Kronecker coefficients for one hook shape

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Irreducible representations of the symmetric group

- Let $\mathcal{S}_n$ be the symmetric group on n letters.
- A *partition* λ of n is a weakly decreasing sequence $(\lambda_1, \ldots, \lambda_l)$ of positive integers such that $\sum \lambda_i = n$.
- Irreducible reps. of $\mathcal{S}_n$ are parameterized by partitions of n.
- M_λ denotes the irreducible rep. corresponding to the partition λ.

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- The *defining representation* V of $\mathcal{S}_n$ is the permutation representation with basis $\{v_1, \ldots, v_n\}$ where $\mathcal{S}_n$ acts by permuting the indices.
- Fact: $V \cong M_{(n)} \oplus M_{(n-1,1)}$, where
- $M_{(n)}$ is spanned by $v_1 + v_2 + \cdots + v_n$.
- $M_{(n-1,1)}$ is spanned by $v_1 - v_2$, $v_2 - v_3$, $\ldots$, $v_{n-1} - v_n$.
- $M_{(n)}$ is the *trivial representation*.
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Kronecker problem

Find a positive combinatorial formula for the Kronecker coefficients $g_{\lambda\mu\nu}$.

The archetypal example of a positive combinatorial formula is the *Littlewood-Richardson rule*, which gives the multiplicities for decomposing a tensor products of two irreps of GL_n.

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The Kronecker problem

Known special cases

- λ and μ are hook shapes: Lascoux (1980), Remmel (1989), Rosas (2001).
- λ and μ have two rows: Remmel-Whitehead (1994), Rosas (2001), piecewise quadratic quasipolynomial Briand-Orellana-Rosas (2008), crystal bases B-Mulmuley-Sohoni (2011).
- λ has two rows, μ a hook shape: Remmel (1992), Rosas (2001).

Other related work

- Two zigzag skew shapes: Gessel (1984), Garsia-Remmel (1985).
- Results on reduced Kronecker coefficients: Ballantine-Orellana (2005), Briand-Orellana-Rosas (2008, 2009).
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Motivation for the Kronecker problem

Geometric complexity theory is an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. The Kronecker problem appears in this approach.

Is there a polynomial time algorithm to test whether a Kronecker coefficient is nonzero?

This is expected to be true and difficult and therefore important for understanding the class **P**.

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The Kronecker problem contains as a special case the plethysm problem of decomposing a symmetric power of an $\mathfrak{sl}_2$-irreducible into irreducibles.

This problem can be phrased combinatorially:

$\mathfrak{sl}_2$ plethysm problem for symmetric powers

Let m_d be the number of partitions of d that fit in an $r \times c$ rectangle. The sequence $m_0, m_1, \ldots, m_{rc}$ is known to be symmetric unimodal. Find a positive combinatorial formula for $|m_{d+1} - m_d|$.

This problem has been intensively studied since nineteenth-century invariant theory, yet no positive combinatorial formula for these coefficients is known.

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Outline

We give a combinatorial formula for $g_{\lambda\,\mu(d)\,\nu}$, where $\mu(d)$ is the hook shape $(n-d, 1^d)$.

This improves and generalizes all known formulas for the Kronecker coefficients $g_{\lambda\mu\nu}$, except for formulas for λ and μ having two rows.

Part I: combinatorial formula for $g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu}$.

- Tensoring with a permutation module
- Colored tableaux
- Colored Yamanouchi tableaux

Part II: combinatorial formula for $g_{\lambda\,\mu(d)\,\nu}$.

- Conversion
- The main theorem
- The proof

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Tensoring with a permutation module

- Let $N_{(n-d,d)}$ be the permutation representation given by the action of $\mathcal{S}_n$ on subsets of $\{1, \ldots, n\}$ of size d.
- Equivalently, $N_{(n-d,d)} := \mathrm{Ind}_{\mathcal{S}_d \times \mathcal{S}_{n-d}}^{\mathcal{S}_n} \mathrm{triv}$.
- Define $N_{\mu(d)} := \mathrm{Ind}_{\mathcal{S}_d \times \mathcal{S}_{n-d}}^{\mathcal{S}_n} \mathrm{sgn} \boxtimes \mathrm{triv}$.

- Decomposing $M_\lambda \otimes N_{(n-d,d)}$ into irreducibles is much easier than decomposing $M_\lambda \otimes M_{(n-d,d)}$.
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- Fact: $N_{\mu(d)} \cong M_{\mu(d)} \oplus M_{\mu(d-1)}$.
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The alphabet $\mathcal{A}$ of barred and unbarred letters

- $\{1, 2, \ldots\}$ is the *alphabet of unbarred letters* or *ordinary letters*.
- $\{\overline{1}, \overline{2}, \ldots\}$ is the *alphabet of barred letters*.
- Define $\mathcal{A} := \{\overline{1}, \overline{2}, \ldots\} \cup \{1, 2, \ldots\}$.
- An *ordinary word* is a sequence of ordinary letters.
- A *colored word* is a sequence of elements of $\mathcal{A}$.

- We will work with the following two orders on $\mathcal{A}$:

$$\text{the } \textit{natural order} \quad \overline{1} < 1 < \overline{2} < 2 \cdots$$

$$\text{the } \textit{small bar order} \quad \overline{1} \prec \overline{2} \prec \overline{3} \prec \cdots \prec 1 \prec 2 \cdots .$$

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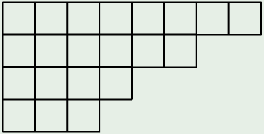
- The *Ferrers diagram* or *shape* of λ is the left-justified array of square cells with λ_i cells in row i.

Example



Ferrers diagram of $(8, 6, 4, 3)$

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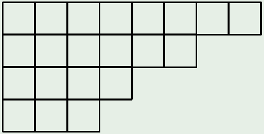

Ferrers diagram of $(8, 6, 4, 3)$

Tableaux

A *tableau of shape* λ is the Ferrers diagram of λ together with a letter in each cell.

Example

$\bar{3}$	2	1	1	1	$\bar{2}$	3
7	$\bar{4}$	2	$\bar{3}$	8	3	
$\bar{6}$	2	$\bar{3}$	1			
$\bar{4}$	$\bar{3}$	3				

A tableau of shape $(8, 6, 4, 3)$

Colored tableaux

A *semistandard colored tableau* or *colored tableau* is a tableau with entries in $\mathcal{A}$ such that

- letters weakly increase north to south and west to east,
- unbarred letters strictly increase from north to south in each column,
- unbarred letter weakly increase from west to east in each row,
- barred letters weakly increase from north to south in each column,
- barred letters strictly increase from west to east in each row.

Example

$\overline{1}$	$\overline{2}$	$\overline{3}$	1	1	2	2	3
$\overline{2}$	$\overline{3}$	1	2	3	3		
$\overline{2}$	$\overline{3}$	2	3				
$\overline{3}$	1	3					

A colored tableau for the order $\prec$

$\overline{1}$	1	1	1	2	2	$\overline{3}$	3
1	$\overline{2}$	2	$\overline{3}$	3	3		
$\overline{2}$	2	$\overline{3}$	3				
$\overline{2}$	$\overline{3}$	3					

A colored tableau for the order $<$

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$\bar{1}$	$\bar{2}$	$\bar{3}$	1	1	2	2	3
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A colored tableau for the order $<$

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Shape $(8, 6, 4, 3)$

Total color 8

Content $(5, 7, 9)$

- *Total color*: the number of barred letters.
- *Content*: remove bars and count number of 1's, number of 2's, etc.

Colored tableaux

Example

$\bar{1}$	1	1	1	2	2	$\bar{3}$	3
1	$\bar{2}$	2	$\bar{3}$	3	3		
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Yamanouchi words

Definition

An ordinary word $y = y_1 \cdots y_n$ is *Yamanouchi* if every terminal subword $y_k y_{k+1} \cdots y_n$ has partition content.

Example

$$y = 1\,2\,1\,3\,2\,1\,2\,1$$

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content (1)

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Example

$$y \quad = \quad 1\;2\;1\;3\;2\;1\;2\;1$$

$$\text{content} \quad (1,1)$$

Yamanouchi words

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Example

$$y \quad = \quad 1\ 2\ 1\ 3\ 2\ 1\ 2\ 1$$

content $(2, 1)$

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Example

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Example

$$y = 1\ 2\ 1\ 3\ 2\ 1\ 2\ 1$$

Hence y is Yamanouchi.

Colored Yamanouchi tableaux

A colored tableau for the order $\prec$ is *Yamanouchi* if the following word is Yamanouchi:

- Considering only barred letters, read rows from right to left, starting with the top row.
- Remove the bars.
- Considering only unbarred letters, read rows from left to right, starting with the bottom row.

Example (A colored Yamanouchi tableau for the order $\prec$)

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$\overline{3}\ \overline{2}\ \overline{1}\ \overline{4}\ \overline{3}\ \overline{1}\ \overline{2}$

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3 2 1 4 3 1 2 3 5 1 2 4 1 1 3 2 1

Sum of two Kronecker coefficients

Proposition

Let $CYT^{\prec}_{\lambda,d}(\nu)$ be the set of colored Yamanouchi tableaux for the order $\prec$ of content λ, total color d, and shape ν.

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = |CYT^{\prec}_{\lambda,d}(\nu)|.$$

Corollary

$$(1+t)\sum_{d=0}^{n-1} g_{\lambda\,\mu(d)\,\nu}\, t^d = \sum_{d=0}^{n} |CYT^{\prec}_{\lambda,d}(\nu)|\, t^d.$$

Goal: combinatorially divide by $1+t$.

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Representation theoretic interpretations:

- As we saw before,

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = \text{multiplicity of } M_\nu \text{ in } M_\lambda \otimes N_{\mu(d)}.$$

- The *hook Schur function* or super Schur function $HS_\nu(\mathbf{x}, \mathbf{y})$ of Berele-Regev is the character of an irreducible representation of the general linear Lie superalgebra.

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Mixed insertion and conversion

Mixed insertion is a generalization of Schensted insertion to colored words developed independently by Haiman and Berele-Regev. Its chief advantage for this work is that it is simultaneously compatible with any ordering of colored letters in which $1 < 2 < \cdots$ and $\overline{1} < \overline{2} < \cdots$. Haiman gives a beautiful connection between mixed insertion and an operation on colored tableaux called conversion.

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Conversion

Conversion is an algorithm that turns a colored tableau for the order $\prec$ into one for the order $<$ (and vice versa).

Consequence of this algorithm:

Proposition (Haiman)

There is a bijection

$$CT^{\prec}_{\lambda,d}(\nu) \cong CT_{\lambda,d}(\nu),$$

where $CT_{\lambda,d}(\nu)$ ($CT^{\prec}_{\lambda,d}(\nu)$) denotes the set of colored tableaux for the order $<$ ($\prec$) of content λ, total color d, and shape ν.

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Conversion

Definition (Conversion)

Let T be a colored tableau for the order $\prec$. The *conversion of T*

$$\text{from } \bar{1} \prec \bar{2} \prec \bar{3} \prec \cdots \prec 1 \prec 2 \cdots$$
$$\text{to } \bar{1} < 1 < \bar{2} < 2 \cdots ,$$

denoted $T(\prec\to<)$, is defined as follows:

Let β be the largest barred letter of T. Repeatedly *exchange* β with the lesser (or only) one of its neighbors below or to the right until its neighbors below and to the right are both $> \beta$.

Repeat with the second largest barred letter of T, then the third largest, etc. until the tableau is a colored tableau for the order $<$.

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Conversion

Example (Converting from the small bar order $\prec$ to the natural order $<$)

$$\begin{array}{|c|c|c|c|}\hline \bar{1} & \bar{2} & \bar{3} & 1 \\ \hline \bar{1} & \bar{3} & \bar{4} & 2 \\ \hline \bar{2} & 1 & 1 & 3 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}(\prec \to <) \ =$$

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$\bar{1}$	$\bar{2}$	$\bar{3}$	1
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1	2	4	
3	5		

$(\prec \to <) \quad =$

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Conversion

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$$
\begin{array}{|c|c|c|c|}
\hline
\bar{1} & \bar{2} & \bar{3} & 1 \\
\hline
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\hline
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\hline
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\hline
1 & \bar{3} & 4 \\
\cline{1-3}
3 & 5 \\
\cline{1-2}
\end{array}
$$

Conversion

Example (Converting from the small bar order $\prec$ to the natural order $<$)

$$\begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{2} & \bar{3} & 1 \\ \hline \bar{1} & \bar{3} & \bar{4} & 2 \\ \hline \bar{2} & 1 & 1 & 3 \\ \hline 1 & 2 & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array} (\prec \to <) \quad = \quad \begin{array}{|c|c|c|c|} \hline \bar{1} & \bar{2} & 1 & 1 \\ \hline \bar{1} & 1 & \bar{3} & 2 \\ \hline \bar{2} & 2 & 3 & \bar{4} \\ \hline 1 & \bar{3} & 4 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array}$$

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\hline
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\hline
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\hline
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\hline
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\cline{1-3}
3 & 5 \\
\cline{1-2}
\end{array}
(\prec \to <) \; = \;
\begin{array}{|c|c|c|c|}
\hline
\bar{1} & \bar{2} & \color{blue}{1} & 1 \\
\hline
\bar{1} & 1 & \color{blue}{2} & \color{red}{\bar{3}} \\
\hline
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Colored Yamanouchi tableaux

A *colored Yamanouchi tableau* is a colored tableau T for the order $<$ such that $T(<\to\prec)$ is Yamanouchi.

Example

$\overline{1}$	$\overline{2}$	$\overline{3}$	1
$\overline{1}$	$\overline{3}$	$\overline{4}$	2
$\overline{2}$	1	1	3
1	2	4	
3	5		

A colored Yamanouchi tableau for the order $\prec$

$\overline{1}$	1	1	1
$\overline{1}$	$\overline{2}$	2	$\overline{3}$
1	2	3	$\overline{4}$
$\overline{2}$	$\overline{3}$	4	
3	5		

A colored Yamanouchi tableau

Hook Kronecker Rule

Definition

Let λ be a partition of n and $d \in \{0, 1, \ldots, n\}$.

- Recall $\mu(d) = (n - d, 1^d)$.
- $\mathrm{CYT}_{\lambda,d}$ is the set of colored Yamanouchi tableaux of content λ and total color d.
- $\mathrm{CYT}^{-}_{\lambda,d}(\nu)$ is the set of colored Yamanouchi tableaux of content λ, total color d, and shape ν having unbarred southwest corner.

Theorem (Hook Kronecker Rule)

$$g_{\lambda\,\mu(d)\,\nu} = |CYT^{-}_{\lambda,d}(\nu)|.$$

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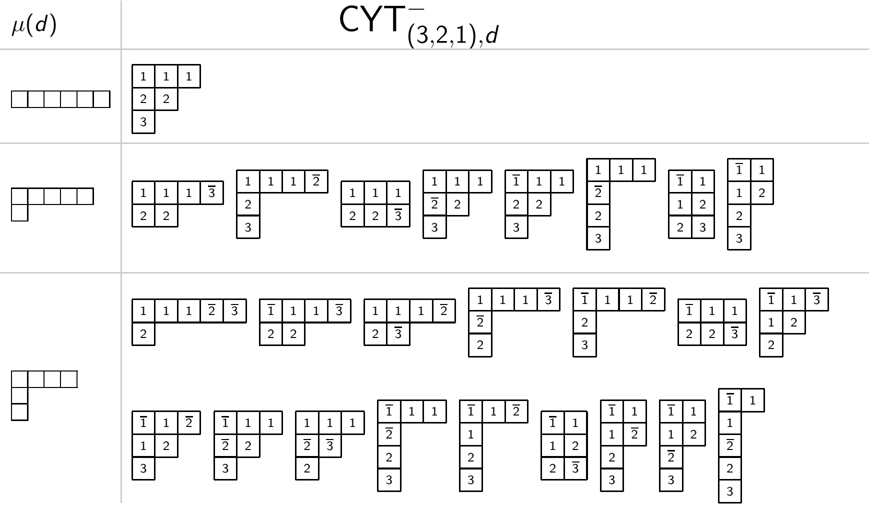
Hook Kronecker Rule



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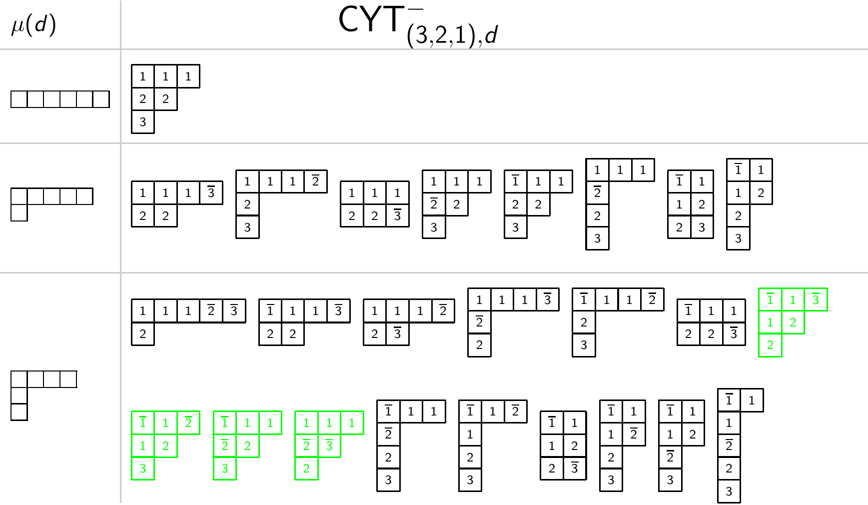
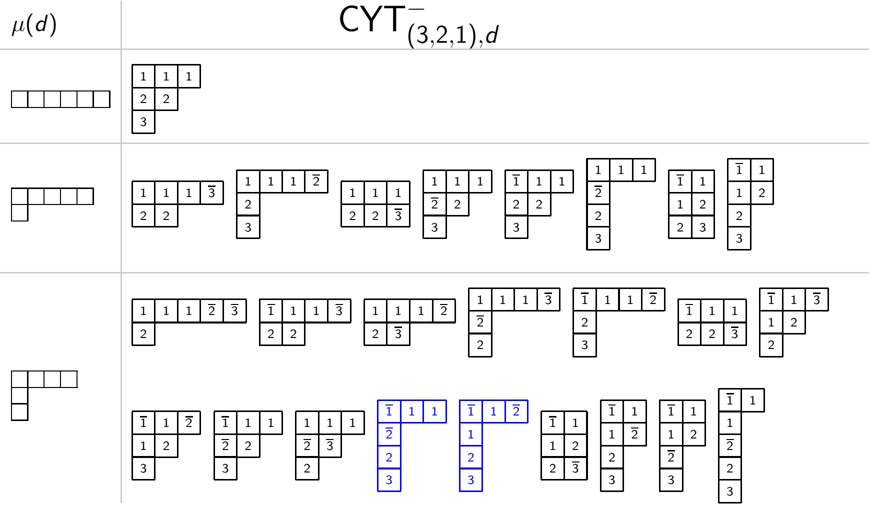


$g_{(3,2,1)\,(4,1,1)\,(3,2,1)} = 4$

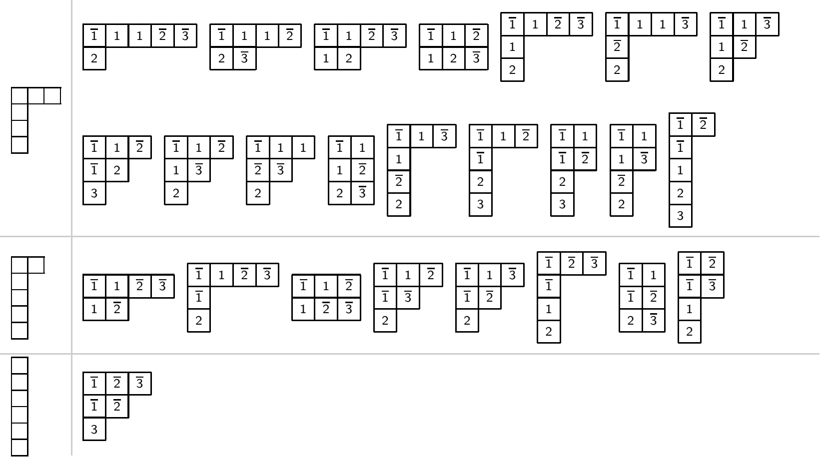
Hook Kronecker Rule



$g_{(3,2,1)\,(4,1,1)\,(3,1,1,1)} = 2$

Hook Kronecker Rule

$\mu(d)$ | $\mathrm{CYT}^{-}_{(3,2,1),d}$



Proof of the Hook Kronecker Rule

A colored tableau for the order $<$ is *color lowerable (raisable)* if its southwest entry is barred (unbarred). Hence unbarring the southwest entry of a color lowerable tableau is a bijection between color lowerable tableaux and color raisable tableaux, which we call the *color lowering operator* C_-.

Example

$$C_-\left(\begin{array}{|c|c|c|}\hline \bar{1} & 1 & \bar{2} \\ \hline \bar{1} & \bar{2} & 2 \\ \hline \bar{2} & 2 & 3 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|}\hline \bar{1} & 1 & \bar{2} \\ \hline \bar{1} & \bar{2} & 2 \\ \hline 2 & 2 & 3 \\ \hline \end{array}.$$

Theorem (B)

A color lowerable tableau T is Yamanouchi if and only if $C_-(T)$ is Yamanouchi.

The Hook Kronecker Rule follows easily from this and the corollary.

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Colored words

Let w be a colored word.

- *Total color*: number of barred letters in w.
- *Content*: remove bars and count number of 1's, number of 2's, etc.
- The ordinary word w^{blft} is formed from w by shuffling the barred letters to the left and then removing their bars.

Example

$$
\begin{aligned}
w &= \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1 \\
& \qquad\qquad\quad 2\,1\,2\,1 \\
& \quad 3\,1\,2\,2\,1 \\
w^{\mathrm{blft}} &= 3\,1\,2\,2\,1\,2\,1\,2\,1
\end{aligned}
$$

The colored word w has total color 5 and content $(4, 4, 1)$.

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- An ordinary word $y = y_1 \cdots y_n$ is *Yamanouchi* if every terminal subword $y_k y_{k+1} \cdots y_n$ has partition content.
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$$\begin{aligned} w &= \bar{1}\,2\,1\,\bar{2}\,\bar{1}\,2\,\bar{3}\,1 \\ w^{\mathrm{blft}} &= 1\,2\,1\,3\,2\,1\,2\,1 \end{aligned}$$

w^{blft} is Yamanouchi, hence w is Yamanouchi.

$w = \bar{3}\,\bar{1}\,2\,1\,\bar{2}\,\bar{2}\,\bar{1}\,2\,1$ is not Yamanouchi because w^{blft} ends in $2\,2\,1\,2\,1\,2\,1$, which has content $(3,4)$.

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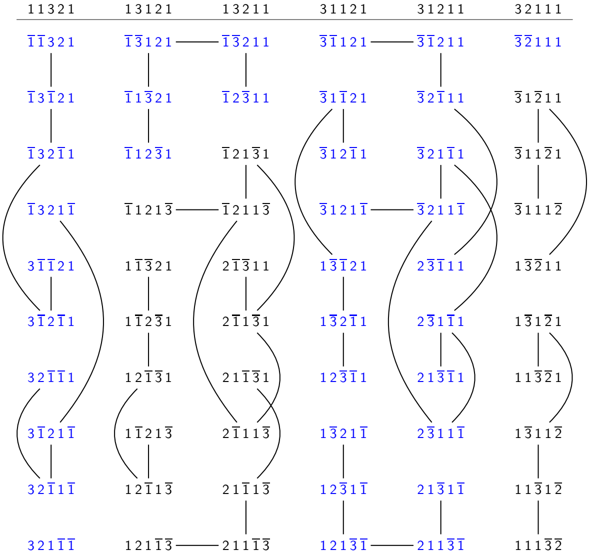
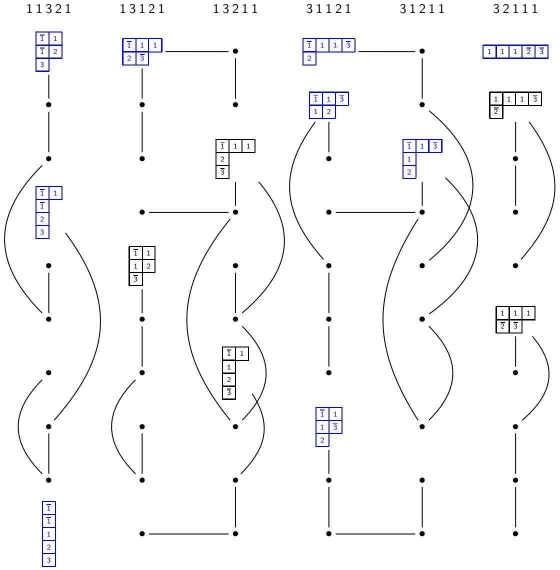


The set $\mathrm{CYW}_{(3,1,1),2}$. The subset $\mathrm{CYW}^{-}_{(3,1,1),2}$ is shown in blue. Edges are Knuth transformations of the words obtained by applying $^{\mathrm{neg}}$. Column labels correspond to applying $^{\mathrm{blft}}$ and the positions of the barred letters are constant along rows.



The mixed insertion tableaux of the words in the previous figure (which are constant on connected components). This set of tableaux is $\mathrm{CYT}_{(3,1,1),2}$ and the tableaux in blue are those with unbarred southwest corner ($\mathrm{CYT}^{-}_{(3,1,1),2}$).