THE DARK SIDE OF GENERALIZED DEMAZURE CRYSTALS

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Abstract. Naoi [20] showed that tensor products of perfect Kirillov-Reshetikhin crystals are isomorphic to certain generalized Demazure crystals. We extend Naoi's results to address distinguished subsets of these tensor products. In type A, these are naturally described in terms of katabolizable tableaux which was key to resolving conjectures of Shimozono-Weyman [25] and Chen-Haiman [2] in [1].

1. Introduction

Naoi [20] showed that tensor products of perfect Kirillov-Reshetikhin (KR) crystals are isomorphic to certain generalized Demazure crystals introduced by Lakshmibai-Littelmann-Magyar [14]. From this he obtained a Demazure operator formula for their characters using the well-developed theory of Demazure crystals [5, 11, 14, 17]. This formed a key step in his resolution of the $X = M$ conjecture [19] in type $D_n^{(1)}$.

We extend Naoi's result to match a larger class of generalized Demazure crystals with certain subsets of tensor products of perfect KR crystals, termed Kirillov-Reshetikhin affine Demazure (DARK) crystals. Our result follows directly from techniques of [20], but the deep combinatorial consequences shown for type A in [1] motivate this presentation of the results in the full generality of any nonexceptional type.

Naoi's work encompasses several earlier results connecting Demazure and KR crystals [4, 23, 24]. The emphasis in these works is on providing a model for KR crystals using the well-developed theory of Demazure crystals, whereas here we are interested in using KR crystals to understand generalized Demazure crystals. While there are combinatorial models of highest weight crystals for affine type [7, 8, 9, 16, 18] which lead to explicit descriptions of generalized Demazure crystals, our explorations suggest that the combinatorics afforded by DARK crystals is simpler. This is possible because the isomorphism between generalized Demazure and DARK crystals is combinatorially nontrivial, roughly analogous to the different models for the $U_q(\mathfrak{gl}_n)$-highest weight crystal $B(\nu)$ afforded by semistandard tableaux of shape ν versus those provided by an embedding $B(\nu) \hookrightarrow B(\lambda) \otimes B(\mu)$.

2. Background on crystals

We follow [20] almost completely and review the notation we will need, emphasizing conventions which may not be well known.

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2.1. **Affine Kac-Moody Lie algebras.** Let $\mathfrak{g}$ be a complex affine Kac-Moody Lie algebra of nonexceptional type (i.e., of type $A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$, or $D_{n+1}^{(2)}$). Let $I = \{0, 1, \ldots, n\}$ be the Dynkin nodes and $A = (a_{ij})_{i,j\in I}$ the Cartan matrix. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra, which has a basis consisting of the simple coroots $\{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$ together with the scaling element $d \in \mathfrak{h}$. We have the linearly independent simple roots $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$, with pairings $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ and $\langle d, \alpha_i \rangle = \delta_{i0}$ $(i, j \in I)$. Let $(a_0, \ldots, a_n)$ (resp. $(a_0^\vee, \ldots, a_n^\vee)$) be the unique tuple of relatively prime positive integers that give a linear dependence relation among the columns (resp. rows) of A.

Choose $N \in \mathbb{Z}_{\geq 1}$ and fundamental weights $\{\Lambda_i \mid i \in I\} \subset \mathfrak{h}^*$ such that $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ and $\langle d, \Lambda_j \rangle \in N^{-1}\mathbb{Z}$ for $i, j \in I$, the choices to be discussed further below. Let $\delta = \sum_{i\in I} a_i \alpha_i$ be the null root. Note that $\{\Lambda_i \mid i \in I\} \sqcup \{\delta\}$ is a basis of $\mathfrak{h}^*$ and $\langle \alpha_i^\vee, \delta \rangle = 0$ for $i \in I$ and $\langle d, \delta \rangle = a_0$. Let $P = \big\{\mu \in \mathfrak{h}^* \mid \langle \alpha_i^\vee, \mu \rangle \in \mathbb{Z} \text{ for } i \in I, \langle d, \mu \rangle \in N^{-1}\mathbb{Z}\big\} = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\frac{\delta}{a_0 N} \subset \mathfrak{h}^*$ be the weight lattice and $P^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i + \mathbb{Z}\frac{\delta}{a_0 N}$ the dominant weights.

Let $\mathrm{cl}\colon \mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{C}\delta$ be the canonical projection, and set $P_{\mathrm{cl}} = \mathrm{cl}(P) = \bigoplus_{i\in I} \mathbb{Z}\,\mathrm{cl}(\Lambda_i)$. Let $\mathrm{aff}\colon \mathfrak{h}^*/\mathbb{C}\delta \to \mathfrak{h}^*$ be the section of cl satisfying $\langle d, \mathrm{aff}(\mu) \rangle = 0$ for all $\mu \in h^*/\mathbb{C}\delta$. Set $\varpi_i = \mathrm{aff}(\mathrm{cl}(\Lambda_i - a_i^\vee \Lambda_0))$ for $i \in I_0 := I \setminus \{0\}$.

The *affine Weyl group* W can be realized as the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections s_i $(i \in I)$, where s_i acts by $s_i(\mu) = \mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i$. Let W_0 be the subgroup generated by s_i for $i \in I_0$.

Let $c_i = \max\{1, a_i/a_i^\vee\}$ for $i \in I_0$, and define $\widetilde{M} = \bigoplus_{i\in I_0} \mathbb{Z}c_i\varpi_i \subset P$. For $\mu \in \widetilde{M}$, define the translation $t_\mu \in GL(\mathfrak{h}^*)$ as in [6, Equation 6.5.2] and set $T = \{t_\mu \mid \mu \in \widetilde{M}\}$. These satisfy $t_\mu t_\lambda = t_{\mu+\lambda}$ and $w t_\mu w^{-1} = t_{w(\mu)}$ for $w \in W_0$ and $\mu, \lambda \in \widetilde{M}$. Thus $\widetilde{W} = W_0 \ltimes T$ is a subgroup of $GL(\mathfrak{h}^*)$, called the *extended affine Weyl group*.

Let $\Sigma \subset \widetilde{W}$ denote the subgroup which takes the set $\{\alpha_i \mid i \in I\}$ to itself. Thus each element $\tau \in \Sigma$ yields a permutation of I, which we also denote by τ; it is an automorphism of the Dynkin diagram, meaning that $a_{ij} = a_{\tau(i)\tau(j)}$ for all $i, j \in I$. (See [20, §2.2, §5.2] for the explicit description of Σ as a set of permutations in each type.) There holds $\widetilde{W} \cong W \rtimes \Sigma$. As discussed in [20, §2.2], we can choose N and $\langle d, \Lambda_i \rangle$ so that for all $\tau \in \Sigma$, $\tau(\Lambda_j) = \Lambda_{\tau(j)}$ for all $j \in I$ and $\tau(\delta) = \delta$. Note that this implies $\widetilde{W}$ preserves P.

2.2. **Crystals.** Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra (as in [10]) specified by the above data and the symmetric bilinear form $(\cdot,\cdot)\colon P \times P \to \mathbb{Q}$ defined by $(\alpha_i, \alpha_j) = a_i^\vee a_i^{-1} a_{ij}$, $(\alpha_i, \Lambda_0) = a_0^{-1}\delta_{i0}$, $(\Lambda_0, \Lambda_0) = 0$. It is generated by e_i, f_i, $i \in I$, and q^h, $h \in P^* := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$. Let $U_q'(\mathfrak{g}) \subset U_q(\mathfrak{g})$ be the subalgebra generated by e_i, f_i, $i \in I$, and q^h, $h \in P_{\mathrm{cl}}^* = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$. A $U_q(\mathfrak{g})$-*crystal* (resp. $U_q'(\mathfrak{g})$-crystal) is a set B equipped with a *weight function* $\mathrm{wt}\colon B \to P$ (resp. $\mathrm{wt}\colon B \to P_{\mathrm{cl}}$) and *crystal operators* $\tilde{e}_i, \tilde{f}_i\colon B \sqcup \{0\} \to B \sqcup \{0\}$ $(i \in I)$ such that for all $i \in I$ and $b \in B$, there holds $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$ and

$$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \text{ whenever } \tilde{e}_i b \neq 0, \quad \text{and} \quad \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \text{ whenever } \tilde{f}_i b \neq 0;$$

$$\varepsilon_i(b) := \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\} < \infty, \quad \phi_i(b) := \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\} < \infty;$$

$$\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = \phi_i(b) - \varepsilon_i(b);$$

$\tilde{f}_i(\tilde{e}_i b) = b$ whenever $\tilde{e}_i b \neq 0$, and $\tilde{e}_i(\tilde{f}_i b) = b$ whenever $\tilde{f}_i b \neq 0$.

This agrees with the notion of a seminormal crystal in [12, §7]. We use the term *crystal* to mean either a $U_q(\mathfrak{g})$-crystal or $U_q'(\mathfrak{g})$-crystal.

A *strict embedding* of a crystal B into a crystal B' is an injective map $\Psi\colon B \sqcup \{0\} \to B' \sqcup \{0\}$ such that $\Psi(0) = 0$ and Ψ commutes with wt, ε_i, ϕ_i, $\tilde{e}_i$, and $\tilde{f}_i$ for all $i \in I$. It is necessarily an isomorphism from B onto a disjoint union of connected components of B'.

For a $U_q(\mathfrak{g})$-crystal B with weight function $\mathrm{wt}\colon B \to P$, its $U_q'(\mathfrak{g})$*-restriction* is the $U_q'(\mathfrak{g})$-crystal with the same edges as B and weight function $\mathrm{cl} \circ \mathrm{wt}\colon B \to P_{\mathrm{cl}}$.

For two crystals B_1 and B_2, their tensor product $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ is the crystal with weight function $\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2)$ and crystal operators

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \phi_i(b_1) < \varepsilon_i(b_2). \end{cases} \tag{2.1}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{2.2}$$

Kirillov-Reshetikhin modules $W^{r,s}$ are finite-dimensional $U_q'(\mathfrak{g})$-modules parameterized by $(r,s) \in I_0 \times \mathbb{Z}_{\geq 1}$. For nonexceptional $\mathfrak{g}$, the $W^{r,s}$ have crystal pseudobases [9, 21, 22], and these yield $U_q'(\mathfrak{g})$-crystals $B^{r,s}$ known as KR crystals. We are interested in the subclass of perfect KR crystals (see [8]); we will not work with the definition directly, but only need the following from [3]: a KR crystal $B^{r,s}$ is perfect if and only if $c_r = \max\{1, a_r/a_r^\vee\}$ divides s.

2.3. **Dynkin diagram automorphisms and crystals.** For $\tau \in \Sigma$ and $U_q(\mathfrak{g})$-crystals (resp. $U_q'(\mathfrak{g})$-crystals) B, B', a bijection of sets $z\colon B \to B'$ is a τ*-twist* if

$$\tau(\mathrm{wt}(b)) = \mathrm{wt}(z(b)), \quad \text{and}$$
$$z(\tilde{e}_i b) = \tilde{e}_{\tau(i)} z(b), \quad z(\tilde{f}_i b) = \tilde{f}_{\tau(i)} z(b) \ \text{ for all } i \in I, \ \text{ where } z(0) := 0.$$

Since $\tau(P) = P$ and $\tau(\delta) = \delta$, τ yields automorphisms of P and P_{cl}, and thus $\tau(\mathrm{wt}(b))$ belongs to P (resp. P_{cl}).

Proposition 2.1 ([23, Lemma 6.5], [20, Proposition 5.5])**.** *For any KR crystal B and $\tau \in \Sigma$, there exists a unique τ-twist of $U_q'(\mathfrak{g})$-crystals $\mathcal{F}_\tau^B\colon B \to B$.*

There is also a unique τ-twist $\mathcal{F}_\tau^\Lambda\colon B(\Lambda) \to B(\tau(\Lambda))$ for any $\Lambda \in P^+$, where $B(\Lambda)$ is the $U_q(\mathfrak{g})$-crystal of the irreducible $U_q(\mathfrak{g})$-module of highest weight Λ [10, 12].

It is easily verified that if $z_1\colon B_1 \to B_1'$ and $z_2\colon B_2 \to B_2'$ are τ-twists, then so is $z_1 \otimes z_2\colon B_1 \otimes B_2 \to B_1' \otimes B_2'$. Thus the tensor product of maps $\mathcal{F}_\tau^{\Lambda^1} \otimes \mathcal{F}_\tau^{\Lambda^2}$ is the natural choice of τ-twist from any tensor product $B(\Lambda^1) \otimes B(\Lambda^2)$ of highest weight $U_q(\mathfrak{g})$-crystals, $\Lambda^1, \Lambda^2 \in P^+$. Using in addition Proposition 2.1, a similar *τ-twist exists from any tensor product of KR crystals and highest weight crystals to another such product, and we denote it $\mathcal{F}_\tau$* (these are the only crystals we will consider in this paper); for example, for a KR crystal B, we denote by $\mathcal{F}_\tau$ the map $\mathcal{F}_\tau^{\Lambda_0} \otimes \mathcal{F}_\tau^B\colon B(\Lambda_0) \otimes B \to B(\Lambda_{\tau(0)}) \otimes B$, where $B(\Lambda_0)$ and $B(\Lambda_{\tau(0)})$ are regarded as $U_q'(\mathfrak{g})$-crystals by restriction.

2.4. **Generalized Demazure crystals.** For a crystal B, $S \subset B$, and $i \in I$, define

$$\mathcal{F}_i S := \{\tilde{f}_i^k b \mid b \in S, k \geq 0\} \setminus \{0\} \subset B.$$

For a reduced expression $w = s_{i_1} \cdots s_{i_m} \in W$, we write $\mathcal{F}_w S$ for $\mathcal{F}_{i_1} \cdots \mathcal{F}_{i_m} S$ when this is well defined, i.e., does not depend on the choice of reduced expression. A *Demazure crystal* is a subset of some highest weight $U_q(\mathfrak{g})$-crystal $B(\Lambda)$ of the form $B_w(\Lambda) := \mathcal{F}_w\{u_\Lambda\}$ for some $w \in W$, where u_Λ is the highest weight element of $B(\Lambda)$; it is well defined by [11].

A *generalized Demazure crystal* is a subset of a tensor product of highest weight crystals of the form $\mathcal{F}_{w_1}\mathcal{F}_{\tau_1}\big(u_{\Lambda^1} \otimes \mathcal{F}_{w_2}\mathcal{F}_{\tau_2}\big(u_{\Lambda^2} \otimes \cdots \mathcal{F}_{w_p}\mathcal{F}_{\tau_p}(\{u_{\Lambda^p}\}) \cdots\big)\big)$ for some $\Lambda^1, \ldots, \Lambda^p \in P^+$, $w_1, \ldots, w_p \in W$, and $\tau_1, \ldots, \tau_p \in \Sigma$. The combinatorial excellent filtration theorem [14], [5] states that $u_{\Lambda^1} \otimes \mathcal{F}_{w_2}\{u_{\Lambda^2}\} \subset B(\Lambda^1) \otimes B(\Lambda^2)$ is a disjoint union of Demazure crystals. It follows that (see [20, Lemma 4.3])

Theorem 2.2. *Any generalized Demazure crystal is a disjoint union of Demazure crystals (and the above expression is well defined).*

3. Matching generalized Demazure and DARK crystals

Lemma 3.1 ([10])**.** *For any $\Lambda, \Lambda' \in P^+$, there is a strict embedding of $U_q(\mathfrak{g})$-crystals $B(\Lambda + \Lambda') \hookrightarrow B(\Lambda) \otimes B(\Lambda')$ determined by $u_{\Lambda+\Lambda'} \mapsto u_\Lambda \otimes u_{\Lambda'}$; it maps $B(\Lambda + \Lambda')$ isomorphically onto a connected component of $B(\Lambda) \otimes B(\Lambda')$.*

Proof. It is well known [10] that $B(\Lambda) \otimes B(\Lambda')$ is isomorphic to a disjoint union of highest weight crystals. Since $\tilde{e}_i(u_\Lambda \otimes u_{\Lambda'}) = 0$ for all $i \in I$, it is the highest weight element of a connected component isomorphic to $B(\Lambda + \Lambda')$. □

The following result gives a beautiful connection between Demazure and KR crystals; part (i) is due to [8] and (ii) to [23, Theorem 6.1]. Let w_0 be the longest element of W_0.

Theorem 3.2. *Let $B = B^{r,c_r s}$ be a perfect KR crystal. There is a unique element $\mathsf{b}^{r,s} \in B$ satisfying $\varepsilon_0(\mathsf{b}^{r,s}) = s$ and $\varepsilon_i(\mathsf{b}^{r,s}) = 0$ for $i \in I_0$. Put $\mu = c_r w_0(\varpi_r)$ and write $t_\mu = y\tau$ with $y \in W$, $\tau \in \Sigma$.*

(i) *There is a $U'_q(\mathfrak{g})$-crystal isomorphism*

$$\theta\colon B(s\Lambda_0) \otimes B \xrightarrow{\cong} B(s\Lambda_{\tau(0)})$$

which maps $u_{s\Lambda_0} \otimes \mathsf{b}^{r,s} \mapsto u_{s\Lambda_{\tau(0)}}$. Here, $B(s\Lambda_0)$ and $B(s\Lambda_{\tau(0)})$ are regarded as $U'_q(\mathfrak{g})$-crystals by restriction—see §2.2.

(ii) *θ maps the subset $u_{s\Lambda_0} \otimes B$ onto the Demazure crystal $B_y(s\Lambda_{\tau(0)}) \subset B(s\Lambda_{\tau(0)})$.*

Remark 3.3. It is convenient to allow $s = 0$ in Theorem 3.2, which holds (trivially) with $B^{r,0} = \{\mathsf{b}^{r,0}\}$ defined to be the trivial $U'_q(\mathfrak{g})$-crystal, i.e., $\mathrm{wt}(\mathsf{b}^{r,0}) = 0$ and $\tilde{e}_i(\mathsf{b}^{r,0}) = \tilde{f}_i(\mathsf{b}^{r,0}) = 0$ for all $i \in I$. Note that $B(0\Lambda_i) = B(0) = \{u_0\}$ is the trivial $U_q(\mathfrak{g})$-crystal.

Lemma 3.4. *Let A and Z be $U'_q(\mathfrak{g})$-crystals. Let $u \in A$, $z \in Z$, and $j_1, \ldots, j_m \in I$, and suppose that $\mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m}(u \otimes z) \subset u \otimes Z$ in $A \otimes Z$. Then for any $G = \tilde{f}_{j_1}^{d_1} \cdots \tilde{f}_{j_m}^{d_m}$, $d_i \in \mathbb{Z}_{\geq 0}$, $G(u \otimes z) = u \otimes G(z)$.*

Proof. The containment tells us that each application of $\tilde{f}_i$ in computing $G(u \otimes z)$ can only be applied to u if $\tilde{f}_i(u) = 0$, but this would mean $\phi_i(u) = 0$ and $\tilde{f}_i$ is applied on the left tensor factor, which is not allowed by the tensor product rule (2.2). □

Lemma 3.5. *Maintain the notation of Theorem 3.2 and in addition let $w \le y$ in Bruhat order and let $w = s_{j_1} \cdots s_{j_m}$ be a reduced expression for w. Let C be any $U_q'(\mathfrak{g})$-crystal. Then for any $G = \tilde{f}_{j_1}^{d_1} \cdots \tilde{f}_{j_m}^{d_m}$, $d_i \in \mathbb{Z}_{\ge 0}$, and $c \in C$, there holds $G(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s} \otimes c) = u_{s\Lambda_0} \otimes G(\mathsf{b}^{r,s} \otimes c)$ in the $U_q'(\mathfrak{g})$-crystal $B(s\Lambda_0) \otimes B \otimes C$.*

Proof. By Theorem 3.2, $u_{s\Lambda_0} \otimes B = \theta^{-1}(\mathcal{F}_y(u_{s\Lambda_{\tau(0)}})) = \mathcal{F}_y(\theta^{-1}(u_{s\Lambda_{\tau(0)}})) = \mathcal{F}_y(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s})$. Hence $\mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m}(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s}) \subset \mathcal{F}_y(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s}) = u_{s\Lambda_0} \otimes B$ in $B(s\Lambda_0) \otimes B$. This implies $\mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m}(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s} \otimes c) \subset \mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m}(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s}) \otimes \mathcal{F}_{j_1} \cdots \mathcal{F}_{j_m}(c) \subset u_{s\Lambda_0} \otimes B \otimes C$. The result then follows from Lemma 3.4 with $A = B(s\Lambda_0)$, $Z = B \otimes C$. □

Theorem 3.6. *Let $B_j = B^{r_j, c_{r_j}\lambda_j}$ for $j \in [p]$ be perfect KR crystals with $\mathbf{r} = (r_1, \ldots, r_p) \in (I_0)^p$ and $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p \ge 0)$, and set $\lambda^j = \lambda_j - \lambda_{j+1}$ with $\lambda_{p+1} = 0$. Put $\mu_j = c_{r_j}\omega_0(\varpi_{r_j})$ and write $t_{\mu_j} = y_j\tau_j$ with $y_j \in W$ and $\tau_j \in \Sigma$. There is a strict embedding (see §2.2) of $U_q'(\mathfrak{g})$-crystals*

$$\Theta_{\mathbf{r},\lambda} \colon B(\lambda_1\Lambda_0) \otimes B_1 \otimes \cdots \otimes B_p \to B(\lambda^1\Lambda_{\tau_1(0)}) \otimes \cdots \otimes B(\lambda^p\Lambda_{\tau_1\tau_2\cdots\tau_p(0)}). \tag{3.1}$$

Proof. Apply the isomorphism θ of Theorem 3.2 to the left two factors, then the strict embedding of Lemma 3.1, then apply $\mathcal{F}_{\tau_1}\theta\mathcal{F}_{\tau_1}^{-1}$ to the second and third factors, and so on:

$$\begin{aligned}
&B(\lambda_1\Lambda_0) \otimes B_1 \otimes B_2 \otimes \cdots \otimes B_p \\
\xrightarrow{\cong}\ &B(\lambda_1\Lambda_{\tau_1(0)}) \otimes B_2 \otimes \cdots \otimes B_p \\
\hookrightarrow\ &B(\lambda^1\Lambda_{\tau_1(0)}) \otimes B(\lambda_2\Lambda_{\tau_1(0)}) \otimes B_2 \otimes B_3 \otimes \cdots \otimes B_p \\
\xrightarrow{\cong}\ &B(\lambda^1\Lambda_{\tau_1(0)}) \otimes B(\lambda_2\Lambda_{\tau_1\tau_2(0)}) \otimes B_3 \otimes \cdots \otimes B_p \\
\hookrightarrow\ &B(\lambda^1\Lambda_{\tau_1(0)}) \otimes B(\lambda^2\Lambda_{\tau_1\tau_2(0)}) \otimes B(\lambda_3\Lambda_{\tau_1\tau_2(0)}) \otimes B_3 \otimes \cdots \otimes B_p \\
&\cdots \\
\to\ &B(\lambda^1\Lambda_{\tau_1(0)}) \otimes B(\lambda^2\Lambda_{\tau_1\tau_2(0)}) \otimes \cdots \otimes B(\lambda^p\Lambda_{\tau_1\tau_2\cdots\tau_p(0)}).
\end{aligned}$$

□

Theorem 3.7. *Maintain the notation of Theorem 3.6 and in addition let $w_1, \ldots, w_p \in W$ with $w_i \le y_i$ for all i. Then the subset*

$$\mathcal{B} := \mathcal{F}_{w_1}\big(\mathsf{b}^{r_1,\lambda_1} \otimes \mathcal{F}_{\tau_1}\mathcal{F}_{w_2}\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}}\mathcal{F}_{w_p}(\{\mathsf{b}^{r_p,\lambda_p}\}) \cdots\big)\big) \subset B_1 \otimes \cdots \otimes B_p \tag{3.2}$$

is well defined—we call it a DARK crystal—and the image of $u_{\lambda_1\Lambda_0} \otimes \mathcal{B}$ under the strict embedding $\Theta_{\mathbf{r},\lambda}$ is a generalized Demazure crystal:

$$\Theta_{\mathbf{r},\lambda}(u_{\lambda_1\Lambda_0} \otimes \mathcal{B}) = \mathcal{F}_{w_1}\mathcal{F}_{\tau_1}\big(u_{\lambda^1\Lambda_0} \otimes \mathcal{F}_{w_2}\mathcal{F}_{\tau_2}\big(u_{\lambda^2\Lambda_0} \otimes \cdots \mathcal{F}_{w_p}\mathcal{F}_{\tau_p}(\{u_{\lambda^p\Lambda_0}\}) \cdots\big)\big). \tag{3.3}$$

Proof. Choose reduced expressions $w_i = s_{j_{i,1}} \cdots s_{j_{i,m_i}}$ for all $i \in [p]$. For now, interpret $\mathcal{F}_{w_i}$ in (3.2) as $\mathcal{F}_{j_{i,1}} \cdots \mathcal{F}_{j_{i,m_i}}$. Then we can specify an arbitrary element of $u_{\lambda_1\Lambda_0} \otimes \mathcal{B}$ as in (3.4) by choosing arbitrary $G_i = \tilde{f}_{j_{i,1}}^{d_{i,1}} \cdots \tilde{f}_{j_{i,m_i}}^{d_{i,m_i}}$ with $d_{i,1}, \ldots, d_{i,m_i} \in \mathbb{Z}_{\ge 0}$ for $i \in [p]$.

Tracing through the maps making up $\Theta_{\mathbf{r},\lambda}$, we obtain

$$
\begin{aligned}
& u_{\lambda_1\Lambda_0} \otimes G_1\Big(\mathsf{b}^{r_1,\lambda_1} \otimes \mathcal{F}_{\tau_1} G_2\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && (3.4)\\
&= G_1\Big(u_{\lambda_1\Lambda_0} \otimes \mathsf{b}^{r_1,\lambda_1} \otimes \mathcal{F}_{\tau_1} G_2\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && \text{by Lemma 3.5}\\
&\mapsto G_1\Big(u_{\lambda_1\Lambda_{\tau_1(0)}} \otimes \mathcal{F}_{\tau_1} G_2\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && \text{by Theorem 3.2 (i)}\\
&= G_1\mathcal{F}_{\tau_1}\Big(u_{\lambda_1\Lambda_0} \otimes G_2\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && \text{by §2.3}\\
&\mapsto G_1\mathcal{F}_{\tau_1}\Big(u_{\lambda^1\Lambda_0} \otimes u_{\lambda_2\Lambda_0} \otimes G_2\big(\mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && \text{by Lemma 3.1}\\
&= G_1\mathcal{F}_{\tau_1}\Big(u_{\lambda^1\Lambda_0} \otimes G_2\big(u_{\lambda_2\Lambda_0} \otimes \mathsf{b}^{r_2,\lambda_2} \otimes \cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big) && \text{by Lemma 3.5}\\
&\mapsto G_1\mathcal{F}_{\tau_1}\Big(u_{\lambda^1\Lambda_0} \otimes G_2\Big(u_{\lambda_2\Lambda_{\tau_2(0)}} \otimes \mathcal{F}_{\tau_2} G_3\big(\cdots \mathcal{F}_{\tau_{p-1}} G_p(\mathsf{b}^{r_p,\lambda_p})\big)\Big)\Big) && \text{by Theorem 3.2 (i)}\\
&\cdots\\
&\mapsto G_1\mathcal{F}_{\tau_1}\Big(u_{\lambda^1\Lambda_0} \otimes G_2\mathcal{F}_{\tau_2}\Big(u_{\lambda^2\Lambda_0} \otimes G_3\tau_3\big(\cdots G_p\mathcal{F}_{\tau_p}(u_{\lambda^p\Lambda_0})\big)\Big)\Big),
\end{aligned}
$$

which is an arbitrary element of the right side of (3.3). Moreover, by Theorem 2.2, the right side of (3.3) does not depend on the chosen reduced expressions for w_i, so the same goes for $\mathcal{B}$ since $\Theta_{\mathbf{r},\lambda}$ is injective. □

Let $\mathbb{Z}[P]$ denote the group ring of P with $\mathbb{Z}$-basis $\{e^\mu\}_{\mu\in P}$. The *Demazure operators* are linear operators D_i on $\mathbb{Z}[P]$ defined for each $i \in I$ by $D_i(f) = \frac{f - e^{-\alpha_i}\cdot s_i(f)}{1-e^{-\alpha_i}}$, where s_i acts on $\mathbb{Z}[P]$ by $s_i(e^\mu) = e^{s_i(\mu)}$. We also have an action of Σ on $\mathbb{Z}[P]$ given by $\tau(e^\mu) = e^{\tau(\mu)}$. For a reduced expression $w = s_{i_1}\cdots s_{i_m} \in W$, define the operator $D_w = D_{i_1}\cdots D_{i_m}$ on $\mathbb{Z}[P]$; it is independent of the choice of reduced expression [13, Corollary 8.2.10].

Naoi [20, Theorem 7.1] showed that $\Theta_{\mathbf{r},\lambda}$ matches the statistic $\langle d, \mathrm{wt}(b)\rangle$ on $U_q(\mathfrak{g})$-crystals to energy. Combining this with Theorem 2.2 and [20, Corollary 4.6] we obtain

Corollary 3.8. *Maintain the notation of Theorem 3.7. The energy adjusted character of the DARK crystal $\mathcal{B}$ agrees with the character of the generalized Demazure crystal in* (3.3) *(call it $\mathcal{B}'$), and both have the following Demazure operator formula:*

$$e^{\lambda_1\Lambda_0+\delta C}\sum_{b\in\mathcal{B}} e^{\mathrm{aff}(\mathrm{wt}(b))-\delta\frac{D(b)}{a_0}} = \sum_{b\in\mathcal{B}'} e^{\mathrm{wt}(b)} = D_{w_1}\tau_1\big(e^{\lambda^1\Lambda_0}\cdot D_{w_2}\tau_2\big(e^{\lambda^2\Lambda_0}\cdots D_{w_p}\tau_p(e^{\lambda^p\Lambda_0})\big)\big),$$

where $D(b)$ is the energy of b and $C \in \mathbb{Q}$ is a constant which depends only on λ and $\mathbf{r}$.

Remark 3.9. When $w_i = y_i$ for all $i \in [p]$, the DARK crystal $\mathcal{B}$ in (3.2) is equal to $B_1 \otimes \cdots \otimes B_p$ (this follows from $\mathcal{F}_{y_i}\{\mathsf{b}^{r_i,\lambda_i}\} = B_i$ and [20, Lemma 5.15]). Thus Proposition 5.16/Corollary 7.2 of [20] are encompassed by Theorem 3.7/Corollary 3.8. Note that the a_0 appearing in Corollary 3.8 corrects a typo in [20, Corollary 7.2].

Remark 3.10. In Theorem 3.7 and Corollary 3.8, we can more generally allow w_i of the form $w_i = v_i w_i'$ where $v_i \in W_0$ and $w_i' \le y_i$. Indeed, in the setting of Lemma 3.5, for any $j \in I_0$, $\tilde{f}_j G(u_{s\Lambda_0} \otimes \mathsf{b}^{r,s} \otimes c) = \tilde{f}_j(u_{s\Lambda_0} \otimes G(\mathsf{b}^{r,s} \otimes c)) = u_{s\Lambda_0} \otimes \tilde{f}_j G(\mathsf{b}^{r,s} \otimes c)$ as $\tilde{f}_j(u_{s\Lambda_0}) = \tilde{e}_j(u_{s\Lambda_0}) = 0$; hence the lemma holds more generally with $w = vw'$ with $v \in W_0$, $w' \le y$.

For $\mathfrak{g}$ of type $A_n^{(1)}$ and $\mathbf{r} = \mathbf{1} = (1, \dots, 1)$, Theorems 3.6 and 3.7 and Remark 3.10 give

Corollary 3.11. *Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_p \geq 0)$ and set $\lambda^j = \lambda_j - \lambda_{j+1}$ with $\lambda_{p+1} = 0$. Let τ be the Dynkin diagram automorphism given by $j \mapsto j+1 \bmod n+1$. Then there is a strict embedding of $U_q'(\widehat{\mathfrak{sl}}_{n+1})$-crystals*

$$\Theta_{\mathbf{1},\lambda} \colon B(\lambda_1\Lambda_0) \otimes B^{1,\lambda_1} \otimes \cdots \otimes B^{1,\lambda_p} \to B(\lambda^1\Lambda_1) \otimes \cdots \otimes B(\lambda^p\Lambda_p). \tag{3.5}$$

Moreover, for any $w_1, \dots, w_p \in W_0$,

$$\begin{aligned} &u_{\lambda_1\Lambda_0} \otimes \mathcal{F}_{w_1}\big(\mathsf{b}^{1,\lambda_1} \otimes \mathcal{F}_\tau\mathcal{F}_{w_2}\big(\mathsf{b}^{1,\lambda_2} \otimes \cdots \mathcal{F}_\tau\mathcal{F}_{w_p}(\{\mathsf{b}^{1,\lambda_p}\}) \cdots\big)\big) \\ &\xmapsto{\Theta_{\mathbf{1},\lambda}} \mathcal{F}_{w_1}\big(u_{\lambda^1\Lambda_1} \otimes \mathcal{F}_\tau\mathcal{F}_{w_2}\big(u_{\lambda^2\Lambda_1} \otimes \cdots \mathcal{F}_\tau\mathcal{F}_{w_p}(\{u_{\lambda^p\Lambda_1}\}) \cdots\big)\big). \end{aligned}$$

This result is used in [1] to connect the katabolism operations of Lascoux [15] and Shimozono-Weyman [25] to generalized Demazure crystals. The combinatorial significance of Theorem 3.7 for more general $\mathbf{r}$ and in other types remains to be explored.

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