Kronecker coefficients for one hook shape

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The Kronecker problem

Let $\mathcal{S}_n$ be the symmetric group on n letters and M_ν be the irreducible $\mathbb{C}\mathcal{S}_n$-module corresponding to the partition ν.

The *Kronecker coefficient* $g_{\lambda\mu\nu}$ is the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$.

Kronecker problem

Find a positive combinatorial formula for the Kronecker coefficients $g_{\lambda\mu\nu}$.

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The Kronecker problem

Known special cases

- λ and μ are hook shapes: Lascoux (1980), Remmel (1989), Rosas (2001).
- λ and μ have two rows: Remmel-Whitehead (1994), Rosas (2001), piecewise quadratic quasipolynomial Briand-Orellana-Rosas (2008), crystal bases B-Mulmuley-Sohoni (2011).
- λ has two rows, μ a hook shape: Remmel (1992), Rosas (2001).

Other related work

- Two zigzag skew shapes: Gessel (1984), Garsia-Remmel (1985).
- Results on reduced Kronecker coefficients: Ballantine-Orellana (2005), Briand-Orellana-Rosas (2008, 2009).
- Results on which M_ν can appear in $M_\lambda \otimes M_\mu$: Berele-Regev (1987), Berele-Imbo (2001).

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Motivation for the Kronecker problem

Geometric complexity theory is an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. The Kronecker problem appears in this approach.

Is there a polynomial time algorithm to test whether a Kronecker coefficient is nonzero?

This is expected to be true and difficult and therefore important for understanding the class **P**.

The Kronecker problem contains as a special case the plethysm problem of decomposing a symmetric power of an $\mathfrak{sl}_2$-irreducible into irreducibles. This plethysm problem has been intensively studied since nineteenth-century invariant theory, yet no positive combinatorial formula for these coefficients is known.

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Outline

We give a combinatorial formula for $g_{\lambda\,\mu(d)\,\nu}$, where $\mu(d)$ is the hook shape $(n-d, 1^d)$.

Part I: combinatorial formula for $g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu}$.

- Tensoring with a permutation module
- Colored tableaux
- Colored Yamanouchi tableaux

Part II: combinatorial formula for $g_{\lambda\,\mu(d)\,\nu}$.

- Conversion
- The main theorem
- The proof

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Tensoring with a permutation module

- Let $N_{(n-d,d)}$ be the permutation representation given by the action of $\mathcal{S}_n$ on subsets of $\{1, \ldots, n\}$ of size d.
- Equivalently, $N_{(n-d,d)} := \mathrm{Ind}_{\mathcal{S}_d \times \mathcal{S}_{n-d}}^{\mathcal{S}_n} \mathrm{triv}$.
- Define $N_{\mu(d)} := \mathrm{Ind}_{\mathcal{S}_d \times \mathcal{S}_{n-d}}^{\mathcal{S}_n} \mathrm{sgn} \boxtimes \mathrm{triv}$.

- Decomposing $M_\lambda \otimes N_{(n-d,d)}$ into irreducibles is much easier than decomposing $M_\lambda \otimes M_{(n-d,d)}$.
- Decomposing $M_\lambda \otimes N_{\mu(d)}$ into irreducibles is much easier than decomposing $M_\lambda \otimes M_{\mu(d)}$.

- Fact: $N_{\mu(d)} \cong M_{\mu(d)} \oplus M_{\mu(d-1)}$.
- Consequence: $g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} =$ multiplicity of M_ν in $M_\lambda \otimes N_{\mu(d)}$.

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The alphabet $\mathcal{A}$ of barred and unbarred letters

- $\{1, 2, \ldots\}$ is the *alphabet of unbarred letters* or *ordinary letters*.
- $\{\overline{1}, \overline{2}, \ldots\}$ is the *alphabet of barred letters*.
- Define $\mathcal{A} := \{\overline{1}, \overline{2}, \ldots\} \cup \{1, 2, \ldots\}$.
- An *ordinary word* is a sequence of ordinary letters.
- A *colored word* is a sequence of elements of $\mathcal{A}$.
- We will work with the following two orders on $\mathcal{A}$:

$$\text{the } \textit{natural order} \quad \overline{1} < 1 < \overline{2} < 2 \cdots$$

$$\text{the } \textit{small bar order} \quad \overline{1} \prec \overline{2} \prec \overline{3} \prec \cdots \prec 1 \prec 2 \cdots .$$

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Colored tableaux

A *semistandard colored tableau* or *colored tableau* is a tableau with entries in $\mathcal{A}$ such that

- unbarred letters strictly increase from north to south in each column,
- unbarred letter weakly increase from west to east in each row,
- barred letters weakly increase from north to south in each column,
- barred letters strictly increase from west to east in each row.

Example

$\overline{1}$	$\overline{2}$	$\overline{3}$	1	1	2	2	3
$\overline{2}$	$\overline{3}$	1	2	3	3		
$\overline{2}$	$\overline{3}$	2	3				
$\overline{3}$	1	3					

A colored tableau for the order $\prec$

$\overline{1}$	1	1	1	2	2	$\overline{3}$	3
1	$\overline{2}$	2	$\overline{3}$	3	3		
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A colored tableau for the order $<$

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Shape $(8, 6, 4, 3)$

Total color 8

Content $(5, 7, 9)$

- *Total color*: the number of barred letters.
- *Content*: remove bars and count number of 1's, number of 2's, etc.

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Yamanouchi words

Definition

An ordinary word $y = y_1 \cdots y_n$ is *Yamanouchi* if every terminal subword $y_k y_{k+1} \cdots y_n$ has partition content.

Example

$$y = 1\,2\,1\,3\,2\,1\,2\,1$$

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content (1)

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$$\text{content} \quad (1,1)$$

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content $(2, 1)$

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$$y \quad = \quad 1\ 2\ 1\ 3\ 2\ 1\ 2\ 1$$

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Example

$$y \;=\; 1\;2\;1\;3\;2\;1\;2\;1$$

Hence y is Yamanouchi.

Colored Yamanouchi tableaux

A colored tableau for the order $\prec$ is *Yamanouchi* if the following word is Yamanouchi:

- Considering only barred letters, read rows from right to left, starting with the top row.
- Remove the bars.
- Considering only unbarred letters, read rows from left to right, starting with the bottom row.

Example (A colored Yamanouchi tableau for the order $\prec$)

$\bar{1}$	$\bar{2}$	$\bar{3}$	1
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1	2	4	
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Sum of two Kronecker coefficients

Proposition

Let $CYT^{\prec}_{\lambda,d}(\nu)$ be the set of colored Yamanouchi tableaux for the order $\prec$ of content λ, total color d, and shape ν.

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = |CYT^{\prec}_{\lambda,d}(\nu)|.$$

Corollary

$$(1+t)\sum_{d=0}^{n-1} g_{\lambda\,\mu(d)\,\nu}\, t^d = \sum_{d=0}^{n} |CYT^{\prec}_{\lambda,d}(\nu)|\, t^d.$$

Goal: combinatorially divide by $1+t$.

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Let $CYT^{\prec}_{\lambda,d}(\nu)$ be the set of colored Yamanouchi tableaux for the order $\prec$ of content λ, total color d, and shape ν.

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = |CYT^{\prec}_{\lambda,d}(\nu)|.$$

Corollary

$$(1+t)\sum_{d=0}^{n-1} g_{\lambda\,\mu(d)\,\nu}\, t^d = \sum_{d=0}^{n} |CYT^{\prec}_{\lambda,d}(\nu)|\, t^d.$$

Goal: combinatorially divide by $1+t$.

Sum of two Kronecker coefficients

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Sum of two Kronecker coefficients

Representation theoretic interpretations:

- As we saw before,

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = \text{multiplicity of } M_\nu \text{ in } M_\lambda \otimes N_{\mu(d)}.$$

- The *hook Schur function* or super Schur function $HS_\nu(\mathbf{x},\mathbf{y})$ of Berele-Regev is the character of an irreducible representation of the general linear Lie superalgebra.

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Mixed insertion and conversion

Mixed insertion is a generalization of Schensted insertion to colored words developed independently by Haiman and Berele-Regev. Its chief advantage for this work is that it is simultaneously compatible with any ordering of colored letters in which $1 < 2 < \cdots$ and $\overline{1} < \overline{2} < \cdots$. Haiman gives a beautiful connection between mixed insertion and an operation on colored tableaux called conversion.

References:

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Conversion

Definition (Conversion)

Let T be a colored tableau for the order $\prec$. The *conversion of T*

$$\text{from} \quad \overline{1} \prec \overline{2} \prec \overline{3} \prec \cdots \prec 1 \prec 2 \cdots$$
$$\text{to} \quad \overline{1} < 1 < \overline{2} < 2 \cdots ,$$

denoted $T(\prec \to <)$, is defined as follows:

Let β be the largest barred letter of T. Repeatedly *exchange* β with the lesser (or only) one of its neighbors below or to the right until its neighbors below and to the right are both $> \beta$.

Repeat with the second largest barred letter of T, then the third largest, etc. until the tableau is a colored tableau for the order $<$.

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Conversion

Example (Converting from the small bar order $\prec$ to the natural order $<$)

$$
\begin{array}{|c|c|c|c|}
\hline
\bar{1} & \bar{2} & \bar{3} & 1 \\
\hline
\bar{1} & \bar{3} & \bar{4} & 2 \\
\hline
\bar{2} & 1 & 1 & 3 \\
\hline
1 & 2 & 4 & \multicolumn{1}{c}{} \\
\cline{1-3}
3 & 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
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\end{array}
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\hline
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\hline
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Colored Yamanouchi tableaux

A *colored Yamanouchi tableau* is a colored tableau T for the order $<$ such that $T(<\to\prec)$ is Yamanouchi.

Example

$\overline{1}$	$\overline{2}$	$\overline{3}$	1
$\overline{1}$	$\overline{3}$	$\overline{4}$	2
$\overline{2}$	1	1	3
1	2	4	
3	5		

A colored Yamanouchi tableau for the order $\prec$

$\overline{1}$	1	1	1
$\overline{1}$	$\overline{2}$	2	$\overline{3}$
1	2	3	$\overline{4}$
$\overline{2}$	$\overline{3}$	4	
3	5		

A colored Yamanouchi tableau

Hook Kronecker Rule

Definition

Let λ be a partition of n and $d \in \{0, 1, \ldots, n\}$.

- Recall $\mu(d) = (n - d, 1^d)$.
- $\mathrm{CYT}_{\lambda,d}$ is the set of colored Yamanouchi tableaux of content λ and total color d.
- $\mathrm{CYT}^-_{\lambda,d}(\nu)$ is the set of colored Yamanouchi tableaux of content λ, total color d, and shape ν having unbarred southwest corner.

Theorem (Hook Kronecker Rule)

$$g_{\lambda\,\mu(d)\,\nu} = |CYT^-_{\lambda,d}(\nu)|.$$

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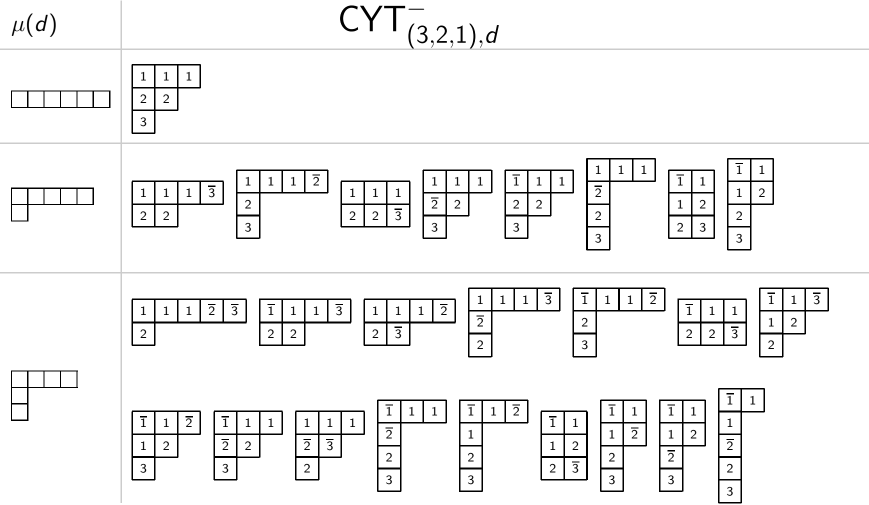
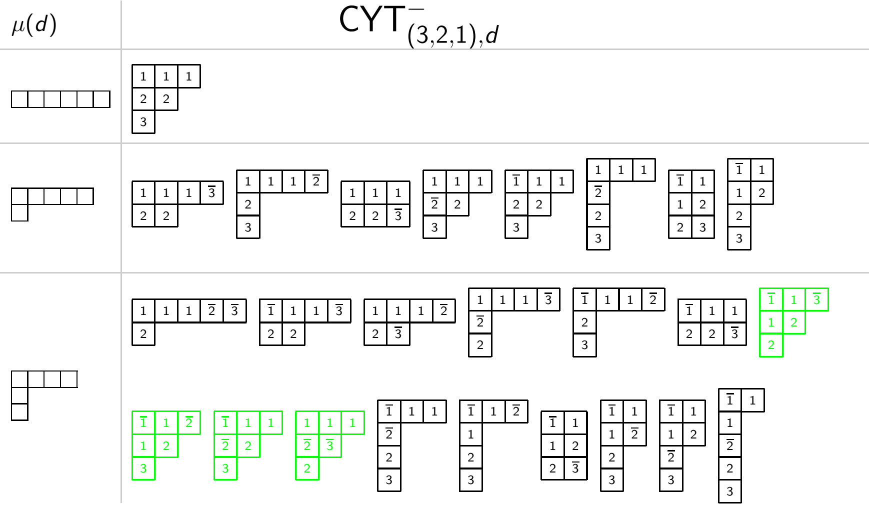
Hook Kronecker Rule

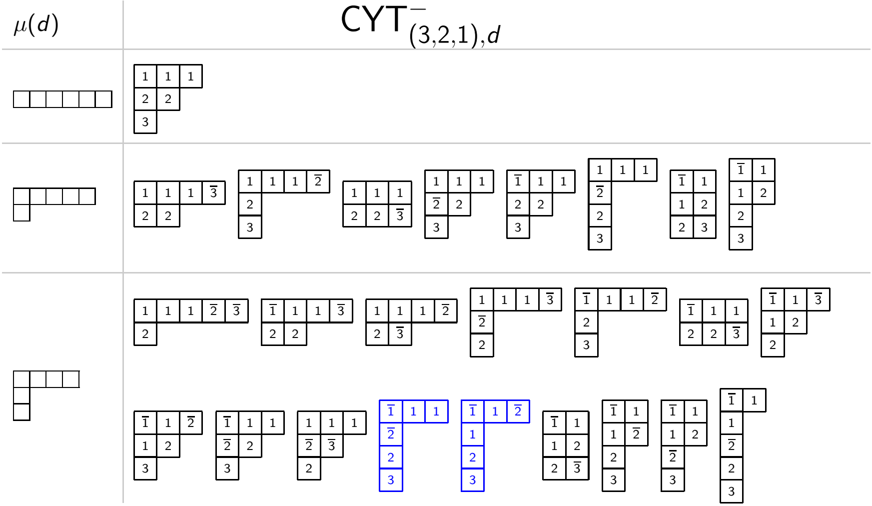


Hook Kronecker Rule



$g_{(3,2,1)\,(4,1,1)\,(3,2,1)} = 4$

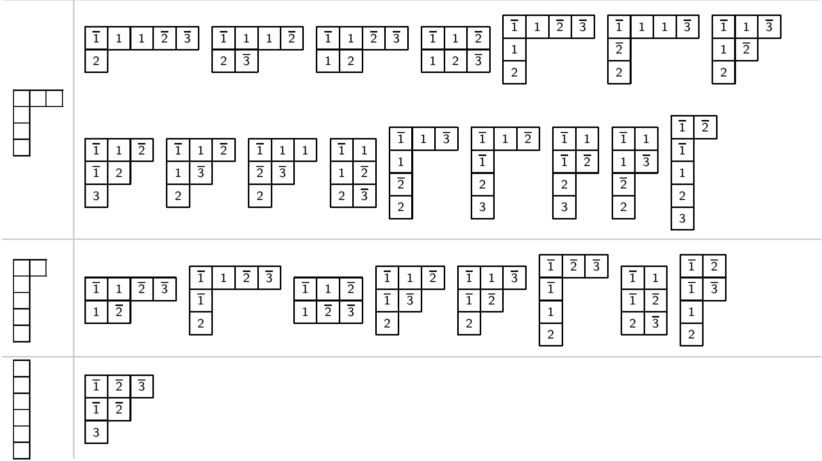
Hook Kronecker Rule



$$g_{(3,2,1)\,(4,1,1)\,(3,1,1,1)} = 2$$

Hook Kronecker Rule

$\mu(d)$	$\mathrm{CYT}^{-}_{(3,2,1),d}$



Proof of the Hook Kronecker Rule

A colored tableau for the order $<$ is *color lowerable (raisable)* if its southwest entry is barred (unbarred). Hence unbarring the southwest entry of a color lowerable tableau is a bijection between color lowerable tableaux and color raisable tableaux, which we call the *color lowering operator* C_-.

Example

$$C_-\left(\begin{array}{|c|c|c|}\hline \bar{1} & 1 & \bar{2} \\ \hline \bar{1} & \bar{2} & 2 \\ \hline \bar{2} & 2 & 3 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|}\hline \bar{1} & 1 & \bar{2} \\ \hline \bar{1} & \bar{2} & 2 \\ \hline 2 & 2 & 3 \\ \hline \end{array}.$$

Theorem (B)

A color lowerable tableau T is Yamanouchi if and only if $C_-(T)$ is Yamanouchi.

The Hook Kronecker Rule follows easily from this and the corollary.

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Mixed insertion

Schensted insertion and mixed insertion (and many other insertion algorithms) use the notion of *inserting a letter into a row or column*.

Definition (Inserting a letter into a row or column)

Let R be a row or column of a colored tableau and $\alpha \in \mathcal{A}$. First assume that the letters of R are distinct and distinct from α. Inserting α into R means that α replaces the least letter $\beta > \alpha$ in R or, if no such β exists, adds a new cell containing α to the end of R. In the former case, we say that α *bumps* β.

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Schensted insertion and mixed insertion (and many other insertion algorithms) use the notion of *inserting a letter into a row or column*.

Definition (Inserting a letter into a row or column)

Let R be a row or column of a colored tableau and $\alpha \in \mathcal{A}$. First assume that the letters of R are distinct and distinct from α. Inserting α into R means that α replaces the least letter $\beta > \alpha$ in R or, if no such β exists, adds a new cell containing α to the end of R. In the former case, we say that α *bumps* β.

Mixed insertion

The case of repeated letters is handled by the following conventions:

- In words, the occurrences of a given letter (barred or unbarred) increase slightly from left to right.
- In tableaux, the occurrences of a given barred letter increase slightly from bottom to top and the occurrences of a given unbarred letter increase slightly from left to right.

Example

Inserting 2 into the row $\boxed{\bar{1}}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{\bar{3}}\boxed{3}$:

Mixed insertion

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$\boxed{\bar{1}}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{\bar{3}}\boxed{3}$ 2

Mixed insertion

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Mixed insertion

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Inserting 2 into the row $\boxed{\bar{1}}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{\bar{3}}\boxed{3}$:

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The 2 bumps the $\bar{3}$.

Mixed insertion

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Example

Inserting 2 into the row

$\overline{1}$	1	1	1	2	2	$\overline{3}$	3

:

$\overline{1}$	1	1	1	2	2	2	3

$\overline{3}$

The 2 bumps the $\overline{3}$.

Mixed insertion

Definition (Mixed insertion)

Let T be a colored tableau and α a colored letter. The *mixed insertion of α into T*, denoted $T \xleftarrow{\mathrm{m}} \alpha$ is computed as follows:

- insert α into the first row of T if α is unbarred,
- insert α into the first column of T if α is barred.

As each subsequent element β of T is bumped by an insertion,

- insert β into the row immediately below if it is unbarred,
- insert β into the column immediately to its right if it is barred.

Continue until an insertion takes place at the end of a row or column.

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Mixed insertion

Example

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\ \hline 1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\ \cline{1-6} \overline{2} & 2 & \overline{3} & 3 \\ \cline{1-4} \overline{2} & \overline{3} & 3 \\ \cline{1-3} \end{array} \overset{\mathrm{m}}{\longleftarrow} \overline{1} =$$

Mixed insertion

Example

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\ \hline 1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\ \cline{1-6} \overline{2} & 2 & \overline{3} & 3 \\ \cline{1-4} \overline{2} & \overline{3} & 3 \\ \cline{1-3} \end{array} \overset{\mathrm{m}}{\longleftarrow} \overline{1} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\ \hline 1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\ \cline{1-6} \overline{2} & 2 & \overline{3} & 3 \\ \cline{1-4} \overline{2} & \overline{3} & 3 \\ \cline{1-3} \end{array} \qquad \overline{1}$$

Mixed insertion

Example

$\bar{1}$	1	1	1	2	2	$\bar{3}$	3
1	$\bar{2}$	2	$\bar{3}$	3	3		
$\bar{2}$	2	$\bar{3}$	3				
$\bar{2}$	$\bar{3}$	3					

$$\overset{\mathrm{m}}{\longleftarrow} \bar{1} =$$

$\bar{1}$	1	1	1	2	2	$\bar{3}$	3
$\bar{1}$	$\bar{2}$	2	$\bar{3}$	3	3		
$\bar{2}$	2	$\bar{3}$	3				
$\bar{2}$	$\bar{3}$	3					

1

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
1 & \bar{2} & 2 & \bar{3} & 3 & 3 \\
\cline{1-6}
\bar{2} & 2 & \bar{3} & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\overset{\mathrm{m}}{\longleftarrow} \bar{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
\color{blue}{\bar{1}} & \bar{2} & 2 & \bar{3} & 3 & 3 \\
\cline{1-6}
\color{red}{1} & 2 & \bar{3} & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\qquad \color{red}{\bar{2}}
$$

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
1 & \bar{2} & 2 & \bar{3} & 3 & 3 \\
\cline{1-6}
\bar{2} & 2 & \bar{3} & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\xleftarrow{\mathrm{m}} \bar{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
\bar{1} & \bar{2} & 2 & \bar{3} & 3 & 3 \\
\cline{1-6}
1 & 2 & \bar{3} & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\qquad \bar{2}
$$

Mixed insertion

Example

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\ \hline 1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\ \cline{1-6} \overline{2} & 2 & \overline{3} & 3 \\ \cline{1-4} \overline{2} & \overline{3} & 3 \\ \cline{1-3} \end{array} \overset{\mathrm{m}}{\longleftarrow} \overline{1} = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\ \hline \overline{1} & \overline{2} & \overline{2} & \overline{3} & 3 & 3 \\ \cline{1-6} 1 & 2 & \overline{3} & 3 \\ \cline{1-4} \overline{2} & \overline{3} & 3 \\ \cline{1-3} \end{array} \qquad 2$$

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
1 & \bar{2} & 2 & \bar{3} & 3 & 3 \\
\cline{1-6}
\bar{2} & 2 & \bar{3} & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\overset{\mathrm{m}}{\longleftarrow} \bar{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 1 & 2 & 2 & \bar{3} & 3 \\
\hline
\bar{1} & \bar{2} & \bar{2} & \bar{3} & 3 & 3 \\
\cline{1-6}
1 & 2 & 2 & 3 \\
\cline{1-4}
\bar{2} & \bar{3} & 3 \\
\cline{1-3}
\end{array}
\qquad \bar{3}
$$

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\
\cline{1-6}
\overline{2} & 2 & \overline{3} & 3 \\
\cline{1-4}
\overline{2} & \overline{3} & 3 \\
\cline{1-3}
\end{array}
\xleftarrow{\mathrm{m}} \overline{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
\color{blue}{\overline{1}} & \color{blue}{\overline{2}} & \color{blue}{\overline{2}} & \overline{3} & 3 & 3 \\
\cline{1-6}
\color{blue}{1} & 2 & \color{blue}{2} & \color{red}{\overline{3}} \\
\cline{1-4}
\overline{2} & \overline{3} & 3 \\
\cline{1-3}
\end{array}
\qquad \color{red}{3}
$$

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
1 & \overline{2} & 2 & \overline{3} & 3 & 3 & \multicolumn{2}{c}{} \\
\cline{1-6}
\overline{2} & 2 & \overline{3} & 3 & \multicolumn{4}{c}{} \\
\cline{1-4}
\overline{2} & \overline{3} & 3 & \multicolumn{5}{c}{} \\
\cline{1-3}
\end{array}
\overset{\mathrm{m}}{\longleftarrow} \overline{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
\overline{1} & \overline{2} & \overline{2} & \overline{3} & 3 & 3 & \multicolumn{2}{c}{} \\
\cline{1-6}
1 & 2 & 2 & \overline{3} & \multicolumn{4}{c}{} \\
\cline{1-4}
\overline{2} & \overline{3} & 3 & 3 & \multicolumn{4}{c}{} \\
\cline{1-4}
\end{array}
$$

Mixed insertion

Example

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
1 & \overline{2} & 2 & \overline{3} & 3 & 3 \\
\cline{1-6}
\overline{2} & 2 & \overline{3} & 3 \\
\cline{1-4}
\overline{2} & \overline{3} & 3 \\
\cline{1-3}
\end{array}
\overset{\mathrm{m}}{\longleftarrow} \overline{1} =
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\overline{1} & 1 & 1 & 1 & 2 & 2 & \overline{3} & 3 \\
\hline
\overline{1} & \overline{2} & \overline{2} & \overline{3} & 3 & 3 \\
\cline{1-6}
1 & 2 & 2 & \overline{3} \\
\cline{1-4}
\overline{2} & \overline{3} & 3 & 3 \\
\cline{1-4}
\end{array}
$$

Mixed insertion

Definition (Mixed insertion and recording tableaux)

Let $w = w_1 \ldots w_n$ be a colored word.

The *mixed insertion tableau of w*, denoted $P_{\mathrm{m}}(w)$, is

$$\varnothing \xleftarrow{\mathrm{m}} w_1 \xleftarrow{\mathrm{m}} w_2 \cdots \xleftarrow{\mathrm{m}} w_n.$$

The *mixed recording tableau of w*, denoted $Q_{\mathrm{m}}(w)$, is the standard Young tableau encoding the sequence of shapes obtained during mixed insertion.

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The *mixed recording tableau of w*, denoted $Q_{\mathrm{m}}(w)$, is the standard Young tableau encoding the sequence of shapes obtained during mixed insertion.

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

$$Q_{\mathrm{m}}(w) \quad =$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

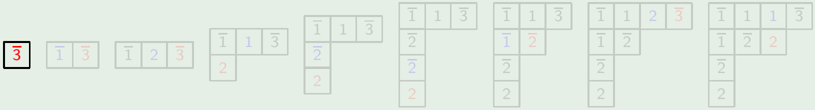
$$Q_{\mathrm{m}}(w) \quad = \qquad \varnothing$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is



$$Q_{\mathrm{m}}(w) = \boxed{1}$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline \end{array} \quad \begin{array}{lll} \overline{1} & 1 & \overline{3} \\ 2 & & \end{array} \quad \begin{array}{lll} \overline{1} & 1 & \overline{3} \\ \overline{2} & & \\ 2 & & \end{array} \quad \begin{array}{lll} \overline{1} & 1 & \overline{3} \\ \overline{2} & & \\ \overline{2} & & \\ 2 & & \end{array} \quad \begin{array}{lll} \overline{1} & 1 & \overline{3} \\ \overline{1} & \overline{2} & \\ \overline{2} & & \\ 2 & & \end{array} \quad \begin{array}{llll} \overline{1} & 1 & 2 & \overline{3} \\ \overline{1} & \overline{2} & & \\ \overline{2} & & & \\ 2 & & & \end{array} \quad \begin{array}{llll} \overline{1} & 1 & 1 & \overline{3} \\ \overline{1} & \overline{2} & 2 & \\ \overline{2} & & & \\ 2 & & & \end{array}$$

$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

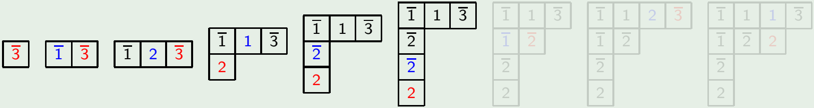
$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is



$Q_{\mathrm{m}}(w) =$

1	2	3
4		
5		
6		

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

Mixed insertion

Example

$$w = \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$\bar{3}$

$\bar{1}$	$\bar{3}$

$\bar{1}$	2	$\bar{3}$

$\bar{1}$	1	$\bar{3}$
2		

$\bar{1}$	1	$\bar{3}$
$\bar{2}$		
2		

$\bar{1}$	1	$\bar{3}$
$\bar{2}$		
$\bar{2}$		
2		

$\bar{1}$	1	$\bar{3}$
$\bar{1}$	$\bar{2}$	
$\bar{2}$		
2		

$\bar{1}$	1	2	$\bar{3}$
$\bar{1}$	$\bar{2}$		
$\bar{2}$			
2			

$\bar{1}$	1	1	$\bar{3}$
$\bar{1}$	$\bar{2}$	2	
$\bar{2}$			
2			

$Q_{\mathrm{m}}(w) =$

1	2	3	8
4	7		
5			
6			

Mixed insertion

Example

$$w = \overline{3}\,\overline{1}\,2\,1\,\overline{2}\,\overline{2}\,\overline{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is

$$\begin{array}{|c|}\hline \overline{3} \\ \hline\end{array}
\quad
\begin{array}{|c|c|}\hline \overline{1} & \overline{3} \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 2 & \overline{3} \\ \hline\end{array}
\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{2} \\ \cline{1-1} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|}\hline \overline{1} & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 2 & \overline{3} \\ \hline \overline{1} & \overline{2} \\ \cline{1-2} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}
\quad
\begin{array}{|c|c|c|c|}\hline \overline{1} & 1 & 1 & \overline{3} \\ \hline \overline{1} & \overline{2} & 2 \\ \cline{1-3} \overline{2} \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$Q_{\mathrm{m}}(w) \quad = \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 8 \\ \hline 4 & 7 & 9 \\ \cline{1-3} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$$

Conversion and mixed insertion

Proposition (Haiman)

Converting between the small bar order and natural order commutes with mixed insertion in the following sense:

$$P_{\mathrm{m}}(w) = P_{\mathrm{m}}^{\prec}(w)(\prec \to <)$$
$$P_{\mathrm{m}}^{\prec}(w) = P_{\mathrm{m}}(w)(< \to \prec)$$
$$Q_{\mathrm{m}}(w) = Q_{\mathrm{m}}^{\prec}(w).$$

Conversion and mixed insertion

Example

$$w = \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is shown on the next line, and below that the sequence for $P_{\mathrm{m}}^{\prec}(w)$.

$$\begin{array}{|c|}\hline \bar3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline \bar1 & \bar3 \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & 2 & \bar3 \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & 1 & \bar3 \\ \hline 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & 1 & \bar3 \\ \hline \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & 1 & \bar3 \\ \hline \bar2 \\ \cline{1-1} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & 1 & \bar3 \\ \hline \bar1 & \bar2 \\ \cline{1-2} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \bar1 & 1 & 2 & \bar3 \\ \hline \bar1 & \bar2 \\ \cline{1-2} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \bar1 & 1 & 1 & \bar3 \\ \hline \bar1 & \bar2 & 2 \\ \cline{1-3} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|}\hline \bar3 \\ \hline\end{array}\quad
\begin{array}{|c|c|}\hline \bar1 & \bar3 \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & \bar3 & 2 \\ \hline\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & \bar3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline \bar1 & \bar3 & 1 \\ \hline \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
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\begin{array}{|c|c|c|}\hline \bar1 & \bar2 & \bar3 \\ \hline \bar1 & 1 \\ \cline{1-2} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \bar1 & \bar2 & \bar3 & 2 \\ \hline \bar1 & 1 \\ \cline{1-2} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|c|}\hline \bar1 & \bar2 & \bar3 & 1 \\ \hline \bar1 & 1 & 2 \\ \cline{1-3} \bar2 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array}$$

The proposition says that each tableau T on the top line is related to the tableau U below it by $U = T(<\to\prec)$.

Conversion and mixed insertion

Example

$$w = \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1$$

The sequence of tableaux produced in computing $P_{\mathrm{m}}(w)$ is shown on the next line, and below that the sequence for $P_{\mathrm{m}}^{\prec}(w)$.

$$\begin{array}{l}\bar{3}\end{array}\quad
\begin{array}{ll}\bar{1} & \bar{3}\end{array}\quad
\begin{array}{lll}\bar{1} & 2 & \bar{3}\end{array}\quad
\begin{array}{lll}\bar{1} & 1 & \bar{3}\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & 1 & \bar{3}\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & 1 & \bar{3}\\ \bar{2}\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & 1 & \bar{3}\\ \bar{1} & \bar{2}\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{llll}\bar{1} & 1 & 2 & \bar{3}\\ \bar{1} & \bar{2}\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{llll}\bar{1} & 1 & 1 & \bar{3}\\ \bar{1} & \bar{2} & 2\\ \bar{2}\\ 2\end{array}$$

$$\begin{array}{l}\bar{3}\end{array}\quad
\begin{array}{ll}\bar{1} & \bar{3}\end{array}\quad
\begin{array}{lll}\bar{1} & \bar{3} & 2\end{array}\quad
\begin{array}{lll}\bar{1} & \bar{3} & 1\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & \bar{3} & 1\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & \bar{3} & 1\\ \bar{2}\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{lll}\bar{1} & \bar{2} & \bar{3}\\ \bar{1} & 1\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{llll}\bar{1} & \bar{2} & \bar{3} & 2\\ \bar{1} & 1\\ \bar{2}\\ 2\end{array}\quad
\begin{array}{llll}\bar{1} & \bar{2} & \bar{3} & 1\\ \bar{1} & 1 & 2\\ \bar{2}\\ 2\end{array}$$

The proposition says that each tableau T on the top line is related to the tableau U below it by $U = T(<\to\prec)$.

Colored words

Let w be a colored word.

- *Total color*: number of barred letters in w.
- *Content*: remove bars and count number of 1's, number of 2's, etc.
- The ordinary word w^{blft} is formed from w by shuffling the barred letters to the left and then removing their bars.

Example

$$w = \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1$$

$$2\,1\,2\,1$$

$$3\,1\,2\,2\,1$$

$$w^{\mathrm{blft}} = 3\,1\,2\,2\,1\,2\,1\,2\,1$$

The colored word w has total color 5 and content $(4, 4, 1)$.

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Example

$$\begin{array}{rcl} w & = & \bar{3}\ \bar{1}\ 2\ 1\ \bar{2}\ \bar{2}\ \bar{1}\ 2\ 1 \\ & & 2\ 1\ 2\ 1 \\ & & 3\ 1\ 2\ 2\ 1 \\ w^{\mathrm{blft}} & = & 3\ 1\ 2\ 2\ 1\ 2\ 1\ 2\ 1 \end{array}$$

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Colored Yamanouchi words

- An ordinary word $y = y_1 \cdots y_n$ is *Yamanouchi* if every terminal subword $y_k y_{k+1} \cdots y_n$ has partition content.
- A colored word w is *Yamanouchi* if w^{blft} is Yamanouchi.

Example

$$\begin{aligned} w &= \bar{1}\ 2\ 1\ \bar{2}\ \bar{1}\ 2\ \bar{3}\ 1 \\ w^{\text{blft}} &= 1\ 2\ 1\ 3\ 2\ 1\ 2\ 1 \end{aligned}$$

w^{blft} is Yamanouchi, hence w is Yamanouchi.

$w = \bar{3}\ \bar{1}\ 2\ 1\ \bar{2}\ \bar{2}\ \bar{1}\ 2\ 1$ is not Yamanouchi because w^{blft} ends in 2 2 1 2 1 2 1, which has content $(3, 4)$.

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Hook Kronecker Rule II

Definition

Let λ be a partition of n and $d \in \{0, 1, \ldots, n\}$.

- $\mathrm{CYW}_{\lambda,d}$ is the set of colored Yamanouchi words of content λ and total color d.
  Fact: the mixed insertion tableaux of these words is $\mathrm{CYT}_{\lambda,d}$.
- $\mathrm{CYW}^{-}_{\lambda,d}$ consists of $w \in \mathrm{CYW}_{\lambda,d}$ such that $P_{\mathrm{m}}(w)$ has unbarred southwest corner.
- $\mathrm{CYW}^{-}_{\lambda,d}(B_\nu)$ consists of $w \in \mathrm{CYW}^{-}_{\lambda,d}$ such that $Q_{\mathrm{m}}(w) = B_\nu$, where B_ν is any standard Young tableau of shape ν.

Theorem (Hook Kronecker Rule II)

$$g_{\lambda\,\mu(d)\,\nu} = |CYW^{-}_{\lambda,d}(B_\nu)|.$$

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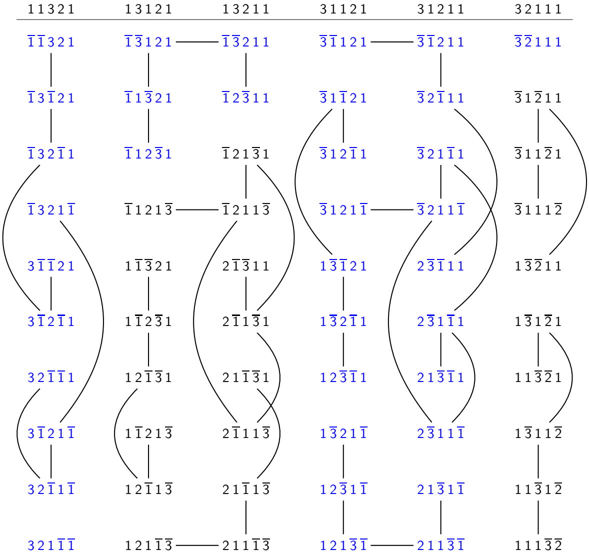
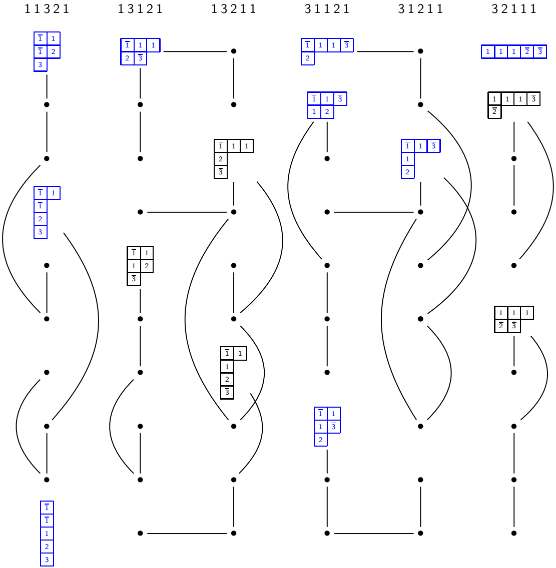


The set $\mathrm{CYW}_{(3,1,1),2}$. The subset $\mathrm{CYW}^{-}_{(3,1,1),2}$ is shown in blue. Edges are Knuth transformations of the words obtained by applying $^{\mathrm{neg}}$. Column labels correspond to applying $^{\mathrm{blft}}$ and the positions of the barred letters are constant along rows.



The mixed insertion tableaux of the words in the previous figure (which are constant on connected components). This set of tableaux is $\mathrm{CYT}_{(3,1,1),2}$ and the tableaux in blue are those with unbarred southwest corner ($\mathrm{CYT}^{-}_{(3,1,1),2}$).