C_Q : Q \in \mathrm{SYT}(\lambda)\}$ and $\{C'_Q : Q \in \mathrm{SYT}(\lambda)\}$ of M_λ, which we refer to as the upper and lower canonical basis of M_λ. These bases are not equivalent, and it appears to be a difficult and interesting question to understand the transition matrix $S(\lambda)$ between them (which is well-defined up to a global scale by the irreducibility of M_λ). It turns out that $S(\lambda)$ is the identity at $u = 0$ and $u = \infty$ (Theorem 7.8) and, though it is not completely clear what it should mean for an element of K to be nonnegative, its entries appear to have some kind of nonnegativity (see Conjecture 7.9).

To compare the upper and lower canonical basis of M_λ, we compare them both to certain seminormal bases of M_λ in the sense of [28]. These bases are compatible with restriction along the chain of subalgebras $\mathscr{H}_1 \subseteq \cdots \subseteq \mathscr{H}_{r-1} \subseteq \mathscr{H}_r$ (see Definition 7.3). Specifically,

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we define an upper (resp. lower) seminormal basis which differs from the upper (resp. lower) canonical basis by a unitriangular transition matrix $T(\lambda)$ (resp. $T'(\lambda)$). It appears that these transition matrices also possess some kind of nonnegativity property. Since the restrictions $\mathscr{H}_{i-1} \subseteq \mathscr{H}_i$ are multiplicity-free, these seminormal bases differ from each other by a diagonal transformation $D(\lambda)$. Hence we have $S(\lambda) = T(\lambda)D(\lambda)T'(\lambda)^{-1}$.

We briefly mention some related investigations in the literature. Other seminormal bases of M_λ have been defined—for instance, Hoefsmit, and later, independently, Ocneanu, and Wenzl construct a Hecke algebra analog of Young's orthogonal basis (see [30]). This basis differs from our upper and lower seminormal bases by a diagonal transformation, but is not equal to either. The recent paper [8] uses an interpretation of the lower seminormal basis in terms of non-symmetric Macdonald polynomials to study $T'(\lambda)$ for λ a two-row shape and gives an explicit formula for a column of this matrix (see Remark 8.4). Along similar lines, the transition matrix between the upper canonical basis at $u = 1$ and Young's natural basis of M_λ is studied by Garsia and McLarnan in [13]; they show that this matrix is unitriangular and has integer entries.

Our investigation further involves projecting the basis element C_w (resp. C'_w) onto the isotypic component corresponding to the two-sided cell containing w. This results in what we call the projected upper (resp. lower) canonical basis; let $\tilde{T}$ (resp. $\tilde{T}'$) denote the transition matrix between the projected and upper (resp. lower) canonical basis. The properties we end up proving about $S(\lambda), T(\lambda), T'(\lambda)$ all follow from properties of $\tilde{T}$ and $\tilde{T}'$. And we are able to get some handle on $\tilde{T}$ and $\tilde{T}'$ using quantum Schur-Weyl duality. Specifically, we use the compatibility between an upper (resp. lower) canonical basis of $V^{\otimes r}$ with the upper (resp. lower) canonical basis of $\mathscr{H}_r$ and well-known results about crystal lattices, where V is the natural representation of $U_q(\mathfrak{gl}_n)$. The results we need are similar to those in [14, 11, 7] and follow easily from results of [25, 22]. Brundan's paper [7] is particularly well adapted to our needs and we follow it closely.

We now return to our original motivation. The type A *nonstandard Hecke algebra* $\check{\mathscr{H}}_r$ is the subalgebra of $\mathscr{H}_r \otimes \mathscr{H}_r$ generated by the elements

$$\mathcal{P}_s := C'_s \otimes C'_s + C_s \otimes C_s, \;\; s \in S,$$

where $S = \{s_1, \ldots, s_{r-1}\}$ is the set of simple reflections of $\mathcal{S}_r$. We think of the inclusion $\check{\mathscr{H}}_r \hookrightarrow \mathscr{H}_r \otimes \mathscr{H}_r$ as a deformation of the coproduct $\mathbb{Z}\mathcal{S}_r \to \mathbb{Z}\mathcal{S}_r \otimes \mathbb{Z}\mathcal{S}_r$, $w \mapsto w \otimes w$. This algebra was constructed by Mulmuley and Sohoni in [26] in an attempt to use canonical bases to understand Kronecker coefficients.

Let $\epsilon_+ = M_{(r)}$, $\epsilon_- = M_{(1^r)}$ be the trivial and sign representations of $K\mathscr{H}_r$. Any representation $M_\lambda \otimes M_\mu$ of $K(\mathscr{H}_r \otimes \mathscr{H}_r)$ is a $K\check{\mathscr{H}}_r$-module by restriction. The trivial and sign representations $\check{\epsilon}_+$ and $\check{\epsilon}_-$ of $K\check{\mathscr{H}}_r$ are the restrictions of $\epsilon_+ \otimes \epsilon_+$ and $\epsilon_+ \otimes \epsilon_-$, respectively. There is a single copy of $\check{\epsilon}_+$ inside $\text{Res}_{K\check{\mathscr{H}}_r} M_\lambda \otimes M_\lambda$ and a single copy of $\check{\epsilon}_-$ inside $\text{Res}_{K\check{\mathscr{H}}_r} M_\lambda \otimes M_{\lambda'}$, where λ' is the conjugate partition of λ. These can be written in terms canonical bases as

$$\check{\epsilon}_+ \cong K \sum_{Q \in \text{SYT}(\lambda)} C_Q \otimes C'_Q, \qquad \check{\epsilon}_- \cong K \sum_{Q \in \text{SYT}(\lambda)} (-1)^{\ell(Q)} C_Q \otimes C_{Q^t}, \tag{1}$$

where Q^t denotes the transpose of the SYT Q and $\ell(Q)$ denotes the distance between Q and some fixed tableau of shape λ in the dual Knuth equivalence graph on $\mathrm{SYT}(\lambda)$.

An important part of understanding the nonstandard Hecke algebra is to understand its trivial and sign representations. If we fix a basis of $K(\mathscr{H}_r \otimes \mathscr{H}_r)$, say $\{C_v \otimes C_w : v, w \in \mathcal{S}_r\}$, then expressing the central idempotent for $\check{\epsilon}_-$ in this basis involves understanding $\tilde{T}$ and expressing the central idempotent for $\check{\epsilon}_+$ involves understanding $\tilde{T}$ and $S(\lambda)$. The same difficulties come up if we choose the basis $\{C_v \otimes C'_w : v, w \in \mathcal{S}_r\}$. Admittedly, $\check{\epsilon}_+ \subseteq \mathrm{Res}_{K\check{\mathscr{H}}_r} M_\lambda \otimes M_\lambda$ and $\check{\epsilon}_- \subseteq \mathrm{Res}_{K\check{\mathscr{H}}_r} M_\lambda \otimes M_{\lambda'}$ both have simple expressions in terms of the Hecke orthogonal basis of [30]. However, we suspect it will be useful to understand $\check{\mathscr{H}}_r$ in terms of a basis like $\{C_v \otimes C_w : v, w \in \mathcal{S}_r\}$. This is somewhat justified by our work in progress [3], joint with Ketan Mulmuley and Milind Sohoni, in which we use canonical bases of quantum groups to give a combinatorial rule for Kronecker coefficients with two two-row shapes (here, we do not need $S(\lambda)$, but the projected upper canonical basis plays an essential role).

This paper is organized as follows. In §2–4 we introduce the necessary background on canonical bases of Hecke algebras and quantum groups. We then use this in §5 to construct canonical bases of $V^{\otimes r}$ and relate them to those of $\mathscr{H}_r$, closely following [7]. Next, in §6, we give several characterizations of projected canonical basis elements, which we then use in §7 to prove that the transition matrices $S(\lambda), T(\lambda)$, and $T'(\lambda)$ are the identity at $u = 0$ and $u = \infty$. Finally, in §8, we compute explicitly a matrix similar to $T'(\lambda)$, for λ a two-row shape, using the $U_q(\mathfrak{sl}_2)$ graphical calculus of [12].

2. Preliminaries and notation

Here we introduce notation for general Coxeter groups and then specialize to the weight lattice and Weyl group of $\mathfrak{gl}_n$. In preparation for quantum Schur-Weyl duality, we introduce notation for words and tableaux. Finally, we define cells in the general setting of modules with basis, rather than only for W-graphs.

2.1. **General notation.** We work primarily over the ground rings $\mathbf{A} = \mathbb{Z}[u, u^{-1}]$ and $K = \mathbb{Q}(u)$. Define K_0 (resp. K_∞) to be the subring of K consisting of rational functions with no pole at $u = 0$ (resp. $u = \infty$).

Let $\bar{\cdot}$ be the involution of K determined by $\overline{u} = u^{-1}$; it restricts to an involution of $\mathbf{A}$. For a nonnegative integer k, the $\bar{\cdot}$-invariant quantum integer is $[k] := \frac{u^k - u^{-k}}{u - u^{-1}} \in \mathbf{A}$ and the quantum factorial is $[k]! := [k][k-1]\ldots[1]$. We also use the notation $[k]$ to denote the set $\{1, \ldots, k\}$, but these usages should be easy to distinguish from context.

Let (W, S) be a Coxeter group with length function ℓ and Bruhat order $<$. If $\ell(vw) = \ell(v) + \ell(w)$, then $vw = v \cdot w$ is a *reduced factorization.* The *right descent set* of $w \in W$ is $R(w) = \{s \in S : ws < w\}$.

For any $J \subseteq S$, the *parabolic subgroup* W_J is the subgroup of W generated by J. Each left (resp. right) coset wW_J (resp. W_Jw) contains a unique element of minimal length called a minimal coset representative. The set of all such elements is denoted W^J (resp. ${}^JW$).

2.2. **Words and tableaux.** Our results depend heavily on quantum Schur-Weyl duality, so we work almost entirely in type A. The *weight lattice* X of the Lie algebra $\mathfrak{gl}_n$ is $\mathbb{Z}^n$ with standard basis $\epsilon_1, \dots, \epsilon_n$. Its dual, $X^\vee$, has basis $\epsilon_1^\vee, \dots, \epsilon_n^\vee$ dual to the standard. The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}, i \in [n-1]$. We write $\lambda \vdash_l r$ for a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of size $r = |\lambda| := \sum_{i=1}^{l} \lambda_i$. A partition $\lambda \vdash_n r$ is identified with the weight $\lambda_1\epsilon_1 + \dots + \lambda_n\epsilon_n \in X$.

For $\zeta = (\zeta_1, \dots, \zeta_l)$ a weak composition of r, let B_j be the interval $[\sum_{i=1}^{j-1} \zeta_i + 1, \sum_{i=1}^{j} \zeta_i]$, $j \in [l]$. Define $J_\zeta = \{s_i : i, i+1 \in B_j \text{ for some } j\}$ so that $(\mathcal{S}_r)_{J_\zeta} \cong \mathcal{S}_{\zeta_1} \times \dots \times \mathcal{S}_{\zeta_l}$.

Let $\mathbf{k} = k_1 k_2 \dots k_r \in [n]^r$ be a word of length r in the alphabet $[n]$. The *content* of $\mathbf{k}$ is the tuple $(\zeta_1, \dots, \zeta_n)$ whose i-th entry ζ_i is the number of i's in $\mathbf{k}$. The notation $\mathbf{k}^\dagger$ denotes the word $k_r k_{r-1} \dots k_1$. The symmetric group $\mathcal{S}_r$ acts on $[n]^r$ on the right by $\mathbf{k} s_i = k_1 \dots k_{i-1}\, k_{i+1}\, k_i\, k_{i+2} \dots k_r$. Define $\text{sort}(\mathbf{k})$ to be the tuple obtained by rearranging the k_j in weakly increasing order. For a word $\mathbf{k}$ of content ζ, define $d(\mathbf{k})$ (resp. $D(\mathbf{k})$) to be the element w of ${}^{J_\zeta}\mathcal{S}_r$ (resp. $(w_0)_{J_\zeta}\, {}^{J_\zeta}\mathcal{S}_r$ where $(w_0)_{J_\zeta}$ is the longest element of $(\mathcal{S}_r)_{J_\zeta}$) such that $\text{sort}(\mathbf{k}) w = \mathbf{k}$.

The set of standard Young tableaux is denoted SYT, those SYT of size r denoted SYT^r, those SYT^r with at most n rows denoted $\text{SYT}^r_{\leq n}$, and those SYT of shape λ denoted $\text{SYT}(\lambda)$. The set of semistandard Young tableaux of size r with entries in $[n]$ is denoted $\text{SSYT}^r_{[n]}$ and the subset of $\text{SSYT}^r_{[n]}$ of shape $\lambda \vdash r$ is $\text{SSYT}^r_{[n]}(\lambda)$. Tableaux are drawn in English notation, so that entries of a SSYT strictly increase from north to south along columns and weakly increase from west to east along rows. For a tableau T, $|T|$ is the number of squares in T and $\text{sh}(T)$ its shape.

We let $P(\mathbf{k}), Q(\mathbf{k})$ denote the insertion and recording tableaux produced by the Robinson-Schensted-Knuth (RSK) algorithm applied to the word $\mathbf{k}$. We abbreviate $\text{sh}(P(\mathbf{k}))$ simply by $\text{sh}(\mathbf{k})$. Let Z_λ be the superstandard tableau of shape and content λ—the tableau whose i-th row is filled with i's. The conjugate partition λ' of a partition λ is the partition whose diagram is the transpose of that of λ and Q^t denotes the transpose of a SYT Q, so that $\text{sh}(Q^t) = \text{sh}(Q)'$. Lastly, $Q^\dagger$ denotes the Schützenberger involution of a SYT Q (see, e.g., [10, A1.2]).

2.3. **Cells.** We define cells in the general setting of modules with basis. Let H be an R-algebra for some commutative ring R. Let M be a left H-module and Γ an R-basis of M. The preorder $\leq_\Gamma$ (also denoted $\leq_M$) on the vertex set Γ is generated by the relations

$$\delta \preceq_\Gamma \gamma \quad \begin{array}{c}\text{if there is an } h \in H \text{ such that } \delta \text{ appears with non-zero} \\ \text{coefficient in the expansion of } h\gamma \text{ in the basis } \Gamma.\end{array} \tag{2}$$

Equivalence classes of $\leq_\Gamma$ are the *left cells* of (M, Γ). The preorder $\leq_M$ induces a partial order on the left cells of M, which is also denoted $\leq_M$.

A *cellular submodule* of (M, Γ) is a submodule of M that is spanned by a subset of Γ (and is necessarily a union of left cells). A *cellular quotient* of (M, Γ) is a quotient of M by a cellular submodule, and a *cellular subquotient* of (M, Γ) is a cellular quotient of a cellular submodule. We denote a cellular subquotient $R\Gamma'/R\Gamma''$ by $R\Lambda$, where $\Gamma'' \subseteq \Gamma' \subseteq \Gamma$ span cellular submodules and $\Lambda = \Gamma' \setminus \Gamma''$. We say that the left cells Λ and Λ' are isomorphic if $(R\Lambda, \Lambda)$ and $(R\Lambda', \Lambda')$ are isomorphic as modules with basis.

Sometimes we speak of the left cells of M, cellular submodules of M, etc. or left cells of Γ, cellular submodules of Γ, etc. if the pair (M, Γ) is clear from context. For a right H-module M, the *right cells*, *cellular submodules*, etc. of M are defined similarly with γh in place of $h\gamma$ in (2). We also use the terminology H-cells, H-cellular submodules, etc. to make it clear that the algebra H is acting, and we omit left and right when they are clear.

3. Hecke algebras and canonical bases

The *Hecke algebra* $\mathscr{H}(W)$ of (W, S) is the free $\mathbf{A}$-module with standard basis $\{T_w : w \in W\}$ and relations generated by

$$\begin{array}{ll} T_v T_w = T_{vw} & \text{if } vw = v \cdot w \text{ is a reduced factorization,} \\ (T_s - u)(T_s + u^{-1}) = 0 & \text{if } s \in S. \end{array} \tag{3}$$

For each $J \subseteq S$, $\mathscr{H}(W)_J$ denotes the subalgebra of $\mathscr{H}(W)$ with $\mathbf{A}$-basis $\{T_w : w \in W_J\}$, which is isomorphic to $\mathscr{H}(W_J)$.

In this section we recall the definition of the Kazhdan-Lusztig basis elements C_w and C'_w of [23] and some of their basic properties. We record some useful results about how they behave under induction and restriction. Then we specialize to type A and review the beautiful connection between cells and the RSK algorithm.

3.1. **The upper and lower canonical basis of $\mathscr{H}(W)$.** The *bar-involution*, $\bar{\cdot}$, of $\mathscr{H}(W)$ is the additive map from $\mathscr{H}(W)$ to itself extending the $\bar{\cdot}$-involution of $\mathbf{A}$ and satisfying $\overline{T_w} = T_{w^{-1}}^{-1}$. Observe that $\overline{T_s} = T_s^{-1} = T_s + u^{-1} - u$ for $s \in S$. Some simple $\bar{\cdot}$-invariant elements of $\mathscr{H}(W)$ are $C'_{\mathrm{id}} := T_{\mathrm{id}}$, $C_s := T_s - u = T_s^{-1} - u^{-1}$, and $C'_s := T_s + u^{-1} = T_s^{-1} + u$, $s \in S$.

Define the lattices $\mathscr{H}(W)_{\mathbb{Z}[u]} := \mathbb{Z}[u]\{T_w : w \in W\}$ and $\mathscr{H}(W)_{\mathbb{Z}[u^{-1}]} := \mathbb{Z}[u^{-1}]\{T_w : w \in W\}$ of $\mathscr{H}(W)$.

(4) For each $w \in W$, there is a unique element $C_w \in \mathscr{H}(W)$ such that $\overline{C_w} = C_w$ and C_w is congruent to $T_w \mod u\mathscr{H}(W)_{\mathbb{Z}[u]}$.

The $\mathbf{A}$-basis $\Gamma_W := \{C_w : w \in W\}$ is the *upper canonical basis* of $\mathscr{H}(W)$ (we use this language to be consistent with that for crystal bases). Similarly,

(5) for each $w \in W$, there is a unique element $C'_w \in \mathscr{H}(W)$ such that $\overline{C'_w} = C'_w$ and C'_w is congruent to $T_w \mod u^{-1}\mathscr{H}(W)_{\mathbb{Z}[u^{-1}]}$.

The $\mathbf{A}$-basis $\Gamma'_W := \{C'_w : w \in W\}$ is the *lower canonical basis* of $\mathscr{H}(W)$.

The coefficients of the lower canonical basis in terms of the standard basis are the *Kazhdan-Lusztig polynomials* $P'_{x,w}$:

$$C'_w = \sum_{x \in W} P'_{x,w} T_x. \tag{6}$$

(Our $P'_{x,w}$ are equal to $q^{(\ell(x)-\ell(w))/2} P_{x,w}$, where $P_{x,w}$ are the polynomials defined in [23] and $q^{1/2} = u$.) Now let $\mu(x, w) \in \mathbb{Z}$ be the coefficient of u^{-1} in $P'_{x,w}$ (resp. $P'_{w,x}$) if $x \leq w$

(resp. $w \leq x$). Then the right regular representation in terms of the canonical bases of $\mathscr{H}(W)$ takes the following simple forms:

$$C'_w C'_s = \begin{cases} [2]C'_w & \text{if } s \in R(w), \\ \displaystyle\sum_{\{w' \in W : s \in R(w')\}} \mu(w', w) C'_{w'} & \text{if } s \notin R(w). \end{cases} \tag{7}$$

$$C_w C_s = \begin{cases} -[2]C_w & \text{if } s \in R(w), \\ \displaystyle\sum_{\{w' \in W : s \in R(w')\}} \mu(w', w) C_{w'} & \text{if } s \notin R(w). \end{cases} \tag{8}$$

The simplicity and sparsity of this action along with the fact that the right cells of Γ_W and Γ'_W often give rise to $\mathbb{C}(u) \otimes_{\mathbf{A}} \mathscr{H}(W)$-irreducibles are among the most amazing and useful properties of canonical bases.

3.2. Induction and restriction of canonical bases.

It will be important for our applications in §5–7 that canonical bases behave well under induction and restriction.

Let $J \subseteq S$. Let $\mathbf{A}\Lambda'$ (resp. $\mathbf{A}\Lambda$) be a right cellular subquotient of Γ'_{W_J} (resp. Γ_{W_J}). The next proposition follows from general results about inducing W-graphs [17, 18] (see [5, Propositions 2.6 and 3.4]). We will only apply this with $\mathbf{A}\Lambda'$ (resp. $\mathbf{A}\Lambda$) the trivial $\mathscr{H}(W)$ representation, which is a cellular submodule (resp. quotient) of Γ'_W (resp. Γ_W).

Proposition 3.1. *The basis $\Gamma'_{\Lambda',J} := \{C'_w : w = v \cdot x, C'_v \in \Lambda', x \in {}^JW\} \subseteq \Gamma'_W$ of $\mathbf{A}\Lambda' \otimes_{\mathscr{H}(W_J)} \mathscr{H}(W)$ can be constructed from the standard basis $\Lambda T' := \{C'_v \otimes_{\mathscr{H}(W_J)} T_x : C'_v \in \Lambda', x \in {}^JW\}$ in the sense of [9]: C'_{vx} is the unique $\bar{\cdot}$-invariant element of $\mathbb{Z}[u^{-1}]\Lambda T'$ congruent to $C'_v \otimes_{\mathscr{H}(W_J)} T_x \mod u^{-1}\mathbb{Z}[u^{-1}]\Lambda T'$. Hence, $\mathbf{A}\Gamma'_{\Lambda',J}$ is a right cellular subquotient of $\mathscr{H}(W)$. The same statement holds with Λ in place of Λ', Γ_W in place of Γ'_W, C's in place of C''s, and u in place of u^{-1}.*

The next result about restricting canonical bases originated in the work of Barbasch and Vogan on primitive ideals [2], and is proven in the generality stated here by Roichman [29] (see also [5, §3.3]).

Proposition 3.2. *Let $J \subseteq S$ and E be the right $\mathscr{H}(W_J)$-module $\operatorname{Res}_{\mathscr{H}(W_J)}\mathscr{H}(W)$. Then for any $x \in W^J$, $E_x := \mathbf{A}\{C'_{xv} : v \in W_J\}$ is a cellular subquotient of (E, Γ'_W) and*

$$E_x \xrightarrow{\cong} \mathscr{H}(W_J), C'_{xv} \mapsto C'_v \tag{9}$$

is an isomorphism of right $\mathscr{H}(W_J)$-modules with basis. In particular, any right cell of (E, Γ'_W) is isomorphic to one occurring in $\mathscr{H}(W_J)$. The same statement holds for (E, Γ_W), with C's replacing C''s.

3.3. Cells in type A.

Let $\mathscr{H}_r = \mathscr{H}(\mathcal{S}_r)$ be the type A Hecke algebra.

It is well known that $K\mathscr{H}_r := K \otimes_{\mathbf{A}} \mathscr{H}_r$ is semisimple and its irreducibles in bijection with partitions of r; let M_λ and $M_\lambda^{\mathbf{A}}$ be the $K\mathscr{H}_r$-irreducible and Specht module of $\mathscr{H}_r$ of shape $\lambda \vdash r$ (hence $M_\lambda \cong K \otimes_{\mathbf{A}} M_\lambda^{\mathbf{A}}$). For any $K\mathscr{H}_r$-module N and partition λ of r, let $N[\lambda]$ be the M_λ-isotypic component of N. Let $s_\lambda^N : N \twoheadrightarrow N[\lambda]$ be the canonical surjection and $i_\lambda^N : N[\lambda] \hookrightarrow N$ the canonical inclusion. Define the projector $p_\lambda^N : N \to N$

by $p_\lambda^N = i_\lambda^N \circ s_\lambda^N$. We also let p_λ denote central idempotent of $K\mathscr{H}_r$ so that the map p_λ^N is given by multiplication by p_λ.

The work of Kazhdan and Lusztig [23] shows that the decomposition of $\Gamma_{\mathcal{S}_r}$ into right cells is $\Gamma_{\mathcal{S}_r} = \bigsqcup_{P \in \mathrm{SYT}^r} \Gamma_P$, where $\Gamma_P := \{C_w : P(w) = P\}$. Moreover, the right cells $\{\Gamma_P : \mathrm{sh}(P) = \lambda\}$ are all isomorphic, and, denoting any of these cells by Γ_λ, $\mathbf{A}\Gamma_\lambda \cong M_\lambda^{\mathbf{A}}$. Similarly, the decomposition of $\Gamma'_{\mathcal{S}_r}$ into right cells is $\Gamma'_{\mathcal{S}_r} = \bigsqcup_{P \in \mathrm{SYT}^r} \Gamma'_P$, where $\Gamma'_P := \{C'_w : P(w)^t = P\}$. Moreover, the right cells $\{\Gamma'_P : \mathrm{sh}(P) = \lambda\}$ are all isomorphic, and, denoting any of these cells by Γ'_λ, $\mathbf{A}\Gamma'_\lambda \cong M_\lambda^{\mathbf{A}}$. A combinatorial discussion of left cells in type A is given in [5, §4].

We refer to the basis Γ_λ of $M_\lambda^{\mathbf{A}}$ as the *upper canonical basis of* M_λ and denote it by $\{C_Q : Q \in \mathrm{SYT}(\lambda)\}$, where C_Q corresponds to C_w for any (every) $w \in \mathcal{S}_r$ with recording tableau Q. Similarly, the basis Γ'_λ of $M_\lambda^{\mathbf{A}}$ is the *lower canonical basis of* M_λ, denoted $\{C'_Q : Q \in \mathrm{SYT}(\lambda)\}$, where C'_Q corresponds to C'_w for any (every) $w \in \mathcal{S}_r$ with recording tableau Q^t. Note that with these labels the action of C_s on the upper canonical basis of M_λ is similar to (8), with $\mu(Q', Q) := \mu(w', w)$ for any w', w such that $P(w') = P(w)$, $Q' = Q(w'), Q = Q(w)$, and right descent sets

$$R(C_Q) = \{s_i : i+1 \text{ is strictly to the south of } i \text{ in } Q\}. \tag{10}$$

Similarly, the action of C'_s on $\{C'_Q : Q \in \mathrm{SYT}(\lambda)\}$ is similar to (7), with $\mu(Q', Q) := \mu(w', w)$ for any w', w such that $P(w')^t = P(w)^t$, $Q' = Q(w')^t, Q = Q(w)^t$, and right descent sets

$$R(C'_Q) = \{s_i : i+1 \text{ is strictly to the east of } i \text{ in } Q\}. \tag{11}$$

Example 3.3. The integers $\mu(Q', Q)$ for both the upper and lower canonical basis of $M_{(3,1)}$ are given by the following graph (μ is 1 if the edge is present and 0 otherwise)

$$\underset{Q_4}{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}} \;\text{——}\; \underset{Q_3}{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}} \;\text{——}\; \underset{Q_2}{\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}}$$

The right action of the C'_s on $(C'_{Q_4}, C'_{Q_3}, C'_{Q_2})$ is given by (the columns of the matrices are $C'_Q C'_s$ in terms of the C'_Q-basis)

$$C'_{s_1} \mapsto \begin{pmatrix} [2] & 0 & 0 \\ 0 & [2] & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad C'_{s_2} \mapsto \begin{pmatrix} [2] & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & [2] \end{pmatrix} \quad C'_{s_3} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & [2] & 0 \\ 0 & 0 & [2] \end{pmatrix}$$

The right action of the C'_s on $(C_{Q_4}, C_{Q_3}, C_{Q_2})$ is given by

$$C'_{s_1} \mapsto \begin{pmatrix} [2] & 0 & 0 \\ 0 & [2] & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad C'_{s_2} \mapsto \begin{pmatrix} [2] & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & [2] \end{pmatrix} \quad C'_{s_3} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & [2] & 0 \\ 0 & 0 & [2] \end{pmatrix}$$

Note that the matrix corresponding to the right action of C'_s on the C'_Q-basis is transpose to the action in the C_Q-basis. This is true in general—it is a consequence of Proposition 7.7 or of [23, Corollary 3.2].

The partial orders $\leq_{\Gamma_{\mathcal{S}_r}}$ and $\leq_{\Gamma'_{\mathcal{S}_r}}$ are not well understood, but there is the following deep result which gives us some understanding. The result follows from Lusztig's a-invariant and the nonnegativity of the structure constants of the C'_w due to Beilinson-Bernstein-Deligne-Gabber [24, §5-6] and results of [2] and [20] on primitive ideals of $\mathbf{U}$ (see the appendix of [15]).

Theorem 3.4. *The partial order on the right cells of* $\Gamma_{\mathcal{S}_r}$ *and* $\Gamma'_{\mathcal{S}_r}$ *is constrained by dominance order: if* $\Gamma_{P'} <_{\Gamma_{\mathcal{S}_r}} \Gamma_P$*, then* $\mathrm{sh}(P') \lhd \mathrm{sh}(P)$*; if* $\Gamma'_{P'} <_{\Gamma'_{\mathcal{S}_r}} \Gamma'_P$*, then* $\mathrm{sh}(P') \rhd \mathrm{sh}(P)$*.*

4. The quantized enveloping algebra and crystal bases

We recall the definition of the quantized enveloping algebra $\mathbf{U} = U_q(\mathfrak{gl}_n)$ following [21, 16]. We then briefly recall the construction of global crystal bases in the sense of [21, 22] and of the similar notion of based modules of [25].

4.1. Definition of $\mathbf{U} = U_q(\mathfrak{gl}_n)$ and basic properties.

The *quantized universal enveloping algebra* $\mathbf{U}$ is the associative K-algebra generated by $q^h, h \in X^\vee$ (set $K_i = q^{\epsilon_i^\vee - \epsilon_{i+1}^\vee}$) and $E_i, F_i, i \in [n-1]$ with relations

$$\begin{array}{ll} q^0 = 1, & q^h q^{h'} = q^{h+h'}, \\ q^h E_i q^{-h} = u^{\langle \alpha_i, h\rangle} E_i, & q^h F_i q^{-h} = u^{-\langle \alpha_i, h\rangle} F_i, \\ E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{u - u^{-1}}, & \\ E_iE_j - E_jE_i = F_iF_j - F_jF_i = 0 & \text{for } |i-j| > 1, \\ E_i^2 E_j - [2] E_i E_j E_i + E_j E_i^2 = 0 & \text{for } |i-j| = 1, \\ F_i^2 F_j - [2] F_i F_j F_i + F_j F_i^2 = 0 & \text{for } |i-j| = 1. \end{array} \tag{12}$$

Remark 4.1. Our notation is related to that of Kashiwara's and Brundan's [7] by $u = q$. We use u instead of q because on the Hecke algebra side, our u is what is usually $q^{1/2}$.

The *bar-involution*, $\bar{\cdot} : \mathbf{U} \to \mathbf{U}$ is the $\mathbb{Q}$-linear automorphism extending the involution $\bar{\cdot}$ on K and satisfying

$$\overline{q^h} = q^{-h}, \ \overline{E_i} = E_i, \ \overline{F_i} = F_i. \tag{13}$$

Let $\varphi : \mathbf{U} \to \mathbf{U}$ be the algebra antiautomorphism determined by

$$\varphi(E_i) = F_i, \quad \varphi(F_i) = E_i, \quad \varphi(K_i) = K_i. \tag{14}$$

The algebra $\mathbf{U}$ is a Hopf algebra with coproduct Δ given by

$$\Delta(q^h) = q^h \otimes q^h, \ \ \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \ \ \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i. \tag{15}$$

This is the same as the coproduct used in [7, 22, 16], and it differs from the coproduct $\tilde{\Delta}$ of [25] by $(\varphi \otimes \varphi) \circ \Delta \circ \varphi$.

The *weight space* N^ζ of a $\mathbf{U}$-module N for the weight $\zeta \in X$ is the K-vector space $\{x \in N : q^h x = u^{\langle \zeta, h\rangle} x\}$. Let $\mathscr{O}^{\geq 0}_{\mathrm{int}}$ be as in [16, Chapter 7], the category of finite-dimensional $\mathbf{U}$-modules such that the weight of any non-zero weight space belongs to $\mathbb{Z}^n_{\geq 0} \subseteq X$. It is semisimple, the simple objects being the highest weight modules V_λ for partitions λ.

For any object N of $\mathscr{O}_{\text{int}}^{\geq 0}$ and partition λ, let $N[\lambda]$ be the V_λ-isotypic component of N. Set $N[\trianglelefteq \lambda] = \bigoplus_{\mu \trianglelefteq \lambda} N[\mu]$, $N[\triangleleft \lambda] = \bigoplus_{\mu \triangleleft \lambda} N[\mu]$, $N[\trianglerighteq \lambda] = \bigoplus_{\mu \trianglerighteq \lambda} N[\mu]$, and $N[\triangleright \lambda] = \bigoplus_{\mu \triangleright \lambda} N[\mu]$. Let $\varsigma_\lambda^N : N \twoheadrightarrow N[\lambda]$ be the canonical surjection and $\iota_\lambda^N : N[\lambda] \hookrightarrow N$ the canonical inclusion. Define the projector $\pi_\lambda^N : N \to N$ by $\pi_\lambda^N = \iota_\lambda^N \circ \varsigma_\lambda^N$.

4.2. **Crystal bases.** A lower crystal basis at $u = 0$ of an object N of $\mathscr{O}_{\text{int}}^{\geq 0}$ is a pair $(\mathscr{L}_0(N), \mathscr{B}')$, where $\mathscr{L}_0(N)$ is a K_0-submodule of N and $\mathscr{B}'$ is a $\mathbb{Q}$-basis of $\mathscr{L}_0(N)/u\mathscr{L}_0(N)$ which satisfy a certain compatibility with the Kashiwara operators $\tilde{E}_i^{\text{low}}, \tilde{F}_i^{\text{low}}$; an upper crystal basis at $u = \infty$ of N is a pair $(\mathscr{L}_\infty(N), \mathscr{B})$, where $\mathscr{L}_\infty(N)$ is a K_∞-submodule of N and $\mathscr{B}$ is a $\mathbb{Q}$-basis of $\mathscr{L}_\infty(N)/u^{-1}\mathscr{L}_\infty(N)$ which satisfy a certain compatibility with the Kashiwara operators $\tilde{E}_i^{\text{up}}, \tilde{F}_i^{\text{up}}$ (see [22, §3.1]).

Kashiwara [22] gives a fairly explicit construction of a lower (resp. upper) crystal basis of V_λ, which we denote by $(\mathscr{L}_0(\lambda), \mathscr{B}'(\lambda))$ (resp. $(\mathscr{L}_\infty(\lambda), \mathscr{B}(\lambda))$). The basis $\mathscr{B}'(\lambda)$ (resp. $\mathscr{B}(\lambda)$) is naturally labeled by $\text{SSYT}_{[n]}(\lambda)$ and we let b'_P (resp. b_P) denote the basis element corresponding to $P \in \text{SSYT}_{[n]}(\lambda)$ (see, for instance, [16, Chapter 7]). A fundamental result of [21, 22] is that a lower (resp. upper) crystal basis is always isomorphic to a direct sum $\bigoplus_j (\mathscr{L}_0(\lambda^j), \mathscr{B}'(\lambda^j))$ (resp. $\bigoplus_j (\mathscr{L}_\infty(\lambda^j), \mathscr{B}(\lambda^j))$).

4.3. **Global crystal bases.** We next define lower based modules and upper based modules, where a lower based module is a based module in the sense of [25, Chapter 27] adapted to our coproduct.

The **A**-*form* $\mathbf{U}_\mathbf{A}$ *of* $\mathbf{U}$ is the **A**-subalgebra of $\mathbf{U}$ generated by $\frac{E_i^m}{[m]!}, \frac{F_i^m}{[m]!}, q^h, \left\{ {q^h \atop m} \right\}$ for $i \in [n-1]$, $m \in \mathbb{Z}_{\geq 0}$, and $h \in X^\vee$, where

$$\left\{ {x \atop m} \right\} := \prod_{k=1}^{m} \frac{u^{1-k}x - u^{k-1}x^{-1}}{u^k - u^{-k}}.$$

We also define the $\mathbb{Q}[u, u^{-1}]$-*form* $\mathbf{U}_\mathbb{Q}$ *of* $\mathbf{U}$ to be $\mathbb{Q} \otimes_\mathbb{Z} \mathbf{U}_\mathbf{A}$.

Definition 4.2. A *lower based module* is a pair (N, B), where N is an object of $\mathscr{O}_{\text{int}}^{\geq 0}$ and B is a K-basis of N such that

(a) $B \cap N^\zeta$ is a basis of N^ζ, for any $\zeta \in X$;
(b) Define $N_\mathbf{A} := \mathbf{A}B$. The $\mathbb{Q}[u, u^{-1}]$-submodule $\mathbb{Q} \otimes_\mathbb{Z} N_\mathbf{A}$ of N is stable under $\mathbf{U}_\mathbb{Q}$;
(c) the $\mathbb{Q}$-linear involution $\bar{\cdot} : N \to N$ defined by $\overline{ab} = \overline{a}b$ for all $a \in K$ and all $b \in B$ intertwines the $\bar{\cdot}$-involution of $\mathbf{U}$, i.e. $\overline{fn} = \overline{f}\overline{n}$ for all $f \in \mathbf{U}, n \in N$;
(d) Set $\mathscr{L}_0(N) = K_0 B$ and let $\mathscr{B}$ denote the image of B in $\mathscr{L}_0(N)/u\mathscr{L}_0(N)$. Then $(\mathscr{L}_0(N), \mathscr{B})$ is a lower crystal basis of N at $u = 0$.

Definition 4.3. An *upper based module* is the same as a lower based module except with condition (d) replaced by

Set $\mathscr{L}_\infty(N) = K_\infty B$ and let $\mathscr{B}$ denote the image of B in $\mathscr{L}_\infty(N)/u^{-1}\mathscr{L}_\infty(N)$. Then $(\mathscr{L}_\infty(N), \mathscr{B})$ is an upper crystal basis of N at $u = \infty$.

The $\bar{\cdot}$-*involution* of the lower (resp. upper) based module is the involution on N defined in (c). The *balanced triple* of a lower (resp. upper) based module is $(\mathbb{Q}[u,u^{-1}]B, K_0B, K_\infty B)$.

Remark 4.4. For simplicity and to be consistent with the treatment of upper global crystal bases in [22], we have used the $\mathbb{Q}[u,u^{-1}]$-form $\mathbf{U}_\mathbb{Q}$ from [22] rather than the $\mathbf{A}$-form of $\dot{\mathbf{U}}$ defined in [25].

Remark 4.5. In the language of Kashiwara [22], the basis B in the definitions above is a lower or upper *global crystal basis.* Kashiwara defines a triple $(N_\mathbb{Q}, \mathscr{L}_0, \mathscr{L}_\infty)$ to be *balanced* if the canonical surjection $N_\mathbb{Q} \cap \mathscr{L}_0 \cap \mathscr{L}_\infty \to \mathscr{L}_0/u\mathscr{L}_0$ is an isomorphism, where N is any K-vector space, and $N_\mathbb{Q}$, $\mathscr{L}_0$, $\mathscr{L}_\infty$ are any $\mathbb{Q}[u,u^{-1}]$-submodule, K_0-submodule, and K_∞-submodule of N, respectively. To define global lower crystal bases, Kashiwara first defines a balanced triple $(\mathbb{Q}\otimes_\mathbb{Z} N_\mathbf{A}, \mathscr{L}_0(N), \overline{\mathscr{L}_0(N)})$ and a basis $\mathscr{B} \subseteq \mathscr{L}_0/u\mathscr{L}_0$ and then defines B to be the inverse image of $\mathscr{B}$ under the isomorphism

$$\mathbb{Q}\otimes_\mathbb{Z} N_\mathbf{A} \cap \mathscr{L}_0(N) \cap \overline{\mathscr{L}_0(N)} \xrightarrow{\cong} \mathscr{L}_0/u\mathscr{L}_0.$$

Global upper canonical bases are defined similarly.

Let η_λ be a highest weight vector of V_λ. The $\bar{\cdot}$-involution on V_λ is defined by setting $\overline{\eta_\lambda} = \eta_\lambda$ and requiring that it intertwines the $\bar{\cdot}$-involution of $\mathbf{U}$. The $\mathbb{Q}[u,u^{-1}]$-forms of V_λ of [22] are denoted $V_\lambda^{\mathbb{Q}\text{ low}}$ and $V_\lambda^{\mathbb{Q}\text{ up}}$; $V_\lambda^{\mathbb{Q}\text{ low}}$ is defined to be $\mathbf{U}_\mathbb{Q}\eta_\lambda$ and $V_\lambda^{\mathbb{Q}\text{ up}}$ is defined by dualizing $V_\lambda^{\mathbb{Q}\text{ low}}$ by a symmetric form on V_λ. We can now state the fundamental result about the existence of global crystal bases and based modules for V_λ.

Theorem 4.6 (Kashiwara [21, 22])**.**

(i) *The triple* $(V_\lambda^{\mathbb{Q}\ low}, \mathscr{L}_0(\lambda), \overline{\mathscr{L}_0(\lambda)})$ *is balanced. Then, letting* G'_λ *be the inverse of the canonical isomorphism*

$$V_\lambda^{\mathbb{Q}\ low} \cap \mathscr{L}_0(\lambda) \cap \overline{\mathscr{L}_0(\lambda)} \xrightarrow{\cong} \mathscr{L}_0(\lambda)/u\mathscr{L}_0(\lambda),$$

$B'(\lambda) := G'_\lambda(\mathscr{B}'(\lambda))$ *is the* lower global crystal basis of V_λ *and* $(V_\lambda, B'(\lambda))$ *is a lower based module.*

(ii) *The triple* $(V_\lambda^{\mathbb{Q}\ up}, \overline{\mathscr{L}_\infty(\lambda)}, \mathscr{L}_\infty(\lambda))$ *is balanced. Then, letting* G_λ *be the inverse of the canonical isomorphism*

$$V_\lambda^{\mathbb{Q}\ up} \cap \overline{\mathscr{L}_\infty(\lambda)} \cap \mathscr{L}_\infty(\lambda) \xrightarrow{\cong} \mathscr{L}_\infty(\lambda)/u^{-1}\mathscr{L}_\infty(\lambda),$$

$B(\lambda) := G_\lambda(\mathscr{B}(\lambda))$ *is the* upper global crystal basis of V_λ *and* $(V_\lambda, B(\lambda))$ *is an upper based module.*

Note that Kashiwara proves that the triples are balanced and the conclusions about based modules follow easily (see [25, 27.1.4] or [16, Theorem 6.2.2]). We may now define integral forms $V_\lambda^{\mathbf{A}\text{ low}} := \mathbf{A}B'(\lambda)$ and $V_\lambda^{\mathbf{A}\text{ up}} := \mathbf{A}B(\lambda)$ of V_λ.

We wish to make use of some of the facts established about lower based modules in [25, Chapter 27] and their corresponding statements for upper based modules. It is shown in [25, Chapter 27] that if (N, B) is a lower based module, then so are $(N[\trianglerighteq \lambda], B[\trianglerighteq \lambda])$ and $(N[\trianglerighteq \lambda]/N[\triangleright \lambda], B[\trianglerighteq \lambda] - B[\triangleright \lambda])$, where $B[\trianglerighteq \lambda] = N[\trianglerighteq \lambda] \cap B$, etc. Moreover, this last based module is isomorphic to a direct sum of copies of V_λ with their lower global canonical bases. The analogous statements for upper based modules are true with $\trianglerighteq$ replaced by $\trianglelefteq$ and are shown in [22, §5.2].

4.4. **Tensor products of based modules.** Let $(N, B), (N', B')$ be lower (resp. upper) based modules. There is a basis $B \diamond B'$ (resp. $B \heartsuit B'$) which makes $N \otimes N'$ into a lower (resp. upper) based module. However, first, we need an involution on $N \otimes N'$ that intertwines the $\bar{\cdot}$-involution on $\mathbf{U}$. This definition is not obvious and requires Lusztig's quasi-R-matrix, but adapted to our coproduct as in [7]: let $\Theta = (\varphi \otimes \varphi)(\tilde{\Theta}^{-1})$ where $\tilde{\Theta}$ is exactly Lusztig's quasi-$\mathcal{R}$-matrix from [25, 4.1.2]. It is an element of a certain completion $(\mathbf{U} \otimes \mathbf{U})^\wedge$ of the algebra $\mathbf{U} \otimes \mathbf{U}$. Then the involution $\bar{\cdot} : N \otimes N' \to N \otimes N'$ is defined by $\overline{n \otimes n'} = \Theta(\overline{n} \otimes \overline{n'})$. (This involution is denoted Ψ in [25].)

Theorem 4.7 (Lusztig [25, Theorem 27.3.2])**.** *Maintain the notation above with $(N, B), (N', B')$ lower based modules and set $(N \otimes N')_{\mathbb{Z}[u]} = \mathbb{Z}[u] B \otimes B'$. For any $(b, b') \in B \times B'$, there is a unique element $b \diamond b' \in (N \otimes N')_{\mathbb{Z}[u]}$ such that $\overline{b \diamond b'} = b \diamond b'$ and $(b \diamond b') - b \otimes b' \in u(N \otimes N')_{\mathbb{Z}[u]}$.*

Set $B \diamond B' = \{b \diamond b' : b \in B, b' \in B'\}$. Then the pair $(N \otimes N', B \diamond B')$ is a lower based module.

There is a similar theorem for upper based modules, as the proof of Theorem 4.7 adapts easily. This is discussed in [12] in the $n = 2$ case, and we use the notation $\heartsuit$ for this product as is done there.

Theorem 4.8. *Maintain the notation above with $(N, B), (N', B')$ upper based modules and set $(N \otimes N')_{\mathbb{Z}[u^{-1}]} = \mathbb{Z}[u^{-1}] B \otimes B'$. For any $(b, b') \in B \times B'$, there is a unique element $b \heartsuit b' \in (N \otimes N')_{\mathbb{Z}[u^{-1}]}$ such that $\overline{b \heartsuit b'} = b \heartsuit b'$ and $(b \heartsuit b') - b \otimes b' \in u^{-1}(N \otimes N')_{\mathbb{Z}[u^{-1}]}$.*

Set $B \heartsuit B' = \{b \heartsuit b' : b \in B, b' \in B'\}$. Then the pair $(N \otimes N', B \heartsuit B')$ is an upper based module.

Moreover, the products $\diamond$ and $\heartsuit$ are associative ([25, 27.3.6]).

5. Quantum Schur-Weyl duality and canonical bases

Write V for the natural representation V_{ϵ_1} of $\mathbf{U}$. The action of $\mathbf{U}$ on the weight basis $v_1, \dots, v_n$ of V is given by $q^{\epsilon_i^\vee} v_j = u^{\delta_{ij}} v_j$, $F_i v_i = v_{i+1}$, $F_i v_j = 0$ for $i \neq j$, and $E_i v_{i+1} = v_i$, $E_i v_j = 0$ for $j \neq i + 1$.

We recall the commuting actions of $\mathbf{U}$ and $\mathscr{H}_r$ on $\mathbf{T} := V^{\otimes r}$ as described in [19, 14, 27, 11, 7] and give several characterizations of the lower and upper canonical basis of $\mathbf{T}$; we closely follow [7] and are consistent with its conventions.

5.1. **Commuting actions on $\mathbf{T} = V^{\otimes r}$.** The action of $\mathbf{U}$ on $\mathbf{T}$ is determined by the coproduct Δ (15). The commuting action of $\mathscr{H}_r$ on $\mathbf{T}$ comes from a $\mathbf{U}$-isomorphism $\mathcal{R}_{V,V} : V \otimes V \to V \otimes V$ determined by the universal R-matrix; this isomorphism can also be defined using the quasi-$\mathcal{R}$-matrix [25, 32.1.5] (see also [7, §3]). The $\mathscr{H}_r$ action is given explicitly on generators as follows: for a word $\mathbf{k} = k_1 \dots k_r \in [n]^r$, let $\mathbf{v_k} =$

$v_{k_1} \otimes v_{k_2} \otimes \ldots \otimes v_{k_r}$ be the corresponding tensor monomial. Recall from §2.2 the right action of $\mathcal{S}_r$ on words of length r. Then

$$\mathbf{v_k} T_i^{-1} = \begin{cases} \mathbf{v}_{\mathbf{k}\, s_i} & \text{if } k_i < k_{i+1}, \\ u^{-1}\mathbf{v_k} & \text{if } k_i = k_{i+1}, \\ (u^{-1} - u)\mathbf{v_k} + \mathbf{v}_{\mathbf{k}\, s_i} & \text{if } k_i > k_{i+1}. \end{cases} \tag{16}$$

Remark 5.1. This convention for the action of $\mathscr{H}_r$ on $\mathbf{T}$ is consistent with that in [7, 27, 26, 3] and [11, Proposition 2.1′], but not with that in [14] and [11, Proposition 2.1]. Note that $\mathbf{v_k}, T_i^{-1}$ are denoted M_α, H_i respectively in [7].

We can now state the beautiful quantum version of Schur-Weyl duality, originally due to Jimbo [19].

Theorem 5.2. *As a $(\mathbf{U}, K\mathscr{H}_r)$-bimodule, $\mathbf{T}$ decomposes into irreducibles as*

$$\mathbf{T} \cong \bigoplus_{\lambda \vdash_n r} V_\lambda \otimes M_\lambda.$$

As an $\mathscr{H}_r$-module, $\mathbf{T}$ decomposes into a direct sum of weight spaces: $\mathbf{T} \cong \bigoplus_{\zeta \in X} \mathbf{T}^\zeta$. The weight space $\mathbf{T}^\zeta$ is the K-vector space spanned by $\mathbf{v_k}$ such that $\mathbf{k}$ has content ζ. Let $\epsilon_+ := M^{\mathbf{A}}_{(r)}$ be the trivial $\mathscr{H}_r$-module, i.e. the one-dimensional module identified with the map $\mathscr{H}_r \to \mathbf{A}$, $T_i \mapsto u$. It is not difficult to prove using (16) (see [7, §4])

Proposition 5.3. *The map $\mathbf{T}^\zeta_{\mathbf{A}} \to \epsilon_+ \otimes_{\mathscr{H}_{J_\zeta}} \mathscr{H}_r$ given by $\mathbf{v_k} \mapsto \epsilon_+ \otimes_{\mathscr{H}_{J_\zeta}} \overline{T}_{d(\mathbf{k})}$ is an isomorphism of right $\mathscr{H}_r$-modules.*

Here $\mathbf{T_A}$ is the integral form of $\mathbf{T}$, defined below.

5.2. **Lower canonical basis of T.** We now apply the general theory of §4 to construct global crystal bases of $\mathbf{T}$. Recall from §4.4 that there is a $\bar{\cdot}$-involution on $\mathbf{T}$ defined using the quasi-$\mathcal{R}$-matrix. The $\bar{\cdot}$-involution on $\mathscr{H}_r$ intertwines that of $\mathbf{T}$, i.e.

$$\overline{vh} = \overline{v}\ \overline{h}, \text{ for any } v \in \mathbf{T},\ h \in \mathscr{H}_r. \tag{17}$$

This follows easily from the identity $\tilde{\Theta}^{-1} = \overline{\tilde{\Theta}}$ from [25]; see [7].

Let $V_{\mathbf{A}} = \mathbf{A}\{v_i : i \in [n]\}$, which is the same as the integral forms $V_{\epsilon_1}^{\mathbf{A}\text{ low}} = V_{\epsilon_1}^{\mathbf{A}\text{ up}}$ from §4.3. By Theorem 4.7 and associativity of the $\diamond$ product, $(\mathbf{T}, B')$ is a lower based module with balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{T_A}, \mathscr{L}_0, \overline{\mathscr{L}_0})$, where

$$\begin{array}{rcl} \mathscr{L}_0 & := & \mathscr{L}_0(\epsilon_1) \otimes_{K_0} \ldots \otimes_{K_0} \mathscr{L}_0(\epsilon_1), \\ \mathscr{B}' & := & \mathscr{B}'(\epsilon_1) \times \cdots \times \mathscr{B}'(\epsilon_1) \subseteq \mathscr{L}_0/u\mathscr{L}_0, \\ \mathbf{T}_{\mathbb{Z}[u]} & := & \mathbb{Z}[u]\{\mathbf{v_k} : \mathbf{k} \in [n]^r\}, \\ \mathbf{T_A} & := & V_{\mathbf{A}} \otimes_{\mathbf{A}} \ldots \otimes_{\mathbf{A}} V_{\mathbf{A}} = \mathbf{A} \otimes_{\mathbb{Z}} \mathbf{T}_{\mathbb{Z}[u]}, \\ B' & := & B'(\epsilon_1) \diamond \cdots \diamond B'(\epsilon_1). \end{array} \tag{18}$$

We call B' the *lower canonical basis* of $\mathbf{T}$ and, for each $\mathbf{k} \in [n]^r$, we write $c'_{\mathbf{k}}$ for the element $v_{k_1} \diamond \cdots \diamond v_{k_r} \in B'$ and $b'_{\mathbf{k}} \in \mathscr{B}'$ for its image in $\mathscr{L}_0/u\mathscr{L}_0$. Figure 1 gives the lower canonical basis in terms of the monomial basis for $r = 3, n = 2$.

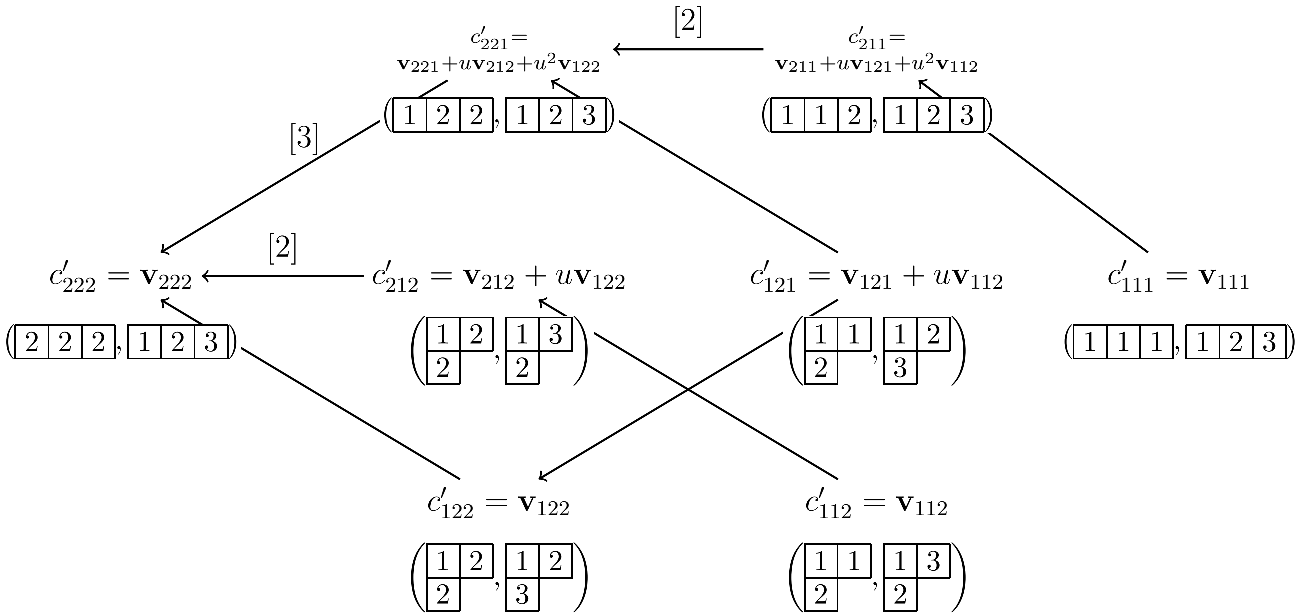


Figure 1: An illustration of Corollary 5.5 for $r = 3, n = 2$. The pairs of tableaux are of the form $(P(\mathbf{k}^\dagger), Q(\mathbf{k}^\dagger))$. The arrows and their coefficients give the action of F_1 on the lower canonical basis.

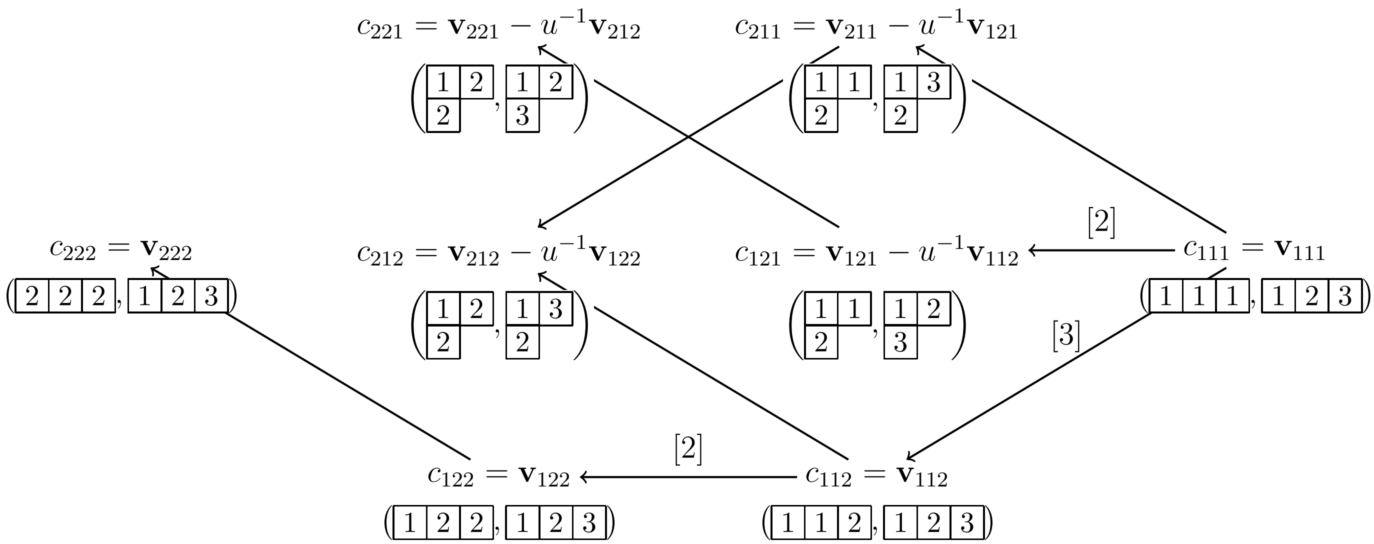


Figure 2: An illustration of Corollary 5.7 for $r = 3, n = 2$. The pairs of tableaux are of the form $(P(\mathbf{k}), Q(\mathbf{k}))$. The arrows and their coefficients give the action of F_1 on the upper canonical basis.

We assemble some equivalent descriptions of the lower canonical basis of $\mathbf{T}$, which are also shown in [7] and appear in a slightly different form in [14, 11].

Theorem 5.4. *The lower canonical basis element* $c'_{\mathbf{k}}$, $\mathbf{k} \in [n]^r$, *has the following equivalent descriptions*

(i) *the unique* $\bar{\cdot}$*-invariant element of* $\mathbf{T}_{\mathbb{Z}[u]}$ *congruent to* $\mathbf{v}_{\mathbf{k}} \mod u\mathbf{T}_{\mathbb{Z}[u]}$;
(ii) $v_{k_1} \diamond \cdots \diamond v_{k_r}$;
(iii) $G'(b'_{\mathbf{k}})$, *where* G' *is the inverse of the canonical isomorphism*

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{T_A}) \cap \mathscr{L}_0 \cap \overline{\mathscr{L}_0} \xrightarrow{\cong} \mathscr{L}_0/u\mathscr{L}_0;$$

(iv) *The image of* $C'_{D(\mathbf{k})}$ *under the isomorphism in Proposition 5.3 (*$D(\mathbf{k})$ *is a maximal coset representative, defined in §2.2).*

Proof. Description (i) is the definition of (ii) (Theorem 4.7) and the element in (iii) is easily seen to satisfy the conditions in (i) (see Remark 4.5 and Theorem 4.6). The element in (iv) satisfies the conditions in (i) by Proposition 3.1 and the combination of Proposition 5.3 and (17). Note that we are actually applying an easy modification of Proposition 3.1 with u in place of u^{-1} and $C'_v \otimes_{\mathscr{H}(W_J)} \overline{T}_x$ in place of $C'_v \otimes_{\mathscr{H}(W_J)} T_x$. □

Proposition 3.1 and the discussion in §3.3, results of [25, Chapter 27] (see the discussion at the end of §4.3), and the well-known combinatorics of the crystal basis $\mathscr{B}'$ (see e.g. [16, Chapter 7]) allow us to determine the $\mathscr{H}_r$- and $\mathbf{U}$-cells of $(\mathbf{T}, B')$.

Corollary 5.5.

(i) *The* $\mathscr{H}_r$*-module with basis* $(\mathbf{T}, B')$ *decomposes into* $\mathscr{H}_r$*-cells as* $B' = \bigsqcup_{T \in SSYT^r_{[n]}} \Gamma'_T$, *where* $\Gamma'_T = \{c'_{\mathbf{k}} : P(\mathbf{k}^\dagger) = T\}$.
(ii) *The* $\mathscr{H}_r$*-cell* Γ'_T *of* $\mathbf{T}$ *is isomorphic to* $\Gamma'_{\mathrm{sh}(T)}$.
(iii) *The* $\mathbf{U}$*-module with basis* $(\mathbf{T}, B')$ *decomposes into* $\mathbf{U}$*-cells as* $B' = \bigsqcup_{T \in SYT^r_{\leq n}} \Lambda'_T$, *where* $\Lambda'_T = \{c'_{\mathbf{k}} : Q(\mathbf{k}^\dagger) = T\}$.
(iv) *The* $\mathbf{U}$*-cell* Λ'_T *is isomorphic to* $B'(\mathrm{sh}(T))$.

5.3. **Upper canonical basis of T.** By Theorem 4.8 and associativity of the ♡ product, $(\mathbf{T}, B)$ is an upper based module with balanced triple $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbf{T_A}, \overline{\mathscr{L}_\infty}, \mathscr{L}_\infty)$, where

$$\begin{array}{rcl} \mathscr{L}_\infty & := & \mathscr{L}_\infty(\epsilon_1) \otimes_{K_\infty} \ldots \otimes_{K_\infty} \mathscr{L}_\infty(\epsilon_1), \\ \mathscr{B} & := & \mathscr{B}(\epsilon_1) \times \cdots \times \mathscr{B}(\epsilon_1) \subseteq \mathscr{L}_\infty/u^{-1}\mathscr{L}_\infty, \\ \mathbf{T}_{\mathbb{Z}[u^{-1}]} & := & \mathbb{Z}[u^{-1}]\{\mathbf{v}_{\mathbf{k}} : \mathbf{k} \in [n]^r\}, \\ B & := & B(\epsilon_1) \heartsuit \ldots \heartsuit B(\epsilon_1). \end{array} \tag{19}$$

We call B the *upper canonical basis* of $\mathbf{T}$ and, for each $\mathbf{k} \in [n]^r$, we write $c_{\mathbf{k}}$ for the element $v_{k_1} \heartsuit \ldots \heartsuit v_{k_r} \in B$ and $b_{\mathbf{k}} \in \mathscr{B}$ for its image in $\mathscr{L}_\infty/u^{-1}\mathscr{L}_\infty$. Figure 2 gives the upper canonical basis in terms of the monomial basis for $r = 3, n = 2$.

We assemble some equivalent descriptions of the upper canonical basis of $\mathbf{T}$. The proof is similar to the corresponding Theorem 5.4 for the lower canonical basis.

Theorem 5.6. *The upper canonical basis element* $c_{\mathbf{k}}$, $\mathbf{k} \in [n]^r$, *has the following equivalent descriptions*

(i) *the unique* $\bar{\cdot}$*-invariant element of* $\mathbf{T}_{\mathbb{Z}[u^{-1}]}$, *congruent to* $\mathbf{v}_{\mathbf{k}} \mod u^{-1}\mathbf{T}_{\mathbb{Z}[u^{-1}]}$;

(ii) $v_{k_1}\heartsuit\dots\heartsuit v_{k_r}$;
(iii) $G(b_{\mathbf{k}})$, *where* G *is the inverse of the canonical isomorphism*

$$(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbf{T_A})\cap\overline{\mathscr{L}_\infty}\cap\mathscr{L}_\infty\xrightarrow{\cong}\mathscr{L}_\infty/u^{-1}\mathscr{L}_\infty;$$

(iv) *The image of* $C_{d(\mathbf{k})}$ *under the isomorphism in Proposition 5.3.*

We also have the upper canonical basis version of Corollary 5.5.

Corollary 5.7.

(i) *The* $\mathscr{H}_r$*-module with basis* $(\mathbf{T}, B)$ *decomposes into* $\mathscr{H}_r$*-cells as* $B=\bigsqcup_{T\in SSYT^r_{[n]}}\Gamma_T$, *where* $\Gamma_T=\{c'_{\mathbf{k}}:P(\mathbf{k})=T\}$.
(ii) *The* $\mathscr{H}_r$*-cell* Γ_T *of* $\mathbf{T}$ *is isomorphic to* $\Gamma_{\mathrm{sh}(T)}$.
(iii) *The* $\mathbf{U}$*-module with basis* $(\mathbf{T}, B)$ *decomposes into* $\mathbf{U}$*-cells as* $B=\bigsqcup_{T\in SYT^r_{\leq n}}\Lambda_T$, *where* $\Lambda_T=\{c'_{\mathbf{k}}:Q(\mathbf{k})=T\}$.
(iv) *The* $\mathbf{U}$*-cell* Λ_T *is isomorphic to* $B(\mathrm{sh}(T))$.

5.4. **A symmetric bilinear form on T.** There is a bilinear form $(\cdot,\cdot)$ on $\mathbf{T}$ under which the upper and lower canonical basis are dual and satisfies several other nice properties. Let $\dagger$ (resp. $\dagger^{\mathrm{op}}$) be the automorphism (resp. antiautomorphism) of $\mathscr{H}_r$ determined by $T_i^\dagger=T_{n-i}$ (resp. $T_i^{\dagger^{\mathrm{op}}}=T_{n-i}$).

Proposition-Definition 5.8. [7] *There is a unique symmetric bilinear form* $(\cdot,\cdot)$ *on* $\mathbf{T}$ *satisfying*

(i) $(x\mathbf{v},\mathbf{v}')=(\mathbf{v},\varphi(x)\mathbf{v}')$ *for any* $x\in\mathbf{U}$, $\mathbf{v},\mathbf{v}'\in\mathbf{T}$,
(ii) $(\mathbf{v}h,\mathbf{v}')=(\mathbf{v},\mathbf{v}'h^{\dagger^{op}})$ *for any* $h\in\mathscr{H}_r$, $\mathbf{v},\mathbf{v}'\in\mathbf{T}$,
(iii) $(\mathbf{v_k},\overline{\mathbf{v}}_{\mathbf{l}^\dagger})=\delta_{\mathbf{k},\mathbf{l}}$,
(iv) $(c_{\mathbf{k}},c'_{\mathbf{l}^\dagger})=\delta_{\mathbf{k},\mathbf{l}}$.

6. Projected canonical bases

Here we give several equivalent definitions of the projected counterparts of the lower and upper canonical basis of $\mathbf{T}$. This will be used in the next section to help us understand the transition matrices discussed in the introduction. Note that by quantum Schur-Weyl duality (Theorem 5.2), $\varsigma^{\mathbf{T}}_\lambda=s^{\mathbf{T}}_\lambda$, $\iota^{\mathbf{T}}_\lambda=i^{\mathbf{T}}_\lambda$, and $\pi^{\mathbf{T}}_\lambda=p^{\mathbf{T}}_\lambda$.

6.1. **Projected upper canonical basis.** For some of our descriptions of the projected upper canonical basis, we need an integral form that is different from $\mathbf{T_A}$. First note that by Corollary 5.7 and [22, §5.2],

$$\mathbf{T}[\trianglelefteq\lambda]=K\{c_{\mathbf{k}}:\mathrm{sh}(\mathbf{k})\trianglelefteq\lambda\}\text{ and }\mathbf{T}[\triangleleft\lambda]=K\{c_{\mathbf{k}}:\mathrm{sh}(\mathbf{k})\triangleleft\lambda\}.\tag{20}$$

Further, applying $\varsigma^{\mathbf{T}}_\lambda$ to the upper based module $(\mathbf{T}[\trianglelefteq\lambda],\{c_{\mathbf{k}}:\mathrm{sh}(\mathbf{k})\trianglelefteq\lambda\})$ yields the upper based module $(\mathbf{T}[\lambda],\{\varsigma^{\mathbf{T}}_\lambda(c_{\mathbf{k}}):\mathrm{sh}(\mathbf{k})=\lambda\})$ with balanced triple

$$(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbf{T_A}[\trianglelefteq\lambda]/\mathbf{T_A}[\triangleleft\lambda],\overline{\mathscr{L}_\infty[\trianglelefteq\lambda]}/\overline{\mathscr{L}_\infty[\triangleleft\lambda]},\mathscr{L}_\infty[\trianglelefteq\lambda]/\mathscr{L}_\infty[\triangleleft\lambda]),\tag{21}$$

where $\mathbf{T_A}[\trianglelefteq \lambda]$ and $\mathscr{L}_\infty[\trianglelefteq \lambda]$ (resp. $\mathbf{T_A}[\triangleleft\lambda]$ and $\mathscr{L}_\infty[\triangleleft\lambda]$) are the $\mathbf{A}$- and K_∞- span of $\{c_\mathbf{k} : \mathrm{sh}(\mathbf{k}) \trianglelefteq \lambda\}$ (resp. $\{c_\mathbf{k} : \mathrm{sh}(\mathbf{k}) \triangleleft \lambda\}$). Finally, define

$$\begin{array}{rcl} (\mathbf{T_A})_\lambda & := & \pi^\mathbf{T}_\lambda(\mathbf{T_A}[\trianglelefteq \lambda]), \\ \mathscr{L}_{\infty\lambda} & := & \pi^\mathbf{T}_\lambda(\mathscr{L}_\infty[\trianglelefteq \lambda]), \\ \tilde{\mathbf{T}}_\mathbf{A} & := & \bigoplus_\lambda (\mathbf{T_A})_\lambda. \end{array} \tag{22}$$

Our next theorem is similar to results in [14] and [7, §7]; those of [14] are proved using geometric methods, and those of [7, §7] using results of [22, 25] as is done here.

Theorem 6.1. *Maintain the notation above and let* $\mathbf{l} \in [n]^r$ *and* $\lambda = \mathrm{sh}(\mathbf{l})$. *Set* $\mathbf{j} = RSK^{-1}(Z_\lambda, Q(\mathbf{l}))$, *where* Z_λ *is the superstandard tableau of shape* λ *(see §2.2). Let* $V_{Q(\mathbf{l})} = \mathbf{U}c_\mathbf{j}$ *and* $V_{Q(\mathbf{l})}^{\mathbb{Q}\ up}$ *be the* $\mathbb{Q}[u, u^{-1}]$*-form of* $V_{Q(\mathbf{l})}$ *as in §4.3. Then the triples in (b) and (c) are balanced and the projected upper canonical basis element* $\tilde{c}_\mathbf{l}$ *has the following descriptions*

(a) *the unique* $\bar{\cdot}$*-invariant element of* $\tilde{\mathbf{T}}_\mathbf{A}$ *congruent to* $\mathbf{v_l} \mod u^{-1}\mathscr{L}_\infty$,

(b) $\tilde{G}(b_\mathbf{l})$, *where* $\tilde{G}$ *is the inverse of the canonical isomorphism*

$$(\mathbb{Q} \otimes_\mathbb{Z} \tilde{\mathbf{T}}_\mathbf{A}) \cap \overline{\mathscr{L}_\infty} \cap \mathscr{L}_\infty \xrightarrow{\cong} \mathscr{L}_\infty / u^{-1}\mathscr{L}_\infty,$$

(c) $\tilde{G}_\lambda(\pi_\lambda(b_\mathbf{l}))$, *where* $\tilde{G}_\lambda$ *is the inverse of the canonical isomorphism*

$$(\mathbb{Q} \otimes_\mathbb{Z} (\mathbf{T_A})_\lambda) \cap \overline{\mathscr{L}_{\infty\lambda}} \cap \mathscr{L}_{\infty\lambda} \xrightarrow{\cong} \mathscr{L}_{\infty\lambda} / u^{-1}\mathscr{L}_{\infty\lambda},$$

(d) *the global crystal basis element* $G_\lambda(b_{P(\mathbf{l})})$ *of* $V_{Q(\mathbf{l})}$,

(e) $\pi^\mathbf{T}_\lambda(c_\mathbf{l})$,

(f) $p^\mathbf{T}_\lambda(c_\mathbf{l})$.

Then $\tilde{B} := \{\tilde{c}_\mathbf{k} : \mathbf{k} \in [n]^r\}$ *is the* projected upper canonical basis of $\mathbf{T}$ *and* $(\mathbf{T}, \tilde{B})$ *is an upper based module. Its* $\mathbf{U}$*- and* $\mathscr{H}_r$*-cells are given by Corollary 5.7 with* $\tilde{c}$ *in place of* c.

Proof. The triple in (c) is just the injective image of the triple in (21) under $\iota^\mathbf{T}_\lambda$, hence the triple in (c) is balanced and the elements in (c) and (e) are the same. As noted earlier, $\pi^\mathbf{T}_\lambda = p^\mathbf{T}_\lambda$, so the elements in (e) and (f) are the same.

By [22, §5.2] (see the discussion at the end of §4.3) and Corollary 5.7, there is an isomorphism of upper based modules

$$(\mathbf{T}[\lambda], \{\varsigma^\mathbf{T}_\lambda(c_\mathbf{k}) : \mathrm{sh}(\mathbf{k}) = \lambda\}) \cong \bigoplus_{Q \in \mathrm{SYT}(\lambda)} (V^Q_\lambda, B^Q(\lambda)), \quad \varsigma^\mathbf{T}_\lambda(c_\mathbf{k}) \mapsto G^{Q(\mathbf{k})}_\lambda(b^{Q(\mathbf{k})}_{P(\mathbf{k})}),$$

where $V^Q_\lambda, B^Q(\lambda)$, etc. denote copies of $V_\lambda, B(\lambda)$, etc. indexed by Q. It follows that the elements in (c) and (d) are the same.

To see that the triple in (b) is balanced and that the elements in (b) and (c) are the same, we must show that the triple in (b) is the direct sum $\bigoplus_{\mu \vdash_n r}(\mathbb{Q} \otimes_\mathbb{Z} (\mathbf{T_A})_\mu, \overline{\mathscr{L}_{\infty\mu}}, \mathscr{L}_{\infty\mu})$ of triples of the form in (c). This amounts to showing the equality of upper crystal bases (we need that this is an equality, not just an isomorphism)

$$(\mathscr{L}_\infty, \mathscr{B}) = \bigoplus_{\mu \vdash_n r} (\mathscr{L}_{\infty\mu}, \{\pi_\mu(b_\mathbf{k}) : \mathrm{sh}(\mathbf{k}) = \mu\}). \tag{23}$$

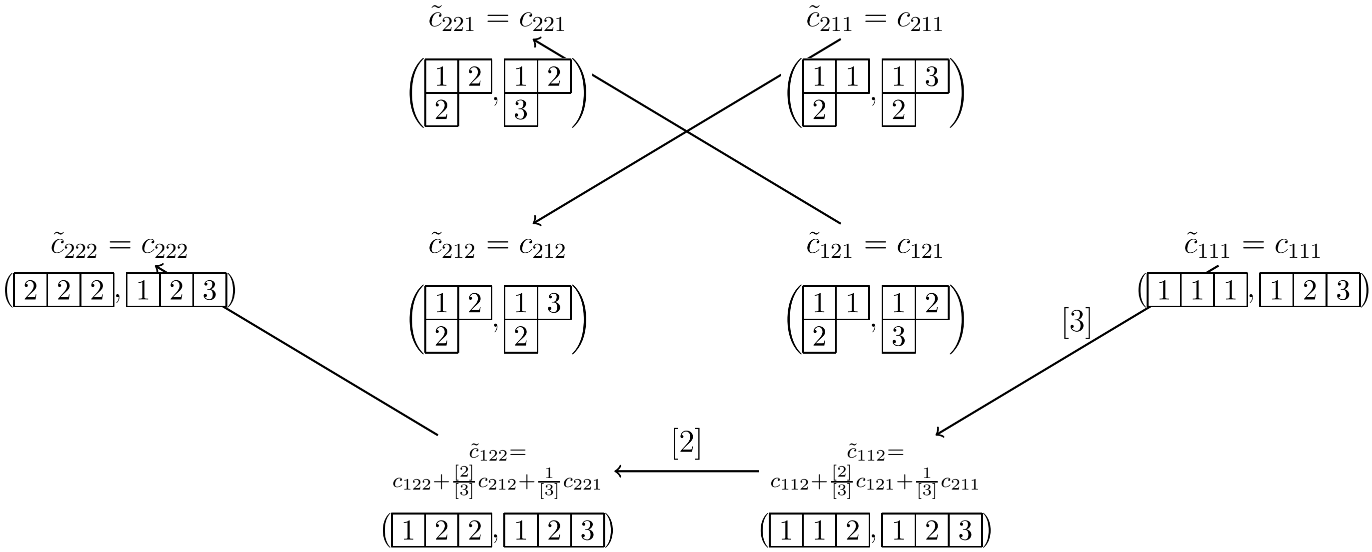


Figure 3: The projected upper canonical basis elements of Theorem 6.1 for $r = 3, n = 2$. The pairs of tableaux are of the form $(P(\mathbf{k}), Q(\mathbf{k}))$. The arrows and their coefficients give the action of F_1 on the projected upper canonical basis.

This follows from the uniqueness of upper crystal bases and the fact that the restriction of both sides of (23) to $\{x \in \mathbf{T}^\mu : E_i x = 0 \text{ for all } i \in [n-1]\}$ is $(K_\infty\{c_{\mathbf{k}} : P(\mathbf{k}) = Z_\mu\}, \{b_{\mathbf{k}} : P(\mathbf{k}) = Z_\mu\})$.

Finally, we can show that the element in (a) is the same as the other descriptions. The minimal central idempotent p_λ is $\bar{\cdot}$-invariant. This follows from the $\bar{\cdot}$-invariance of the upper canonical basis of $\mathscr{H}_r$ and the fact that an algebra involution must yield an involution of minimal central idempotents. Thus $p_\lambda^{\mathbf{T}}(c_{\mathbf{l}})$ is $\bar{\cdot}$-invariant and, by description (b), it satisfies the other requirements of (a). The uniqueness in (a) was not clear a priori but is now clear because the triple in (b) is balanced.

The statements about $(\mathbf{T}, \tilde{B})$ are clear from (e), (f) and Corollary 5.7. □

6.2. **Projected lower canonical basis.** Our equivalent descriptions of the projected lower canonical basis are similar to those for the upper, with some minor changes. By Corollary 5.5 and [25, Proposition 27.1.8],

$$\mathbf{T}[\trianglerighteq \lambda] = K\{c'_{\mathbf{k}} : \mathrm{sh}(\mathbf{k}^\dagger) \trianglerighteq \lambda\} \text{ and } \mathbf{T}[\triangleright \lambda] = K\{c'_{\mathbf{k}} : \mathrm{sh}(\mathbf{k}^\dagger) \triangleright \lambda\}. \tag{24}$$

Let $\mathbf{T_A}[\trianglerighteq \lambda]$ and $\mathscr{L}_0[\trianglerighteq \lambda]$ be the $\mathbf{A}$- and K_0- span of $\{c'_{\mathbf{k}} : \mathrm{sh}(\mathbf{k}^\dagger) \trianglerighteq \lambda\}$ and define

$$\begin{array}{rcl} (\mathbf{T_A})'_\lambda & := & \pi_\lambda^{\mathbf{T}}(\mathbf{T_A}[\trianglerighteq \lambda]), \\ \mathscr{L}_{0\lambda} & := & \pi_\lambda^{\mathbf{T}}(\mathscr{L}_0[\trianglerighteq \lambda]), \\ \tilde{\mathbf{T}}'_{\mathbf{A}} & := & \bigoplus_\lambda (\mathbf{T_A})'_\lambda. \end{array} \tag{25}$$

Theorem 6.2. *Maintain the notation above and let* $\mathbf{l} \in [n]^r$ *and* $\lambda = \mathrm{sh}(\mathbf{l}^\dagger)$. *Set* $\mathbf{j} = RSK^{-1}(Z_\lambda, Q(\mathbf{l}^\dagger))$. *Let* $V_{Q(\mathbf{l}^\dagger)} = \mathbf{U}\pi_\lambda^{\mathbf{T}}(c'_{\mathbf{j}})$ *and* $V_{Q(\mathbf{l}^\dagger)}^{\mathbb{Q}\ low}$ *be the* $\mathbb{Q}[u, u^{-1}]$*-form of* $V_{Q(\mathbf{l}^\dagger)}$ *as in*

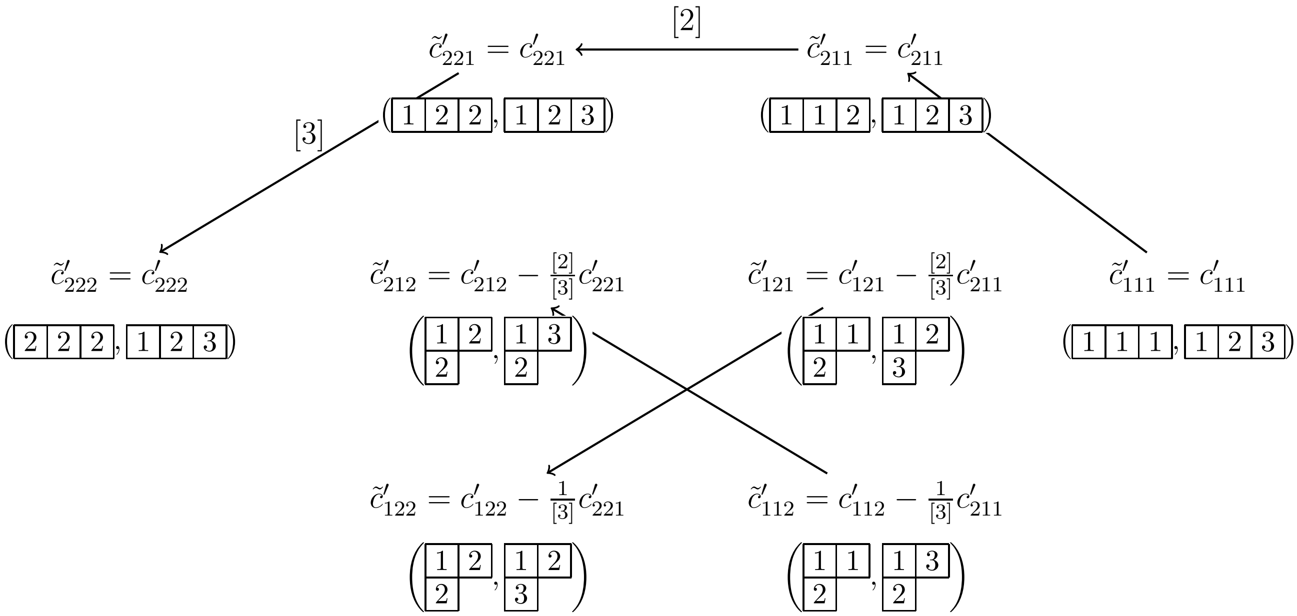


Figure 4: The projected lower canonical basis elements of Theorem 6.2 for $r = 3, n = 2$. The pairs of tableaux are of the form $(P(\mathbf{k}^\dagger), Q(\mathbf{k}^\dagger))$. The arrows and their coefficients give the action of F_1 on the projected lower canonical basis.

§4.3. Then the triples in (b) and (c) are balanced and the projected lower canonical basis element $\tilde{c}'_{\mathbf{l}}$ *has the following descriptions*

(a) *the unique* $\bar{\cdot}$*-invariant element of* $\tilde{\mathbf{T}}'_{\mathbf{A}}$ *congruent to* $\mathbf{v_l} \mod u\mathscr{L}_0$,

(b) $\tilde{G}'(b_{\mathbf{l}})$*, where* $\tilde{G}'$ *is the inverse of the canonical isomorphism*

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \tilde{\mathbf{T}}'_{\mathbf{A}}) \cap \mathscr{L}_0 \cap \overline{\mathscr{L}_0} \xrightarrow{\cong} \mathscr{L}_0/u\mathscr{L}_0,$$

(c) $\tilde{G}'_\lambda(\pi_\lambda(b_{\mathbf{l}}))$*, where* $\tilde{G}'_\lambda$ *is the inverse of the canonical isomorphism*

$$(\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbf{T_A})'_\lambda) \cap \mathscr{L}_{0\lambda} \cap \overline{\mathscr{L}_{0\lambda}} \xrightarrow{\cong} \mathscr{L}_{0\lambda}/u\mathscr{L}_{0\lambda},$$

(d) *the global crystal basis element* $G'_\lambda(b_{P(\mathbf{l}^\dagger)})$ *of* $V_{Q(\mathbf{l}^\dagger)}$,

(e) $\pi^{\mathbf{T}}_\lambda(c'_{\mathbf{l}})$,

(f) $p^{\mathbf{T}}_\lambda(c'_{\mathbf{l}})$.

Then $\tilde{B}' := \{\tilde{c}'_{\mathbf{k}} : \mathbf{k} \in [n]^r\}$ *is the* projected lower canonical basis of $\mathbf{T}$ *and* $(\mathbf{T}, \tilde{B}')$ *is a lower based module. Its* $\mathbf{U}$*- and* $\mathscr{H}_r$*-cells are given by Corollary 5.5 with* $\tilde{c}'$ *in place of* c'.

Proof. The proof is similar to that of Theorem 6.1, using results of [25, Chapter 27] in place of [22, §5.2]. Slightly more care is needed to prove that

$$\mathscr{L}_0 = \bigoplus_{\mu \vdash_n r} \mathscr{L}_{0\mu} \tag{26}$$

since $\pi^{\mathbf{T}}_\lambda(c_{\mathbf{j}}) = c_{\mathbf{j}}$, whereas $\pi^{\mathbf{T}}_\lambda(c'_{\mathbf{j}}) \neq c'_{\mathbf{j}}$ in general. However, uniqueness of lower crystal lattices is still enough: uniqueness means that both sides of (26) are determined by their intersection with $\mathbf{T}^{\mu}_{\text{hw}} := \{x \in \mathbf{T}^{\mu} : E_i x = 0 \text{ for all } i \in [n-1]\}$ for all $\mu \vdash_n r$. Since $\varsigma^{\mathbf{T}}_{\mu}$ restricts to an isomorphism $\mathbf{T}^{\mu}_{\text{hw}} \xrightarrow{\cong} \mathbf{T}[\mu]^{\mu}$ and

$$\varsigma^{\mathbf{T}}_{\mu}(\mathscr{L}_0 \cap \mathbf{T}^{\mu}_{\text{hw}}) = K_0\{\varsigma^{\mathbf{T}}_{\mu}(c'_{\mathbf{k}}) : P(\mathbf{k}^{\dagger}) = Z_{\mu}\} = \varsigma^{\mathbf{T}}_{\mu}(\mathscr{L}_{0\mu} \cap \mathbf{T}^{\mu}_{\text{hw}}), \tag{27}$$

we have $\mathscr{L}_0 \cap \mathbf{T}^{\mu}_{\text{hw}} = \mathscr{L}_{0\mu} \cap \mathbf{T}^{\mu}_{\text{hw}}$. The equality (26) follows. □

The most interesting part of Theorems 6.1 and 6.2 for us is that, though the integral form needed for the upper (resp. lower) canonical basis and projected upper (resp. lower) canonical basis differ, the upper (resp. lower) crystal lattices are the same. This has the following consequence.

Corollary 6.3. *The transition matrix from the projected upper (resp. lower) canonical basis to the upper (resp. lower) canonical basis is unitriangular and is the identity at $u = 0$ and $u = \infty$. Precisely,*

$$\begin{aligned} \tilde{c}_{\mathbf{k}} &= c_{\mathbf{k}} + \textstyle\sum_{\text{sh}(\mathbf{k}') \lhd \text{sh}(\mathbf{k})} t_{\mathbf{k}'\mathbf{k}} c_{\mathbf{k}'}, \\ \tilde{c}'_{\mathbf{k}} &= c'_{\mathbf{k}} + \textstyle\sum_{\text{sh}(\mathbf{k}'^{\dagger}) \rhd \text{sh}(\mathbf{k}^{\dagger})} t'_{\mathbf{k}'\mathbf{k}} c'_{\mathbf{k}'}, \end{aligned}$$

where the coefficients $t_{\mathbf{k}'\mathbf{k}}, t'_{\mathbf{k}'\mathbf{k}}$ are $\bar{\cdot}$-invariant and belong to $uK_0 \cap u^{-1}K_\infty$.

Proof. The constraints on dominance order follow from (20) and (24) using the expressions (e) of Theorems 6.1 and 6.2 for the projected canonical basis elements. By the expression (a) of Theorem 6.1, $c_{\mathbf{k}} \equiv \tilde{c}_{\mathbf{k}} \mod u^{-1}\mathscr{L}_\infty$. Thus $t_{\mathbf{k}'\mathbf{k}} \in u^{-1}K_\infty$. Since the upper canonical basis and projected upper canonical basis are $\bar{\cdot}$-invariant, so are the entries of the transition matrix between them. This further implies $t_{\mathbf{k}'\mathbf{k}} \in uK_0 \cap u^{-1}K_\infty$. The proof for the lower canonical basis is similar. □

6.3. **Projected canonical bases are dual under the bilinear form.**

Proposition 6.4. *The projected upper and lower canonical basis of $\mathbf{T}$ are dual under $(\cdot,\cdot)$: there holds $(\tilde{c}_{\mathbf{k}}, \tilde{c}'_{\mathbf{l}^{\dagger}}) = \delta_{\mathbf{k},\mathbf{l}}$ for all $\mathbf{k}, \mathbf{l} \in [n]^r$.*

Proof. Let $\mathbf{k}, \mathbf{l} \in [n]^r$ and $\lambda = \text{sh}(\mathbf{k}), \mu = \text{sh}(\mathbf{l})$. If $\lambda = \mu$, then by the unitriangularities established in Corollary 6.3 together with the fact that the upper canonical basis is dual to the lower canonical basis, we have $(\tilde{c}_{\mathbf{k}}, \tilde{c}'_{\mathbf{l}^{\dagger}}) = \delta_{\mathbf{k},\mathbf{l}}$. In the case $\lambda \neq \mu$, we use Proposition-Definition 5.8 (ii) to conclude

$$(\tilde{c}_{\mathbf{k}}, \tilde{c}'_{\mathbf{l}^{\dagger}}) = (\tilde{c}_{\mathbf{k}} p_\lambda, \tilde{c}'_{\mathbf{l}^{\dagger}} p_\mu) = (\tilde{c}_{\mathbf{k}}, \tilde{c}'_{\mathbf{l}^{\dagger}} p_\mu p_\lambda^{\dagger^{\text{op}}}) = (\tilde{c}_{\mathbf{k}}, \tilde{c}'_{\mathbf{l}^{\dagger}} p_\mu p_\lambda) = 0. \tag{28}$$

□

7. Consequences for the canonical bases of M_λ

We use the results of the previous section to understand projected canonical bases of $\mathscr{H}_r$ and the relation between the upper and lower canonical basis of M_λ. We will come across several transition matrices whose entries lie in K_0 and are the identity at $u = 0$. Define, for an element $f \in uK_0$, the *leading coefficient* of f, denoted $\mu(f)$, to be the coefficient of u in the power series expansion of f. It turns out that the leading coefficients of many of these transition matrix entries coincide with the $\mathcal{S}_r$-graph edge weights $\mu(v, w)$.

7.1. **Projected canonical bases of $\mathscr{H}_r$.** Here we define projected canonical bases of $\mathscr{H}_r$, which are essentially a special case of the projected canonical bases in the previous section. For each $w \in \mathcal{S}_r$, the projected upper (resp. lower) canonical basis element $\tilde{C}_w \in \mathscr{H}_r$ (resp. $\tilde{C}'_w$) is defined to be $C_w p_\lambda$ (resp. $C'_w p_\lambda$), where $\lambda = \text{sh}(w)$ (resp. $\lambda = \text{sh}(w^\dagger)$).

Corollary 7.1. *The transition matrix $\tilde{T} = (\tilde{T}_{w'w})_{w',w\in\mathcal{S}_r}$ (resp. $\tilde{T}' = (\tilde{T}'_{w'w})_{w',w\in\mathcal{S}_r}$) expressing the projected upper (resp. lower) canonical basis of $\mathscr{H}_r$ in terms of the upper (resp. lower) canonical basis of $\mathscr{H}_r$*

(i) *is unitriangular: $\tilde{C}_w = C_w + \sum_{\text{sh}(w')\lhd\text{sh}(w)} \tilde{T}_{w'w} C_{w'}$ (resp. $\tilde{C}'_w = C'_w + \sum_{\text{sh}(w'^\dagger)\rhd\text{sh}(w^\dagger)} \tilde{T}'_{w'w} C'_{w'}$),*

(ii) *has entries that are $\bar{\cdot}$-invariant and belong to $K_0 \cap K_\infty$,*

(iii) *is the identity at $u = 0$ and $u = \infty$,*

(iv) *satisfies: $\mu(\tilde{T}_{w'w}) = \mu(w', w)$ (resp. $\mu(\tilde{T}'_{w'w}) = -\mu(w', w)$) for w', w such that $P(w') \neq P(w)$ and $R(w') \setminus R(w) \neq \emptyset$.*

Proof. Choose $n = r$ and set $\varepsilon = \epsilon_1 + \cdots + \epsilon_r$. Then by Proposition 5.3, Theorem 5.4 (iv), and Theorem 5.6 (iv), $\mathscr{H}_r \cong \mathbf{T}^\varepsilon_{\mathbf{A}}$ as right $\mathscr{H}_r$-modules and under this isomorphism canonical bases are sent to canonical bases and projected canonical bases are sent to projected canonical bases. Thus (i)-(iii) are a special case of Corollary 6.3.

To prove (iv), we compute $\tilde{C}_w C_s$, for $w \in \mathcal{S}_r$ such that $s \notin R(w)$, in terms of the upper canonical basis in two different ways:

$$\tilde{C}_w C_s = (\sum_{w'} \tilde{T}_{w'w} C_{w'}) C_s = -[2] \sum_{\{w' : s\in R(w')\}} \tilde{T}_{w'w} C_{w'} + \sum_{\substack{w',w'' \\ s\notin R(w'), s\in R(w'')}} \tilde{T}_{w'w} \mu(w'', w') C_{w''}. \tag{29}$$

On the other hand, since $\mathbf{A}\{\tilde{C}_{w'} : P(w') = P(w)\}$ and the cellular subquotient $\mathbf{A}\Gamma_{P(w)}$ are isomorphic as modules with basis,

$$\tilde{C}_w C_s = \sum_{\left\{w' : \substack{s\in R(w'), \\ P(w')=P(w)}\right\}} \mu(w', w) \tilde{C}_{w'} = \sum_{\left\{w' : \substack{s\in R(w'), \\ P(w')=P(w)}\right\}} \mu(w', w) \sum_{w''} \tilde{T}_{w''w'} C_{w''}. \tag{30}$$

Then for any w'' such that $s \in R(w'')$ and $P(w'') \neq P(w)$, equating coefficients of $C_{w''}$ yields

$$0 = \sum_{\left\{w' : \substack{s\in R(w'), \\ P(w')=P(w)}\right\}} \tilde{T}_{w''w'} \mu(w', w) + [2]\tilde{T}_{w''w} - \sum_{\{w' : s\notin R(w')\}} \mu(w'', w') \tilde{T}_{w'w} \equiv \mu(\tilde{T}_{w''w}) - \mu(w'', w),$$

where the equivalence is mod uK_0 and uses (iii). This proves (iv) for the upper canonical basis. The proof for the lower canonical basis is similar. □

Remark 7.2. It is sensible to ask whether Corollary 7.1 and other results in this section hold for other finite Coxeter groups W in place of $\mathcal{S}_r$ (perhaps with a slight modification if right cells do not correspond to $\mathbb{C}(u) \otimes_{\mathbf{A}} \mathscr{H}(W)$-irreducibles). We have not investigated this, but note that our proof will not extend easily as it depends on Schur-Weyl duality.

	$\tilde{C}_{1234}$	$\tilde{C}_{1324}$	$\tilde{C}_{2134}$	$\tilde{C}_{1243}$	$\tilde{C}_{1423}$	$\tilde{C}_{1342}$	$\tilde{C}_{2314}$	$\tilde{C}_{3124}$	$\tilde{C}_{2143}$	$\tilde{C}_{2413}$	$\tilde{C}_{4123}$	$\tilde{C}_{2341}$	$\tilde{C}_{3142}$	$\tilde{C}_{3412}$
C_{1234}	1	0	0	0	0	0	0	0	0	0	0	0	0	0
C_{1324}	$\frac{[2]^2}{[4]}$	1	0	0	0	0	0	0	0	0	0	0	0	0
C_{2134}	$\frac{[3]}{[4]}$	0	1	0	0	0	0	0	0	0	0	0	0	0
C_{1243}	$\frac{[3]}{[4]}$	0	0	1	0	0	0	0	0	0	0	0	0	0
C_{1423}	$\frac{[2]}{[4]}$	0	0	0	1	0	0	0	0	0	0	0	0	0
C_{1342}	$\frac{[2]}{[4]}$	0	0	0	0	1	0	0	0	0	0	0	0	0
C_{2314}	$\frac{[2]}{[4]}$	0	0	0	0	0	1	0	0	0	0	0	0	0
C_{3124}	$\frac{[2]}{[4]}$	0	0	0	0	0	0	1	0	0	0	0	0	0
C_{2143}	$\frac{[2]^3}{[3][4]}$	0	$\frac{1}{[2]}$	$\frac{1}{[2]}$	0	0	0	0	1	0	$\frac{1}{[2]}$	$\frac{1}{[2]}$	0	0
C_{2413}	$\frac{[2]^2}{[3][4]}$	0	0	0	$\frac{1}{[2]}$	0	$\frac{1}{[2]}$	0	0	1	0	0	0	0
C_{4123}	$\frac{1}{[4]}$	0	0	0	0	0	0	0	0	0	1	0	0	0
C_{2341}	$\frac{1}{[4]}$	0	0	0	0	0	0	0	0	0	0	1	0	0
C_{3142}	$\frac{[2]^2}{[3][4]}$	0	0	0	0	$\frac{1}{[2]}$	0	$\frac{1}{[2]}$	0	0	0	0	1	0
C_{3412}	$\frac{[2]}{[3][4]}$	$\frac{1}{[2]}$	0	0	0	0	0	0	0	0	0	0	0	1
C_{1432}	$\frac{[2]^2}{[3][4]}$	$\frac{-1}{[2][4]}$	0	$-\frac{[2]}{[4]}$	$\frac{[3]}{[4]}$	$\frac{[3]}{[4]}$	0	0	0	0	0	0	0	$\frac{1}{[2]}$
C_{3214}	$\frac{[2]^2}{[3][4]}$	$\frac{-1}{[2][4]}$	$-\frac{[2]}{[4]}$	0	0	0	$\frac{[3]}{[4]}$	$\frac{[3]}{[4]}$	0	0	0	0	0	$\frac{1}{[2]}$
C_{2431}	$\frac{[2]}{[3][4]}$	0	0	$-\frac{1}{[4]}$	$\frac{[3]}{[2][4]}$	0	$\frac{-1}{[2][4]}$	0	0	$\frac{1}{[2]}$	0	$\frac{[3]}{[4]}$	0	0
C_{4132}	$\frac{[2]}{[3][4]}$	0	0	$-\frac{1}{[4]}$	0	$\frac{[3]}{[2][4]}$	0	$\frac{-1}{[2][4]}$	0	0	$\frac{[3]}{[4]}$	0	$\frac{1}{[2]}$	0
C_{4213}	$\frac{[2]}{[3][4]}$	0	$-\frac{1}{[4]}$	0	$\frac{-1}{[2][4]}$	0	$\frac{[3]}{[2][4]}$	0	0	$\frac{1}{[2]}$	$\frac{[3]}{[4]}$	0	0	0
C_{3241}	$\frac{[2]}{[3][4]}$	0	$-\frac{1}{[4]}$	0	0	$\frac{-1}{[2][4]}$	0	$\frac{[3]}{[2][4]}$	0	0	0	$\frac{[3]}{[4]}$	$\frac{1}{[2]}$	0
C_{4312}	$\frac{1}{[3][4]}$	$\frac{[3]}{[2][4]}$	0	0	$-\frac{1}{[4]}$	0	0	$-\frac{1}{[4]}$	0	0	$\frac{[2]}{[4]}$	0	0	$\frac{1}{[2]}$
C_{3421}	$\frac{1}{[3][4]}$	$\frac{[3]}{[2][4]}$	0	0	0	$-\frac{1}{[4]}$	$-\frac{1}{[4]}$	0	0	0	0	$\frac{[2]}{[4]}$	0	$\frac{1}{[2]}$
C_{4231}	$\frac{1}{[3][4]}$	0	$\frac{-1}{[2][4]}$	$\frac{-1}{[2][4]}$	0	0	0	0	$\frac{1}{[2]}$	0	$\frac{[3]}{[2][4]}$	$\frac{[3]}{[2][4]}$	0	0
C_{4321}	$\frac{1}{[2][3][4]}$	$\frac{1}{[4]}$	0	0	$\frac{-1}{[2][4]}$	$\frac{-1}{[2][4]}$	$\frac{-1}{[2][4]}$	$\frac{-1}{[2][4]}$	$\frac{1}{[3]}$	$\frac{-1}{[2][3]}$	$\frac{1}{[4]}$	$\frac{1}{[4]}$	$\frac{-1}{[2][3]}$	$\frac{1}{[3]}$

Figure 5: Some of the projected upper canonical basis elements $\tilde{C}_w$ in terms of the upper canonical basis $\{C_v\}_{v\in S_4}$.

7.2. **Seminormal bases.** We wish to use the results about projected canonical bases to understand the transition matrix between the lower canonical basis of M_λ and the upper canonical basis of M_λ. To do this, we relate both to seminormal bases in the sense of [28]. The transition matrices between the canonical bases of M_λ and their corresponding seminormal bases also appear to be quite interesting—see the positivity conjectures in the next subsection.

Definition 7.3. Given a chain of split semisimple K-algebras $K \cong H_1 \subseteq H_2 \subseteq \cdots \subseteq H_r$ and an H_r-irreducible N_λ, a *seminormal basis* of N_λ is a K-basis B of N_λ compatible with the restrictions in the following sense: there is a partition $B = B_{\mu^1} \sqcup \cdots \sqcup B_{\mu^k}$ such that if $N_{\mu^i} = KB_{\mu^i}$ then $N_\lambda = N_{\mu^1} \oplus \cdots \oplus N_{\mu^k}$ as H_{r-1}-modules. Further, there is a partition of each B_{μ^i} that gives rise to a decomposition of N_{μ^i} into H_{r-2}-irreducibles, and so on, all the way down to H_1.

Note that if the restriction of an H_i-irreducible to H_{i-1} is multiplicity-free, then a seminormal basis is unique up to a diagonal transformation.

To construct seminormal bases corresponding to the upper and lower canonical basis of M_λ, first define, for any $J \subseteq S$, $(\tilde{C}_Q)^J$ to be the projection of C_Q onto the irreducible $K\mathscr{H}_J$-module corresponding to the right cell of $\text{Res}_{K\mathscr{H}_J} K\Gamma_\lambda$ containing C_Q. Define $(\tilde{C}'_Q)^J$ similarly. If $J = \{s_1, \dots, s_{r-2}\}$, then by [5, §4], $(\tilde{C}_Q)^J$ (resp. $(\tilde{C}'_Q)^J$) is equal to $C_Q p_\mu$ (resp. $C'_Q p_\mu$), where $\mu = \text{sh}(Q|_{[r-1]})$. Here, for a tableau Q and set $Z \subseteq \mathbb{Z}$, $Q|_Z$ denotes the subtableau of Q obtained by removing the entries not in Z.

Define a total order $\trianglelefteq$ on $\text{SYT}(\lambda)$ by declaring $Q' \trianglelefteq Q$ if the numbers $k+1, \dots, r$ are in the same positions in Q' and Q and $\text{sh}(Q'|_{[k-1]}) \trianglerighteq \text{sh}(Q|_{[k-1]})$; this k is unique and we refer to it as $k(Q', Q)$. This total order is the reverse of the last letter order defined in [13].

Lemma 7.4. *For $J = \{s_1, \dots, s_{r-2}\}$, the transition matrix expressing the projected basis $\{(\tilde{C}_Q)^J : Q \in SYT(\lambda)\}$ in terms of the upper canonical basis of M_λ is lower-unitriangular, is the identity at $u = 0$ and $u = \infty$, and has $\bar{\cdot}$-invariant entries (i.e. $(\tilde{C}_Q)^J = C_Q + \sum_{Q' \triangleright Q} m_{Q'Q} C_{Q'}$, $m_{Q'Q} \in uK_0$, $\overline{m_{Q'Q}} = m_{Q'Q}$). The transition matrix expressing the projected basis $\{(\tilde{C}'_Q)^J : Q \in SYT(\lambda)\}$ in terms of the lower canonical basis of M_λ satisfies the same properties except is upper-unitriangular instead of lower-unitriangular (i.e. $(\tilde{C}'_Q)^J = C'_Q + \sum_{Q' \triangleleft Q} m'_{Q'Q} C'_{Q'}$, $m'_{Q'Q} \in uK_0$, $\overline{m'_{Q'Q}} = m'_{Q'Q}$).*

Proof. This follows from Corollary 7.1 and Proposition 3.2: identify $M_\lambda^{\mathbf{A}}$ and its upper canonical basis with $\mathbf{A}\Gamma_{Z_\lambda^*}$, where Z_λ^* is the standard tableau with $1, 2, \dots, \lambda_1$ in the first row, $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ in the second row, etc. Then $\text{Res}_{\mathscr{H}_J} \mathbf{A}\Gamma_{Z_\lambda^*}$ has right cells labeled by the result of uninserting an outer corner of Z_λ^* (see [5, §4]). The key point is that all uninsertions of Z_λ^* result in the entry λ_1 being kicked out, and therefore by Proposition 3.2, the $\mathscr{H}_J$-module with basis $\text{Res}_{\mathscr{H}_J} \mathbf{A}\Gamma_{Z_\lambda^*} \subseteq \mathbf{A}\{C'_{xv} : v \in (\mathcal{S}_r)_J\}$ is isomorphic to a cellular subquotient of $\Gamma_{\mathcal{S}_{r-1}}$ (here $x = s_{\lambda_1} s_{\lambda_1+1} \cdots s_{r-1}$). We can then apply Corollary 7.1 with r of the proposition set to $r-1$. Lower-unitriangularity follows from Corollary 3.4. The proof for the lower canonical basis is similar. □

Set $J_i = \{s_1, \dots, s_{i-1}\}$. We now define the upper seminormal basis to be $\{C_Q^{\text{sn}} : Q \in \text{SYT}(\lambda)\}$, where C_Q^{sn} is the result of applying the construction $C_Q \rightsquigarrow (\tilde{C}_Q)^J$ first with $J = J_{r-1}$, then with $J = J_{r-2}$, and so on, finishing with $J = J_1 = \emptyset$. The lower seminormal basis $\{C_Q'^{\text{sn}} : Q \in \text{SYT}(\lambda)\}$ is defined similarly. These bases are seminormal with respect to the chain $\mathscr{H}_{J_1} \subseteq \cdots \subseteq \mathscr{H}_{J_{r-1}} \subseteq \mathscr{H}_r$.

Proposition 7.5. *The transition matrix $T(\lambda) = (T_{Q'Q})_{Q',Q \in SYT(\lambda)}$ (resp. $T'(\lambda) = (T'_{Q'Q})_{Q',Q \in SYT(\lambda)}$) expressing the upper (resp. lower) seminormal basis of M_λ in terms of the upper (resp. lower) canonical basis of M_λ and $T(\lambda)^{-1}$ (resp. $T'(\lambda)^{-1}$)*

(i) *are lower-unitriangular: $C_Q^{sn} = C_Q + \sum_{Q' \triangleright Q} T_{Q'Q} C_{Q'}$ and similarly for $T(\lambda)^{-1}$ (resp. upper-unitriangular: $C_Q'^{sn} = C'_Q + \sum_{Q' \triangleleft Q} T'_{Q'Q} C'_{Q'}$ and similarly for $T'(\lambda)^{-1}$),*

(ii) *have entries that are $\bar{\cdot}$-invariant and belong to $K_0 \cap K_\infty$,*

(iii) *are the identity at $u = 0$ and $u = \infty$,*

(iv) *satisfy: $\mu(T_{Q'Q}) = \mu(Q', Q)$ and $\mu(T^{-1}_{Q'Q}) = -\mu(Q', Q)$ for Q', Q such that $Q' \triangleright Q$ and $(R(C_{Q'}) \setminus R(C_Q)) \cap J_{k(Q',Q)-1} \neq \emptyset$*

(resp. $\mu(T'_{Q'Q}) = -\mu(Q', Q)$ *and* $\mu(T'^{-1}_{Q'Q}) = \mu(Q', Q)$ *for* Q', Q *such that* $Q' \lhd Q$ *and* $(R(C'_{Q'}) \setminus R(C'_Q)) \cap J_{k(Q',Q)-1} \neq \emptyset$*).*

Proof. The transition matrix $T(\lambda)$ is the product $\tilde{M}^{J_{r-1}} \tilde{M}^{J_{r-2}} \cdots \tilde{M}^{J_1}$, where $\tilde{M}^{J_i}$ is a block diagonal matrix, with each block of the form described in Lemma 7.4 with J of the lemma equal to J_i. Properties (i)-(iii) of $T(\lambda)$ then follow because they are preserved under matrix multiplication and diagonally joining blocks.

To prove (iv), we apply the following easy claim

(31) if $M^1, \dots, M^l$ are matrices satisfying (iii) and $M = \prod_{k=1}^{l} M^k$, then $\mu(M_{ij}) = \sum_k \mu(M^k_{ij})$ for $i \neq j$.

to obtain $\mu(T_{Q'Q}) = \sum_{k=1}^{r-1} \mu(\tilde{M}^{J_k}_{Q'Q})$. If $Q' \rhd Q$, then there is exactly one k for which $\tilde{M}^{J_{k-1}}_{Q'Q}$ is non-zero; this k is exactly $k(Q', Q)$. Further, by Corollary 7.1 (iv) and the proof of Lemma 7.4, $\mu(\tilde{M}^{J_{k(Q',Q)-1}}_{Q'Q}) = \mu(Q', Q)$ if $(R(C_{Q'}) \setminus R(C_Q)) \cap J_{k(Q',Q)-1} \neq \emptyset$. This proves (iv) for $T(\lambda)$. The statements for $T'(\lambda)$ are proved similarly and the statements for $T(\lambda)^{-1}$ and $T'(\lambda)^{-1}$ follow easily. □

Example 7.6. Continuing Example 3.3, we give transition matrices between the various bases defined above. The convention is that the columns of the matrix express the basis element at the top of the column in terms of the row labels. The matrices $D(\lambda)$ and $S(\lambda)$ are defined in Theorem 7.8 and its proof (below).

	$C^{\mathrm{sn}}_{Q_4}$	$C^{\mathrm{sn}}_{Q_3}$	$C^{\mathrm{sn}}_{Q_2}$
C_{Q_4}	1	0	0
C_{Q_3}	$\frac{[2]}{[3]}$	1	0
C_{Q_2}	$\frac{1}{[3]}$	$\frac{1}{[2]}$	1

$T((3,1))$

	$C'^{\mathrm{sn}}_{Q_4}$	$C'^{\mathrm{sn}}_{Q_3}$	$C'^{\mathrm{sn}}_{Q_2}$
$C^{\mathrm{sn}}_{Q_4}$	[3]	0	0
$C^{\mathrm{sn}}_{Q_3}$	0	$\frac{[2][4]}{[3]}$	0
$C^{\mathrm{sn}}_{Q_2}$	0	0	$\frac{[4]}{[2]}$

$D((3,1))$

	C'_{Q_4}	C'_{Q_3}	C'_{Q_2}
$C'^{\mathrm{sn}}_{Q_4}$	1	$\frac{[2]}{[3]}$	$\frac{1}{[3]}$
$C'^{\mathrm{sn}}_{Q_3}$	0	1	$\frac{1}{[2]}$
$C'^{\mathrm{sn}}_{Q_2}$	0	0	1

$T'((3,1))^{-1}$

	C'_{Q_4}	C'_{Q_3}	C'_{Q_2}
C_{Q_4}	[3]	[2]	1
C_{Q_3}	[2]	$[2]^2$	[2]
C_{Q_2}	1	[2]	[3]

$S((3,1))$

The more substantial example $\lambda = (4,2)$ is below, where $S(\lambda)$ is scaled as in Conjecture 7.9, so that its entries lie in $\mathbf{A}$, are $\bar{\cdot}$-invariant, and have greatest common divisor 1.

	$\begin{smallmatrix}1&2&3&4\\5&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&3&5\\4&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&4&5\\3&6\end{smallmatrix}$	$\begin{smallmatrix}1&3&4&5\\2&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&3&6\\4&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&4&6\\3&5\end{smallmatrix}$	$\begin{smallmatrix}1&3&4&6\\2&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&5&6\\3&4\end{smallmatrix}$	$\begin{smallmatrix}1&3&5&6\\2&4\end{smallmatrix}$
$\begin{smallmatrix}1&2&3&4\\5&6\end{smallmatrix}$	1	$\frac{[3]}{[4]}$	$\frac{[2]}{[4]}$	$\frac{[1]}{[4]}$	$\frac{[2]}{[4]}$	$\frac{[2]^2}{[4]}$	$\frac{[2]}{[4]}$	$\frac{[2]}{[4][3]}$	$\frac{[2]^2}{[3][4]}$
$\begin{smallmatrix}1&2&3&5\\4&6\end{smallmatrix}$	0	1	$\frac{[2]}{[3]}$	$\frac{1}{[3]}$	$\frac{[2]}{[3]}$	$\frac{[2]^2}{[3]^2}$	$\frac{[2]}{[3]^2}$	$\frac{[2]}{[3]^2}$	$\frac{[2]^2}{[3]^2}$
$\begin{smallmatrix}1&2&4&5\\3&6\end{smallmatrix}$	0	0	1	$\frac{1}{[2]}$	0	$\frac{[2]}{[3]}$	$\frac{1}{[3]}$	$\frac{1}{[3]}$	$\frac{1}{[2][3]}$
$\begin{smallmatrix}1&3&4&5\\2&6\end{smallmatrix}$	0	0	0	1	0	0	$\frac{[2]}{[3]}$	0	$\frac{1}{[3]}$
$\begin{smallmatrix}1&2&3&6\\4&5\end{smallmatrix}$	0	0	0	0	1	$\frac{[2]}{[3]}$	$\frac{1}{[3]}$	$\frac{1}{[3]}$	$\frac{[2]}{[3]}$
$\begin{smallmatrix}1&2&4&6\\3&5\end{smallmatrix}$	0	0	0	0	0	1	$\frac{1}{[2]}$	$\frac{1}{[2]}$	$\frac{1}{[2]^2}$
$\begin{smallmatrix}1&3&4&6\\2&5\end{smallmatrix}$	0	0	0	0	0	0	1	0	$\frac{1}{[2]}$
$\begin{smallmatrix}1&2&5&6\\3&4\end{smallmatrix}$	0	0	0	0	0	0	0	1	$\frac{1}{[2]}$
$\begin{smallmatrix}1&3&5&6\\2&4\end{smallmatrix}$	0	0	0	0	0	0	0	0	1

$T'((4,2))^{-1}$

	$\begin{smallmatrix}1&2&3&4\\5&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&3&5\\4&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&4&5\\3&6\end{smallmatrix}$	$\begin{smallmatrix}1&3&4&5\\2&6\end{smallmatrix}$	$\begin{smallmatrix}1&2&3&6\\4&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&4&6\\3&5\end{smallmatrix}$	$\begin{smallmatrix}1&3&4&6\\2&5\end{smallmatrix}$	$\begin{smallmatrix}1&2&5&6\\3&4\end{smallmatrix}$	$\begin{smallmatrix}1&3&5&6\\2&4\end{smallmatrix}$
$\begin{smallmatrix}1&2&3&4\\5&6\end{smallmatrix}$	$[3][4]$	$[3]^2$	$[2][3]$	$[3]$	$[2][3]$	$[2]^2[3]$	$[2][3]$	$[2]$	$[2]^2$
$\begin{smallmatrix}1&2&3&5\\4&6\end{smallmatrix}$	$[3]^2$	$[2][3]^2$	$[2]^2[3]$	$[2][3]$	$[2]^2[3]$	$2[4]+3[2]$	$2[3]+1$	$[2]^2$	$[2]^3$
$\begin{smallmatrix}1&2&4&5\\3&6\end{smallmatrix}$	$[2][3]$	$[2]^2[3]$	$[2][3]^2$	$[3]^2$	$[2]^3$	$[2]^4$	$[2]^3$	$[2][3]$	$2[3]+1$
$\begin{smallmatrix}1&3&4&5\\2&6\end{smallmatrix}$	$[3]$	$[2][3]$	$[3]^2$	$[3][4]$	$[2]^2$	$[2]^3$	$[3]^2$	$[3]$	$[2][3]$
$\begin{smallmatrix}1&2&3&6\\4&5\end{smallmatrix}$	$[2][3]$	$[2]^2[3]$	$[2]^3$	$[2]^2$	$[2][3]^2$	$[2]^4$	$[2]^3$	$[2][3]$	$[2]^2[3]$
$\begin{smallmatrix}1&2&4&6\\3&5\end{smallmatrix}$	$[2]^2[3]$	$2[4]+3[2]$	$[2]^4$	$[2]^3$	$[2]^4$	$[2]^5$	$[2]^4$	$[2]^2[3]$	$2[4]+3[2]$
$\begin{smallmatrix}1&3&4&6\\2&5\end{smallmatrix}$	$[2][3]$	$2[3]+1$	$[2]^3$	$[3]^2$	$[2]^3$	$[2]^4$	$[2][3]^2$	$[2][3]$	$[2]^2[3]$
$\begin{smallmatrix}1&2&5&6\\3&4\end{smallmatrix}$	$[2]$	$[2]^2$	$[2][3]$	$[3]$	$[2][3]$	$[2]^2[3]$	$[2][3]$	$[3][4]$	$[3]^2$
$\begin{smallmatrix}1&3&5&6\\2&4\end{smallmatrix}$	$[2]^2$	$[2]^3$	$2[3]+1$	$[2][3]$	$[2]^2[3]$	$2[4]+3[2]$	$[2]^2[3]$	$[3]^2$	$[2][3]^2$

$S((4,2))$

For the next proposition, let us clarify a confusing point. Let Γ'_T, $T \in \mathrm{SSYT}_{[n]}(\lambda)$, be the $\mathscr{H}_r$-cell of $\mathbf{T}$ from Corollary 5.5 (i); let Γ'_P be the right cell of $\Gamma'_{\mathcal{S}_r}$ from §3.3, where $P \in \mathrm{SYT}(\lambda)$ is the *standardization* of T, i.e. $P = P(D(\mathbf{k}))^t$ for any $\mathbf{k}$ with $P(\mathbf{k}^\dagger) = T$. The $\mathscr{H}_r$-cells Γ'_P, Γ'_T, and Γ'_λ give rise to isomorphic $\mathscr{H}_r$-modules with basis, the isomorphisms being given by

$$\begin{array}{ccccccccc} \Gamma'_T & \cong & \Gamma'_P, & \Gamma'_P & \cong & \Gamma'_\lambda, & \Gamma'_T & \cong & \Gamma'_\lambda, \\ c'_{\mathbf{k}} & \longleftrightarrow & C'_{D(\mathbf{k})} & C'_w & \longleftrightarrow & C'_{Q(w)^t} & c'_{\mathbf{k}} & \longleftrightarrow & C'_{Q(\mathbf{k}^\dagger)^\dagger} \end{array} \tag{32}$$

where $Q^\dagger$, for Q a SYT, denotes the Schützenberger involution of Q (see, e.g., [10, A1.2]). The left-hand isomorphism is from Theorem 5.4 (iv), the middle from §3.3, and the right is the composition of the two.

Let 1^{op} be the antiautomorphism of $\mathscr{H}_r$ determined by $T_i^{1^{\mathrm{op}}} = T_i$.

Proposition 7.7. *There is a bilinear form $\langle \cdot, \cdot \rangle : M_\lambda \times M_\lambda \to K$ satisfying*

(i) $\langle xh, x'\rangle = \langle x, x'h^{1^{op}}\rangle$ *for any* $h \in \mathscr{H}_r$, $x, x' \in M_\lambda$,
(ii) $\langle C_Q, C'_{Q'}\rangle = \delta_{QQ'}$,
(iii) $\langle (\tilde{C}_Q)^J, (\tilde{C}'_{Q'})^J\rangle = \delta_{QQ'}$,
(iv) $\langle C^{sn}_Q, C'^{sn}_{Q'}\rangle = \delta_{QQ'}$.

Proof. By Proposition 6.4, the inner product on $\mathbf{T}$ restricts to an inner product on $\pi^{\mathbf{T}}_\lambda(K\Gamma_T) \times \pi^{\mathbf{T}}_\lambda(K\Gamma'_T) \to K$ for any $T \in \mathrm{SSYT}_{[n]}(\lambda)$. This yields an inner product on M_λ satisfying $(C_Q, C'_{Q'}) = \delta_{QQ'^\dagger}$ (we have used the right-hand isomorphism of (32)). Letting $M^\dagger_\lambda$ denote the result of twisting M_λ by the automorphism $\dagger$, we have $M_\lambda \cong M^\dagger_\lambda$ via $C'_Q \mapsto C'_{Q^\dagger}$. Applying this isomorphism to the second factor yields the inner product $\langle\cdot,\cdot\rangle$ satisfying (i) and (ii). Given (ii), the proof of (iii) is similar to that of Proposition 6.4. Iterating this argument through the sequence of projections that yields the seminormal bases proves (iv). □

Theorem 7.8. *The transition matrix* $S(\lambda) = (S_{Q'Q})_{Q',Q\in SYT(\lambda)}$ *expressing the lower canonical basis of* M_λ *in terms of the upper canonical basis of* M_λ *(i.e.* $C'_Q = \sum_{Q'\in SYT(\lambda)} S_{Q'Q}C_{Q'}$*) has* $\bar{\cdot}$*-invariant entries that belong to* $K_0 \cap K_\infty$ *and is the identity matrix at* $u = 0$ *and* $u = \infty$.

Proof. First note that the $\bar{\cdot}$-invariance of the lower and upper canonical basis of M_λ shows that the entries of $S(\lambda)$ are $\bar{\cdot}$-invariant. As remarked after Definition 7.3, the upper seminormal and lower seminormal bases differ by a diagonal transformation. Thus $S(\lambda) = T(\lambda)D(\lambda)T'(\lambda)^{-1}$ for some diagonal matrix $D(\lambda)$. Given Proposition 7.5, it suffices to show that $D(\lambda)$ is the identity matrix at $u = 0$.

Let A^s (resp. A'^s) be the matrix that expresses right multiplication by C_s in terms of the upper (resp. lower) seminormal basis of M_λ. Then by definition of $D(\lambda)$, $D(\lambda)A^sD(\lambda)^{-1} = A'^s$. Also, it follows from Proposition 7.7 that $A^s = (A'^s)^t$. Thus $D(\lambda)$ is determined up to a global scale by the equations $\frac{D(\lambda)_{Q'Q'}}{D(\lambda)_{QQ}} = \frac{A^s_{QQ'}}{A^s_{Q'Q}}$ for all $s \in S$, $Q, Q' \in \mathrm{SYT}(\lambda)$ such that $A^s_{Q'Q} \neq 0$ ($D(\lambda)$ must be determined uniquely by these equations up to a global scale because M_λ is irreducible).

Now $A^s = T(\lambda)^{-1}M^sT(\lambda)$, where M^s expresses right multiplication by C_s in terms of the upper canonical basis of M_λ (thus $M^s_{Q'Q} = \mu(Q', Q)$ if $s \in R(C_{Q'}) \setminus R(C_Q)$). Now we apply an easy modification of (31) to the product $-uA^s = T(\lambda)^{-1}(-uM^s)T(\lambda)$ to obtain (assuming $Q' \neq Q$)

$$\mu(-uA^s_{Q'Q}) = \chi\{s \in R(C_Q)\}(\mu(T^{-1}_{Q'Q})) + \mu(-uM^s_{Q'Q}) + \chi\{s \in R(C_{Q'})\}(\mu(T_{Q'Q}))$$

$$= \begin{cases} 0 + \mu(-uM^s_{Q'Q}) + 0 = -\mu(Q', Q) & \text{if } s \in R(C_{Q'}) \setminus R(C_Q) \text{ and } Q' \lhd Q, \\ -\mu(Q', Q) + 0 + 0 \quad = -\mu(Q', Q) & \text{if } s \in R(C_Q) \setminus R(C_{Q'}), \\ & (R(C_{Q'}) \setminus R(C_Q)) \cap J_{k(Q',Q)-1} \neq \emptyset, \text{ and } Q' \rhd Q, \end{cases} \tag{33}$$

where $\chi\{P\}(x)$ is equal to x if P is true and 0 otherwise. Here we have used the lower triangularity of $T(\lambda)$ for the top case and Proposition 7.5 (iv) for the bottom case (note that these cases do not cover all possibilities, and we do not know the answer in general).

To complete the proof, consider the dual Knuth equivalence graph on $\mathrm{SYT}(\lambda)$ as in, for instance, [1]. We say that a dual Knuth transformation $Q' \underset{\text{DKE}}{\longleftrightarrow} Q$ is *initial* if Q' and Q have the entries $i, i+1$ in different positions and $|R(C_{Q'}) \cap \{s_{i-1}, s_i\}| = |R(C_Q) \cap \{s_{i-1}, s_i\}| = 1$. For example, the dual Knuth equivalence $\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix} \underset{\text{DKE}}{\longleftrightarrow} \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix}$ is not initial. It is easy to check that $Q' \rhd Q$ and $Q' \underset{\text{DKE}}{\longleftrightarrow} Q$ initial implies $s_{k(Q',Q)-2} \in (R(C_{Q'}) \setminus R(C_Q)) \cap J_{k(Q',Q)-1}$. Hence by applying (33) to $A^s_{Q'Q}$ and $A^s_{QQ'}$ for any pair Q', Q such that $Q' \rhd Q$ and $Q' \underset{\text{DKE}}{\longleftrightarrow} Q$ is initial, and with $s \in R(C_Q) \setminus R(C_{Q'})$, we conclude $\frac{D(\lambda)_{Q'Q'}}{D(\lambda)_{QQ}} = \frac{A^s_{QQ'}}{A^s_{Q'Q}} \equiv 1 \mod uK_0$. The result then follows from the following combinatorial claim

(34) The graph on $\mathrm{SYT}(\lambda)$ consisting of initial dual Knuth transformations is connected.

The claim is proved by induction on $r = |\lambda|$. Let $\mu^1, \dots, \mu^l$ be the shapes obtained from λ by removing an outer corner. Assume that the graphs for the μ^i are connected. For any distinct $i, j \in [l]$, it is easy to construct $Q', Q \in \mathrm{SYT}(\lambda)$ such that $\mathrm{sh}(Q'|_{[r-1]}) = \mu^i$, $\mathrm{sh}(Q|_{[r-1]}) = \mu^j$, and $Q' \underset{\text{DKE}}{\longleftrightarrow} Q$ is a dual Knuth transformation. Such dual Knuth transformations are always initial, so the claim follows. □

7.3. **Positivity conjectures.** In our computations of many of the matrices discussed above, we have observed positivity properties, which we make precise below. Computing in Magma, we have verified (a) and (b) for all $\lambda \vdash r$, $r \leq 8$ and (c) for $r \leq 6$. Our original motivation for looking for positivity here is that the positivity of $S(\lambda)$ is related to the conjecture in [26] stating that an element spanning $\check{\epsilon}_+ \subseteq K\check{\mathscr{H}}_r \subseteq K\mathscr{H}_r \otimes \mathscr{H}_r$ has nonnegative coefficients when expressed in the basis $\{C_v \otimes C_w : v, w \in \mathcal{S}_r\}$ (see the introduction).

Conjecture 7.9. *For a non-zero matrix M with $\bar{\cdot}$-invariant entries in K, let $D(M)$ be the unique up to sign element of K such that $D(M)M$ has $\bar{\cdot}$-invariant entries in $\mathbf{A}$ and the greatest common divisor of the entries in $D(M)M$ is 1. The matrices $\tilde{T}, \tilde{T}', T(\lambda), T'(\lambda)^{-1}$, and $S(\lambda)$ from Corollary 7.1, Proposition 7.5, and Theorem 7.8 have the following positivity properties.*

(a) *Let M be $T(\lambda), T'(\lambda)^{-1}$, or $S(\lambda)$. After replacing $D(M)$ with $-D(M)$ if needed, all of the entries of $D(M)M$ have nonnegative coefficients.*

(b) *If M is $T(\lambda)$ or $T'(\lambda)^{-1}$, then $D(M)$ belongs to $\mathbf{A}$ and has all nonnegative or all nonpositive coefficients (this is not a sensible conjecture for $S(\lambda)$ because it is only well-defined up to a global scale).*

(c) $\pm D(\tilde{T}) = \pm D(\tilde{T}') = \pm [r]!$.

It follows from Proposition 7.7 that $T'(\lambda)^{-1} = T(\lambda)^t$, so the nonnegativity conjectures for $T(\lambda)$ and $T'(\lambda)^{-1}$ are equivalent. This conjecture, or rather its weakening discussed in the remark below, is supported by Proposition 7.5 (iv) since the $\mathcal{S}_r$-graph edge weights $\mu(Q', Q)$ are known to be nonnegative.

Remark 7.10. It is not completely clear how to define nonnegativity in K. At first, we used the following definition of nonnegativity: $f \in K$ is nonnegative if $f = g/h$,

$g, h \in \mathbf{A}$, g and h have no common factor, and g and h have nonnegative coefficients. To our surprise, we discovered that this is not a good definition because this subset is not a semiring. For example, $[2], [3]$, and $\frac{1}{[6]}$ are all nonnegative by this definition, but $\frac{[2][3]}{[6]} = \frac{u^2}{1-u^2+u^4}$ is not (in fact, this is an entry of $T'((6,2))$).

A strictly weaker definition of nonnegativity that we may adopt instead is: an element $f \in K$ is *nonnegative* if $f(a)$ is defined and nonnegative for all positive real a. With this definition, the set of nonnegative rational functions in u is a semiring and Conjecture 7.9 would imply that the matrices $T(\lambda), T'(\lambda)^{-1}$, and $S(\lambda)$ (after adjusting $S(\lambda)$ by a suitable global scale) have nonnegative entries.

Remark 7.11. It is tempting to conjecture from Figure 7.4 that every entry of $D(\tilde{T})\tilde{T}$ has either all nonnegative coefficients or all nonpositive coefficients. This turns out to be true for $r \leq 5$, but fails for $r = 6$—the only entries of $[6]!\tilde{T}$ without this property are equal to

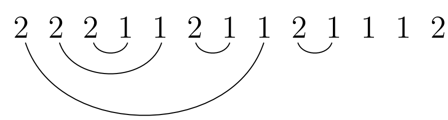
$$[2]^3[5]([3]-3) = u^9 + 2u^7 - 2u^3 - u - u^{-1} - 2u^{-3} + 2u^{-7} + u^{-9}.$$

Despite this failure, the matrices $\tilde{T}, \tilde{T}'$ deserve further investigation as their entries appear combinatorial in nature.

8. The two-row case

We now set $n = 2$ and use the graphical calculus of [12] to compute the transition matrix of Lemma 7.4 explicitly for λ a two-row partition.

Definition 8.1. The *diagram* of a word $\mathbf{k} \in [n]^r$ is the picture obtained from $\mathbf{k}$ by pairing 2s and 1s as left and right parentheses and then drawing an arc between matching pairs as shown below. The word $\mathbf{k}$ is *Yamanouchi* if its diagram has no unpaired 2s.



As shown in [12], diagrams provide a simple and beautiful way to visualize the action of $\mathbf{U}$ and $\mathscr{H}_r$ on the upper canonical basis. The first part of the next theorem is established in [12, §2.3], and the second part is obtained from the first by dualizing with respect to the inner product on $\mathbf{T}$.

Theorem 8.2.

(a) *The action of* F_1 *on the upper canonical basis of* $\mathbf{T}$ *is given by*

$$F_1(c_{\mathbf{k}}) = \sum_{j=1}^{t} [j] c_{\mathscr{F}_{(j)}(\mathbf{k})},$$

where t *is the number of unpaired* 1*s in* $\mathbf{k}$ *and* $\mathscr{F}_{(j)}(\mathbf{k})$ *is the word obtained by replacing the* j*-th unpaired* 1 *in* $\mathbf{k}$ *with a* 2 *(the first unpaired* 1 *means the leftmost unpaired 1 and the* t*-th means the rightmost).*

(b) *the action of* E_1 *on the lower canonical basis of* $\mathbf{T}$ *is given by*

$$E_1(c'_{\mathbf{k}^\dagger}) = \sum_{\mathbf{k}'} [\alpha(\mathbf{k}', \mathbf{k})] c'_{\mathbf{k}'^\dagger},$$

where $\alpha(\mathbf{k}', \mathbf{k})$ *is the positive integer* j *such that* $\mathscr{F}_{(j)}(\mathbf{k}') = \mathbf{k}$ *and 0 if there is no such positive integer.*

We will also make use of the action of the Kashiwara operator $\tilde{F}_1^{\text{up}}$ on the upper crystal basis (we abuse notation by letting the operator act on words rather than the crystal basis elements $b_{\mathbf{k}} \in \mathscr{B}$):

$\tilde{F}_1^{\text{up}}(\mathbf{k})$ is the word obtained by replacing the rightmost unpaired 1 in $\mathbf{k}$ with a 2 and is undefined if there are no unpaired 1s.

We need some notation for the next theorem. Let $\mathbf{k}|_j$ denote the subword $k_1k_2\cdots k_j$ of the word $\mathbf{k} = k_1k_2\cdots k_r$. Let f denote the function on $V^{\otimes r-1}$ given by

$$f(\sum_{\mathbf{j}\in[n]^{r-1}} a_{\mathbf{j}} c'_{\mathbf{j}^\dagger}) = \sum_{\mathbf{j}} a_{\mathbf{j}} c'_{\tilde{F}_1^{\text{up}}(\mathbf{j}^\dagger)},$$

where the $a_{\mathbf{j}}$ belong to K and the sum on the right is over those $\mathbf{j}$ such that $\tilde{F}_1^{\text{up}}(\mathbf{j}^\dagger)$ is defined.

Let λ be a partition of r with two rows and identify the lower canonical basis of M_λ with the $\mathscr{H}_r$-cell Γ'_{Z_λ} of $\mathbf{T}$ (the vertices of this cell are those $c'_{\mathbf{k}^\dagger}$ such that $\mathbf{k}$ is Yamanouchi and has content λ) via the right-hand isomorphism of (32) (with $T = Z_\lambda$). Set $\lambda^1 = (\lambda_1-1, \lambda_2)$ and $\lambda^2 = (\lambda_1, \lambda_2 - 1)$ and $l = \lambda_1 - \lambda_2$. Let $\tilde{F}_1^{\text{up}}(Z_{\lambda^2})$ denote the tableau obtained from Z_{λ^2} by changing the last entry in the first row to a 2.

We will compute the transition matrix of Lemma 7.4 for λ as above. We have found it more convenient to compute the matrix for $J^\dagger = \{s_2, \dots, s_{r-1}\}$ rather than $J = \{s_1, \dots, s_{r-2}\}$ (the matrix for J can then be obtained from that for $J^\dagger$ by conjugating by the permutation matrix corresponding to $C'_Q \mapsto C'_{Q^\dagger}$). Consider the weight space $\mathbf{T}^\lambda$, which is isomorphic to $\epsilon_+ \otimes_{K\mathscr{H}_{J_\lambda}} K\mathscr{H}_r$. Since the intersection of two cellular subquotients is a cellular subquotient, Proposition 3.2 with parabolic subgroup $(\mathcal{S}_r)_{J^\dagger}$ and Theorem 5.4 imply that

$$R : Res_{K\mathscr{H}_{J^\dagger}} K\{c'_{\mathbf{k}^\dagger} \in \mathbf{T}^\lambda : k_r = 1\} \xrightarrow{\cong} (V^{\otimes r-1})^{\lambda^1},\ c'_{\mathbf{k}^\dagger} \mapsto c'_{(\mathbf{k}|_{r-1})^\dagger} \tag{35}$$

is an isomorphism of $K\mathscr{H}_{J^\dagger}$-modules with basis. Quotienting by $\mathscr{H}_r$-cells below Γ'_{Z_λ}, this yields an isomorphism of modules with basis

$$Res_{K\mathscr{H}_{J^\dagger}} K\Gamma'_{Z_\lambda} \xrightarrow{\cong} K\big(\Gamma'_{Z_{\lambda^1}} \sqcup \Gamma'_{\tilde{F}_1^{\text{up}}(Z_{\lambda^2})}\big). \tag{36}$$

Theorem 8.3. *Maintain the notation above. For each* $c'_{\mathbf{k}^\dagger} \in \Gamma'_{Z_\lambda}$*, define the element*

$$(\tilde{c}'_{\mathbf{k}^\dagger})^{J^\dagger} := \begin{cases} c'_{\mathbf{k}^\dagger} - \dfrac{1}{[l+1]} R^{-1}(f(E_1(c'_{(\mathbf{k}|_{r-1})^\dagger}))) & \textit{if } \operatorname{sh}((\mathbf{k}|_{r-1})^\dagger) = \lambda^1, \\ c'_{\mathbf{k}^\dagger} & \textit{if } \operatorname{sh}((\mathbf{k}|_{r-1})^\dagger) = \lambda^2. \end{cases} \tag{37}$$

Applying $s_\lambda^{\mathbf{T}^\lambda}$ to both sides of this definition (s_λ^N is the surjection onto the M_λ-isotypic component of N) then yields $(\tilde{C}'_{Q(\mathbf{k})^\dagger})^{J^\dagger}$ expanded in terms of the lower canonical basis of M_λ.

It is helpful to follow the proof with an example: take $\lambda = (5,2)$ and $\mathbf{k} = 2112111$. Then

$$E_1(c'_{211211^\dagger}) = c'_{111211^\dagger} + [2]c'_{211111^\dagger},$$

$$(\tilde{c}'_{2112111^\dagger})^{J^\dagger} = c'_{2112111^\dagger} - \frac{1}{[4]}(c'_{1112121^\dagger} + [2]c'_{2111121^\dagger}),$$

$$\Big(\tilde{C}'_{\substack{12356\\47}}\Big)^{J^\dagger} = C'_{\substack{12356\\47}} - \frac{1}{[4]}\Big(C'_{\substack{13567\\24}} + [2]C'_{\substack{13456\\27}}\Big),$$

$$\Big(\tilde{C}'_{\substack{13467\\25}}\Big)^{J} = C'_{\substack{13467\\25}} - \frac{1}{[4]}\Big(C'_{\substack{12346\\57}} + [2]C'_{\substack{13456\\27}}\Big).$$

Proof. Assume throughout that $\mathbf{k}$ corresponds to the top case of (37), the arguments needed for the bottom case being easy. The key fact to check is that $E_1(R((\tilde{c}'_{\mathbf{k}^\dagger})^{J^\dagger}))$ is zero mod $V^{\otimes r-1}[\rhd\lambda^2]$. To see that this would prove the theorem, let η be the element of the weight space $(V^{\otimes r-1})^{\lambda^1}$ such that $E_1(\eta) = 0$ and $R((\tilde{c}'_{\mathbf{k}^\dagger})^{J^\dagger}) - \eta \in V^{\otimes r-1}[\rhd\lambda^2]$. Then η is a highest weight vector of weight λ^1, so by quantum Schur-Weyl duality, η belongs to the M_{λ^1}-isotypic component of $V^{\otimes r-1}$. Thus $R((\tilde{c}'_{\mathbf{k}^\dagger})^{J^\dagger})$ and η only differ by lower canonical basis elements outside of $\Gamma'_{Z_{\lambda^1}} \sqcup \Gamma'_{\tilde{F}_1^{\mathrm{up}}(Z_{\lambda^2})}$; so by (36), $(\tilde{c}'_{\mathbf{k}^\dagger})^{J^\dagger}$, regarded as an element of the cellular subquotient $K\Gamma'_{Z_\lambda}$, belongs to the M_{λ^1}-isotypic component of $\mathrm{Res}_{K\mathscr{H}_{J^\dagger}} K\Gamma'_{Z_\lambda}$.

Now, checking the key fact amounts to showing that if

$$(E_1 - \frac{E_1}{[l+1]}fE_1)(c'_{(\mathbf{k}|_{r-1})^\dagger}) = (1 - \frac{E_1}{[l+1]}f)\sum_{\mathbf{j}'}[\alpha(\mathbf{j}', \mathbf{k}|_{r-1})]c'_{\mathbf{j}'^\dagger}$$

is written as $\sum_{\mathbf{j}\in[n]^{r-1}} a_\mathbf{j} c'_{\mathbf{j}^\dagger}$, then $a_\mathbf{j} = 0$ for $\mathbf{j}$ such that $\mathbf{j}$ is Yamanouchi. Here we are using the fact that $(V^{\otimes r-1})^{\lambda^2}[\rhd\lambda^2]$ is spanned by $c'_{\mathbf{j}^\dagger}$ such that $\mathbf{j}$ has content λ^2 and is not Yamanouchi. Now let $\mathbf{j}$ be of content λ^2 and Yamanouchi; then one checks that $c'_{\mathbf{j}^\dagger}$ occurs as a term of $E_1 f c'_{\mathbf{j}'^\dagger}$ expanded in the lower canonical basis if and only if $\mathbf{j} = \mathbf{j}'$, and if it does occur, then its coefficient is $[l+1]$ since $l+1$ is the number of unpaired 1s in $\mathbf{j}$. It follows that $a_\mathbf{j} = 0$, as desired. □

Remark 8.4. The recent paper [8] studies the matrix $T'(\lambda)$ for λ a two-row partition. The lower canonical basis of M_λ is realized in a polynomial representation of $\mathscr{H}_r$ and the lower seminormal basis of M_λ is given by specialized non-symmetric Macdonald polynomials. Let $\lambda = (r/2, r/2)$ and Q be the SYT of shape λ such that the first row of Q has odd entries and the second has even entries. The authors show that the coefficients of C'^{sn}_Q expressed in the lower canonical basis of M_λ (i.e. the last column of $T'(\lambda)$) are all powers of $-\frac{1}{[2]}$ and they give a combinatorial formula for the exponents.

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