

CATALAN FUNCTIONS AND k -SCHUR POSITIVITY

JONAH BLASIAK, JENNIFER MORSE, ANNA PUN, AND DANIEL SUMMERS

ABSTRACT. We prove that graded k -Schur functions are G -equivariant Euler characteristics of vector bundles on the flag variety, settling a conjecture of Chen-Haiman. We expose a new miraculous shift invariance property of the graded k -Schur functions and resolve the Schur positivity and k -branching conjectures in the strongest possible terms by providing direct combinatorial formulas using strong marked tableaux.

1. INTRODUCTION

We resolve conjectures made in [23], [7], and [22] which were inspired by problems on Macdonald polynomials. These remarkable polynomials form a basis for the ring of symmetric functions over the field $\mathbb{Q}(q, t)$. Their study over the last three decades has generated an impressive body of research, a prominent focus being the Macdonald positivity conjecture: the Schur expansion coefficients of the (Garsia) modified Macdonald polynomials $H_\mu(\mathbf{x}; q, t)$ lie in $\mathbb{N}[q, t]$. This was proved by Haiman [13] using geometry of Hilbert schemes, yet many questions arising in this study remain unanswered.

Lapointe, Lascoux, and Morse [23] considerably strengthened the Macdonald positivity conjecture. They constructed a family of functions and conjectured (i) they form a basis for the space $\Lambda^k = \text{span}_{\mathbb{Q}(q,t)}\{H_\mu(\mathbf{x}; q, t)\}_{\mu_1 \leq k}$, (ii) they are Schur positive, and (iii) the expansion of $H_\mu(\mathbf{x}; q, t) \in \Lambda^k$ in this basis has coefficients in $\mathbb{N}[q, t]$. The problem of Schur expanding Macdonald polynomials thus factors into the two positivity problems (ii) and (iii). Because the intricate construction of these functions lacked in mechanism for proof, many conjecturally equivalent candidates have since been proposed. Informally, all these candidates are now called *k-Schur functions*.

At the forefront of k -Schur investigations is the conjectured *branching property*:

$$\text{the } k+1\text{-Schur expansion of a } k\text{-Schur function has coefficients in } \mathbb{N}[t]. \quad (1.1)$$

Since a k -Schur function reduces to a Schur function for large k , the iteration of branching implies (ii). However, every effort to prove that even one of the k -Schur candidates satisfies (i) and (1.1), or even (i) and (ii), over the last decades has failed.

In contrast, the *ungraded* case has been more tractable. The k -Schur concept is interesting even when $t = 1$ despite needing generic t for Macdonald polynomial applications. The existence of an ungraded family of k -Schur functions satisfying (the $t = 1$ versions of) (i), (ii), and (1.1) was established for certain functions $s_\lambda^{(k)}(\mathbf{x})$ defined in [26] using chains in the weak order of the affine symmetric group $\widehat{\mathcal{S}}_{k+1}$. It was proven that they form a basis

Key words and phrases. k -Schur functions, Schur positivity, branching rule, spin, strong tableaux, generalized Kostka polynomials.

Authors were supported by NSF Grants DMS-1600391 (J. B.) and DMS-1833333 (J. M.).

for $\Lambda^k|_{t=1}$ with the Gromov-Witten invariants for the quantum cohomology of the Grassmannian included in their structure constants [27]. Lam established that the basis $s_\lambda^{(k)}(\mathbf{x})$ represents Schubert classes in the homology of the affine Grassmannian $\text{Gr}_{SL_{k+1}}$ of SL_{k+1} [19] and gave a geometric proof of (1.1) at $t = 1$: the branching property reflects that the image of a Schubert class is a positive sum of Schubert classes under an inclusion $H_*(\text{Gr}_{SL_{k+1}}) \rightarrow H_*(\text{Gr}_{SL_{k+2}})$ [20]. Additionally, an algorithm for computing the branching coefficients using an intricate equivalence relation is worked out in [21, 22].

Building off the work on the ungraded case, a candidate $\{s_\lambda^{(k)}(\mathbf{x}; t)\}$ for k -Schur functions was proposed in [21] by attaching a nonnegative integer called spin to *strong marked tableaux*, certain chains in the strong (Bruhat) order of $\widehat{\mathcal{S}}_{k+1}$. It was conjectured that they satisfy the desired properties (i)–(iii) and shown [21] that they are equal to the weak order k -Schur functions $s_\lambda^{(k)}(\mathbf{x})$ at $t = 1$.

In a different vein, Li-Chung Chen and Mark Haiman [7] conjectured that k -Schur functions are a subclass of a family of symmetric functions indexed by pairs (Ψ, γ) consisting of an upper order ideal Ψ of positive roots (of which there are Catalan many) and a weight $\gamma \in \mathbb{Z}^\ell$. These *Catalan (symmetric) functions* can be defined by a Demazure-operator formula, and are equal to GL_ℓ -equivariant Euler characteristics of vector bundles on the flag variety by the Borel-Weil-Bott theorem. Chen-Haiman [7] investigated their Schur expansions and conjectured a positive combinatorial formula when $\gamma_1 \geq \gamma_2 \geq \dots$. Panyushev [30] studied similar questions and proved a cohomological vanishing theorem to establish Schur positivity of a large subclass of Catalan functions.

Catalan functions are the modified Hall-Littlewood polynomials when the ideal of roots Ψ consists of all the positive roots and γ is a partition. Here their Schur expansion coefficients are the *Kostka-Foulkes polynomials* (Lusztig's t -analog of weight multiplicities in type A), which have been extensively studied from algebraic, geometric, and combinatorial perspectives (see, e.g., [29, 14, 8]). In the case that Ψ consists of the roots above a block diagonal matrix, the Schur expansion coefficients are the *generalized Kostka polynomials* investigated by Broer, Shimozono-Weyman, and others [5, 34, 32, 31, 33, 18].

Chen-Haiman constructed an upper order ideal of positive roots associated to each partition λ with $\lambda_1 \leq k$ and conjectured that the associated Catalan functions $\{\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)\}$ are k -Schur functions [7]. A key discovery in our work is a different set of upper order ideals of roots (compare Definitions 2.1 and 10.1) which we show yields the same family $\{\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)\}$. From there, we prove

- (Chen-Haiman conjecture) The polynomials $\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)$ are the k -Schur functions $s_\lambda^{(k)}(\mathbf{x}; t)$.
- (k -Schur branching) The coefficients of (1.1) are $\sum t^{\text{spin}(T)}$ over certain skew strong tableaux T ; this settles Conjecture 1.1 of [22] in the strongest possible terms.
- (Schur positive basis) The functions $\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)$ are a Schur positive basis of Λ^k . This resolves step (ii) in the route for finding the Schur expansion of Macdonald polynomials and finally establishes that a k -Schur candidate satisfies (i), (ii), and (1.1).
- (Dual Pieri rule) The polynomials $\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)$ satisfy a vertical-strip defining rule.
- (Shift invariance) $\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t) = e_\ell^\perp \mathfrak{s}_{\lambda+1^\ell}^{(k+1)}(\mathbf{x}; t)$. This powerful new discovery follows easily from our definition of $\mathfrak{s}_\lambda^{(k)}(\mathbf{x}; t)$ and, together with the dual Pieri rule, it immediately yields our k -Schur branching rule.

2. MAIN RESULTS

We work in the ring $\Lambda = \mathbb{Q}(t)[h_1, h_2, \dots]$ of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$, where $h_d = h_d(\mathbf{x}) = \sum_{i_1 \leq \dots \leq i_d} x_{i_1} \cdots x_{i_d}$. The Schur functions s_λ indexed by partitions λ form a basis for Λ . Schur functions may be defined more generally for any $\gamma \in \mathbb{Z}^\ell$ by the following version of the Jacobi-Trudi formula:

$$s_\gamma = s_\gamma(\mathbf{x}) = \det(h_{\gamma_i+j-i}(\mathbf{x}))_{1 \leq i, j \leq \ell} \in \Lambda, \tag{2.1}$$

where by convention $h_0(\mathbf{x}) = 1$ and $h_d(\mathbf{x}) = 0$ for $d < 0$.

Central to our work is a family of symmetric functions investigated by Chen-Haiman [7] and Panyushev [30]; special cases include all s_γ , as well as the modified Hall-Littlewood polynomials and generalizations thereof studied by Broer and Shimozono-Weyman [5, 34]. They can be described geometrically in terms of cohomology of vector bundles on the flag variety (see Theorem 3.1), or algebraically as follows:

Definition 2.1. Fix a positive integer ℓ . A *root ideal* is an upper order ideal of the poset $\Delta_\ell^+ = \Delta^+ := \{(i, j) \mid 1 \leq i < j \leq \ell\}$ with partial order given by $(a, b) \leq (c, d)$ when $a \geq c$ and $b \leq d$. Given a root ideal $\Psi \subset \Delta_\ell^+$ and $\gamma \in \mathbb{Z}^\ell$, the associated *Catalan function* is

$$H(\Psi; \gamma)(\mathbf{x}; t) := \prod_{(i, j) \in \Psi} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x}) \in \Lambda, \tag{2.2}$$

where the raising operator R_{ij} acts on the subscripts of the s_γ by $R_{ij}s_\gamma = s_{\gamma+\epsilon_i-\epsilon_j}$ (a discussion of raising operators is given in §4.3).

Let $\text{Par}_\ell^k = \{(\mu_1, \dots, \mu_\ell) \in \mathbb{Z}^\ell : k \geq \mu_1 \geq \dots \geq \mu_\ell \geq 0\}$ denote the set of partitions contained in the $\ell \times k$ -rectangle and Par^k the set of partitions μ with $\mu_1 \leq k$. The trailing zeros are a useful bookkeeping device for elements of Par_ℓ^k , whereas for Par^k we adopt the more common convention: each $\mu \in \text{Par}^k$ is identified with $(\mu_1, \dots, \mu_{\ell(\mu)}, 0^i)$ for any $i \geq 0$, where the *length* $\ell(\mu)$ is the number of nonzero parts of μ .

Definition 2.2. For $\mu \in \text{Par}_\ell^k$, define the root ideal

$$\Delta^k(\mu) = \{(i, j) \in \Delta_\ell^+ \mid k - \mu_i + i < j\}, \tag{2.3}$$

and the Catalan function

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) := H(\Delta^k(\mu); \mu)(\mathbf{x}; t) = \prod_{i=1}^{\ell} \prod_{j=k+1-\mu_i+i}^{\ell} (1 - tR_{ij})^{-1} s_\mu(\mathbf{x}). \tag{2.4}$$

We will often use the shorthands $\mathfrak{s}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)}(\mathbf{x}; t)$ and $H(\Psi; \gamma) = H(\Psi; \gamma)(\mathbf{x}; t)$ to reduce clutter. Their $t = 1$ specializations are denoted $\mathfrak{s}_\mu^{(k)}(\mathbf{x}; 1)$ and $H(\Psi; \gamma)(\mathbf{x}; 1)$.

We will soon see (Theorem 2.4) that the $\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t)$ are the k -Schur functions, which proves a conjecture of Chen-Haiman. This is a consequence of four fundamental properties of these Catalan functions described in the next theorem. Their statement requires the definition of strong marked tableaux, which are certain saturated chains in the strong Bruhat order for the affine symmetric group $\widehat{\mathcal{S}}_{k+1}$, together with some extra data. The precise definition is most readily given in terms of partitions arising in modular representation theory called $k + 1$ -cores. Combinatorial examples are provided in §2.2.

The *diagram* of a partition λ is the subset of boxes $\{(r, c) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \mid c \leq \lambda_r\}$ in the plane, drawn in English (matrix-style) notation so that rows (resp. columns) are increasing from north to south (resp. west to east). Each box has a *hook length* which counts the number of boxes below it in its column and weakly to its right in its row. A $k+1$ -*core* is a partition with no box of hook length $k+1$. There is a bijection \mathbf{p} [25] from the set of $k+1$ -cores to Par^k mapping a $k+1$ -core κ to the partition λ whose r -th row λ_r is the number of boxes in the r -th row of κ having hook length $\leq k$.

A *strong cover* $\tau \Rightarrow \kappa$ is a pair of $k+1$ -cores such that $\tau \subset \kappa$ and $|\mathbf{p}(\tau)| + 1 = |\mathbf{p}(\kappa)|$. A *strong marked cover* $\tau \xrightarrow{r} \kappa$ is a strong cover $\tau \Rightarrow \kappa$ together with a positive integer r which is allowed to be the smallest row index of any connected component of the skew shape κ/τ . Let $\eta = (\eta_1, \eta_2, \dots) \in \mathbb{Z}_{\geq 0}^{\infty}$ with $m = |\eta| := \sum_i \eta_i$ finite. A *strong marked tableau* T of *weight* η is a sequence of strong marked covers

$$\kappa^{(0)} \xrightarrow{r_1} \kappa^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_m} \kappa^{(m)}$$

such that $r_{v_i+1} \geq r_{v_i+2} \geq \dots \geq r_{v_i+\eta_i}$ for all $i \geq 1$, where $v_i := \eta_1 + \dots + \eta_{i-1}$. A *vertical strong marked tableau* is defined the same way except we require each subsequence $r_{v_i+1} < r_{v_i+2} < \dots < r_{v_i+\eta_i}$ to be strictly increasing rather than weakly decreasing. We write $\text{inside}(T) = \mathbf{p}(\kappa^{(0)})$ and $\text{outside}(T) = \mathbf{p}(\kappa^{(m)})$. The set of strong marked tableaux (resp. vertical strong marked tableaux) T of weight η with $\text{outside}(T) = \mu$ is denoted $\text{SMT}_{\eta}^k(\mu)$ (resp. $\text{VSMT}_{\eta}^k(\mu)$).

The *spin* of a strong marked cover $\tau \xrightarrow{r} \kappa$ is defined to be $c \cdot (h - 1) + N$, where c is the number of connected components of the skew shape κ/τ , h is the height (number of rows) of each component, and N is the number of components entirely contained in rows $> r$. For a (vertical) strong marked tableau T , $\text{spin}(T)$ is defined to be the sum of the spins of the strong marked covers comprising T .

For $f \in \Lambda$, let f^{\perp} be the linear operator on Λ that is adjoint to multiplication by f with respect to the Hall inner product, i.e., $\langle f^{\perp}(g), h \rangle = \langle g, fh \rangle$ for all $g, h \in \Lambda$.

Theorem 2.3. *Fix a positive integer ℓ . The Catalan functions $\{\mathfrak{s}_{\mu}^{(k)}\}_{k \geq 1, \mu \in \text{Par}_{\ell}^k}$, satisfy the following properties:*

$$\text{(horizontal dual Pieri rule)} \quad h_d^{\perp} \mathfrak{s}_{\mu}^{(k)} = \sum_{T \in \text{SMT}_{(d)}^k(\mu)} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(k)} \quad \text{for all } d \geq 0; \quad (2.5)$$

$$\text{(vertical dual Pieri rule)} \quad e_d^{\perp} \mathfrak{s}_{\mu}^{(k)} = \sum_{T \in \text{VSMT}_{(d)}^k(\mu)} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(k)} \quad \text{for all } d \geq 0; \quad (2.6)$$

$$\text{(shift invariance)} \quad \mathfrak{s}_{\mu}^{(k)} = e_{\ell}^{\perp} \mathfrak{s}_{\mu+1^{\ell}}^{(k+1)}; \quad (2.7)$$

$$\text{(Schur function stability)} \quad \text{if } k \geq |\mu|, \text{ then } \mathfrak{s}_{\mu}^{(k)} = s_{\mu}. \quad (2.8)$$

Theorem 2.3 has several powerful consequences. Foremost is that the $\mathfrak{s}_{\mu}^{(k)}$ are k -Schur functions. More precisely, we adopt the following definition of the k -Schur functions from

[21, §9.3] (see also [1, 22]): for $\mu \in \text{Par}^k$, let

$$s_\mu^{(k)}(\mathbf{x}; t) = \sum_{\eta \in \mathbb{Z}_{\geq 0}^\infty, |\eta|=|\mu|} \sum_{T \in \text{SMT}_\eta^k(\mu)} t^{\text{spin}(T)} \mathbf{x}^\eta. \quad (2.9)$$

Among the several conjecturally equivalent definitions, this one has the advantage that its $t = 1$ specializations $\{s_\mu^{(k)}(\mathbf{x}; 1)\}$ are known to agree [21] with a quite different looking combinatorial definition using weak tableaux from [26], and are Schubert classes in the homology of the affine Grassmannian $\text{Gr}_{SL_{k+1}}$ of SL_{k+1} [19].

Theorem 2.4. *For $\mu \in \text{Par}_\ell^k$, the k -Schur function $s_\mu^{(k)}(\mathbf{x}; t)$ is the Catalan function $\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t)$.*

Proof. The homogeneous symmetric function basis for Λ consists of $h_\lambda = \prod_i h_{\lambda_i}$, as λ ranges over partitions. For any partition λ of size $|\mu|$, (2.5) implies

$$\langle \mathfrak{s}_\mu^{(k)}, h_\lambda \rangle = \langle h_\lambda^\perp \mathfrak{s}_\mu^{(k)}, 1 \rangle = \sum_{T \in \text{SMT}_\lambda^k(\mu)} t^{\text{spin}(T)},$$

where we have also used (2.8) to obtain $\mathfrak{s}_{\text{inside}(T)}^{(k)} = \mathfrak{s}_{0^\ell}^{(k)} = 1$ for every T in the sum. The basis of monomial symmetric functions $\{m_\nu(\mathbf{x})\}$ is dual to the homogenous basis and thus

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) = \sum_{\text{partitions } \lambda \text{ of } |\mu|} \sum_{T \in \text{SMT}_\lambda^k(\mu)} t^{\text{spin}(T)} m_\lambda(\mathbf{x}). \quad (2.10)$$

Let $\eta \in \mathbb{Z}_{\geq 0}^\infty$ with $|\eta| = |\mu|$ and let λ be the partition obtained by sorting η . Since the h_d^\perp pairwise commute, again using (2.5) and (2.8) we obtain

$$\sum_{T \in \text{SMT}_\eta^k(\mu)} t^{\text{spin}(T)} = (h_{\eta_1}^\perp h_{\eta_2}^\perp \cdots) \mathfrak{s}_\mu^{(k)} = (h_{\lambda_1}^\perp h_{\lambda_2}^\perp \cdots) \mathfrak{s}_\mu^{(k)} = \sum_{T \in \text{SMT}_\lambda^k(\mu)} t^{\text{spin}(T)}. \quad (2.11)$$

Thus the right sides of (2.10) and (2.9) agree, as desired. \square

Theorem 2.4 has several important consequences; we give informal statements now, deferring their precise versions for later.

Corollary 2.5.

- (1) *The k -Schur functions $s_\mu^{(k)}(\mathbf{x}; t)$ defined by (2.9) are symmetric functions.*
- (2) *The k -Schur functions $s_\mu^{(k)}(\mathbf{x}; t)$ are equal to certain Catalan functions defined by Chen-Haiman via a different root ideal than our $\Delta^k(\mu)$ (see Section 10).*
- (3) *For $\mu \in \text{Par}_\ell^k$, $s_\mu^{(k)}(\mathbf{x}; t)$ is the GL_ℓ -equivariant Euler characteristic of a vector bundle on the flag variety determined by μ and k .*
- (4) *Homology Schubert classes of $\text{Gr}_{SL_{k+1}}$ are equal to the ungraded ($t = 1$) version of this Euler characteristic.*

Proof. Theorem 2.4 allows us to work interchangeably with $\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t)$ and $s_\mu^{(k)}(\mathbf{x}; t)$ and we do so from now on without further mention. Symmetry follows directly from the definition of the Catalan functions. We match $\mathfrak{s}_\mu^{(k)}$ to the Catalan functions studied by Chen-Haiman in Theorem 10.4 to settle (2). Statement (3) is proved in Theorem 3.1, and this in turn implies (4) using [21, Theorem 4.11] and [19, Theorem 7.1]. \square

The realization of the k -Schur functions as the subclass (2.4) of Catalan functions is highly lucrative. One of the most striking outcomes is Property (2.7); overlooked in prior investigations of k -Schur functions, it fully resolves k -branching:

The coefficients in the $k + 1$ -Schur expansion of a k -Schur function are none other than the structure coefficients appearing in the vertical dual Pieri rule (2.6).

Theorem 2.6 (k -Schur branching rule). *For $\mu \in \text{Par}_\ell^k$, the expansion of the k -Schur function $s_\mu^{(k)}$ into $k + 1$ -Schur functions is given by*

$$s_\mu^{(k)} = \sum_{T \in \text{VSMT}_{(\ell)}^{k+1}(\mu+1^\ell)} t^{\text{spin}(T)} s_{\text{inside}(T)}^{(k+1)}. \quad (2.12)$$

The Schur expansion of a k -Schur function can then be achieved by incrementally iterating (2.12) until k is large enough to apply (2.8). Interestingly, a more elegant formula can be derived by a different combination of (2.6), (2.7), and (2.8).

Theorem 2.7 (k -Schur into Schur). *Let $\mu \in \text{Par}_\ell^k$ and set $m = \max(|\mu| - k, 0)$. The Schur expansion of the k -Schur function $s_\mu^{(k)}$ is given by*

$$s_\mu^{(k)} = \sum_{T \in \text{VSMT}_{(\ell^m)}^{k+m}(\mu+m^\ell)} t^{\text{spin}(T)} s_{\text{inside}(T)}. \quad (2.13)$$

Proof. Apply (2.7) m times to obtain

$$\mathfrak{s}_\mu^{(k)} = (e_\ell^\perp)^m \mathfrak{s}_{\mu+m^\ell}^{(k+m)}. \quad (2.14)$$

The vertical dual Pieri rule (2.6) then gives the $(k + m)$ -Schur function decomposition and (2.8) ensures this is the Schur function decomposition by the careful choice of m . \square

See Example 2.13 for Theorem 2.6 and Example 2.14 and Figure 1 for Theorem 2.7.

Recall from the introduction that the original k -Schur candidate of [23] (as well as subsequent candidates) conjecturally satisfies (i)–(ii), i.e., forms a Schur positive basis for the space $\Lambda^k = \text{span}_{\mathbb{Q}(q,t)} \{H_\mu(\mathbf{x}; q, t)\}_{\mu_1 \leq k}$. We settle this conjecture for the k -Schur functions of (2.9), thereby giving the first proof that a k -Schur candidate satisfies (i)–(ii). This follows from Theorem 2.7 together with the next result which additionally refines statement (i) to give bases for subspaces Λ_ℓ^k which depend on ℓ as well as k .

Theorem 2.8. *For any positive integers ℓ and k , the k -Schur functions $\{s_\mu^{(k)}(\mathbf{x}; t)\}_{\mu \in \text{Par}_\ell^k}$ form a basis for the space $\Lambda_\ell^k = \text{span}_{\mathbb{Q}(t)} \{H_\mu(\mathbf{x}; t) \mid \mu \in \text{Par}_\ell^k\} \subset \Lambda$.*

Here $H_\mu(\mathbf{x}; t) = H_\mu(\mathbf{x}; 0, t)$ are the modified Hall-Littlewood polynomials (denoted $Q'_\mu(\mathbf{x}; t)$ in [29, p. 234]). Note that $\mathbb{Q}(q, t) \otimes_{\mathbb{Q}(t)} (\sum_\ell \Lambda_\ell^k) = \Lambda^k$ since Λ^k has the alternative description $\Lambda^k = \text{span}_{\mathbb{Q}(q,t)} \{H_\mu(\mathbf{x}; t) \mid \mu \in \text{Par}^k\}$; this follows from the fact that the Schur functions and Macdonald's integral forms $J_\mu(\mathbf{x}; q, t)$ are related by a triangular change of basis—see, e.g., Equations 6.6, 2.18, and 2.20 of [24].

We also give three intrinsic descriptions of the k -Schur functions. Studies of ungraded k -Schur functions have been served well by the characterization of $s_\mu^{(k)}(\mathbf{x}; 1)$ as the unique symmetric functions satisfying (2.5) and (2.8) at $t = 1$. We can now show that the k -Schur

functions $s_\mu^{(k)}(\mathbf{x}; t)$ of (2.9) have this characterization (without the $t = 1$ specialization), as well as a new one coming from the shift invariance property (2.7).

Corollary 2.9. *The k -Schur functions $\{s_\mu^{(k)}\}_{k \geq 1, \mu \in \text{Par}_\ell^k}$ are the unique family of symmetric functions satisfying the following subsets of properties (2.5)–(2.8).*

- (1) (2.5) and (2.8);
- (2) (2.6) and (2.8);
- (3) (2.6) for $d = \ell$, (2.7), and (2.8).

Proof. The proof of Theorem 2.4 establishes (1). By a similar argument, (2.6) and (2.8) imply

$$s_\mu^{(k)}(\mathbf{x}; t) = \sum_{\text{partitions } \lambda \text{ of } |\mu|} \sum_{T \in \text{VSMT}_\lambda^k(\mu)} t^{\text{spin}(T)} \omega(m_\lambda(\mathbf{x})), \quad (2.15)$$

where ω is the involution on Λ defined by $\omega(e_d) = h_d$ for $d \geq 0$. This establishes (2). The properties in (3) determine a unique family of symmetric functions by Theorem 2.7. \square

2.1. Outline. The bulk of our paper is devoted to developing machinery to prove the vertical dual Pieri rule (2.6). The remaining results are fairly straightforward; we prove (2.7), (2.8), and Theorem 2.8 in Section 4, and (2.5) in §9.2 as a corollary to (2.6).

Here are some highlights from the proof of the vertical dual Pieri rule, many of which we feel will have further applications beyond this paper. This also serves to give a rough outline of the proof.

- We prove that $e_d^\perp(H(\Psi; \gamma)) = \sum_{S \subset [\ell], |S|=d} H(\Psi; \gamma - \epsilon_S)$ (Lemma 4.11), which is our starting point for evaluating the left side of (2.6).
- To handle the terms $H(\Psi; \gamma - \epsilon_S)$ in this sum, we prove a *k -Schur straightening rule* (Theorem 7.12) which shows that analogs of the $s_\mu^{(k)}$ indexed by nonpartitions are equal to 0 or to a power of t times $s_\nu^{(k)}$ for partition ν .
- Miraculously, the combinatorics arising in this rule exactly matches that of strong covers (Proposition 8.10).
- To prove the k -Schur straightening rule, we develop several tools for working with Catalan functions including a recurrence which expresses a Catalan function as the sum of two such polynomials with similar root ideals (Proposition 5.6).
- To prove (2.6) by induction on d , we must prove a stronger statement in which the right side of (2.6) is replaced by a sum over tableaux which are marked only in rows $\leq m$, and the left side is an algebraically defined generalization of $e_d^\perp s_\mu^{(k)}$.

2.2. Combinatorial examples.

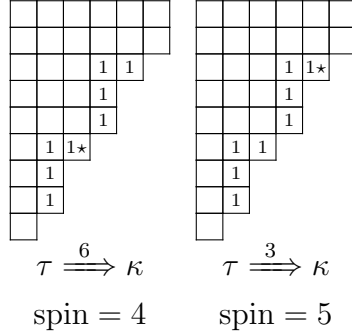
Example 2.10. The 5-core $\kappa = 53221$ and its image $\mathfrak{p}(\kappa) = 32221 \in \text{Par}^4$ are

$$\kappa = \begin{array}{|c|c|c|c|c|} \hline 9 & 7 & 4 & 2 & 1 \\ \hline 6 & 4 & 1 & & \\ \hline 4 & 2 & & & \\ \hline 3 & 1 & & & \\ \hline 1 & & & & \\ \hline \end{array} \quad \mapsto \quad \mathfrak{p}(\kappa) = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array},$$

where the boxes of κ are labeled by their hook lengths.

A (vertical) strong marked tableau $T = (\kappa^{(0)} \xrightarrow{r_1} \kappa^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_m} \kappa^{(m)})$ is drawn by filling each skew shape $\kappa^{(i)}/\kappa^{(i-1)}$ with the entry i and starring the entry in position $(r_i, \kappa_{r_i}^{(i)})$, for all $i \in [m]$. (This is really the standardization of T , but suffices for the examples in this paper as we will always specify the weight η separately.) Strong marked covers are drawn this way too, regarding them as strong marked tableaux of weight (1).

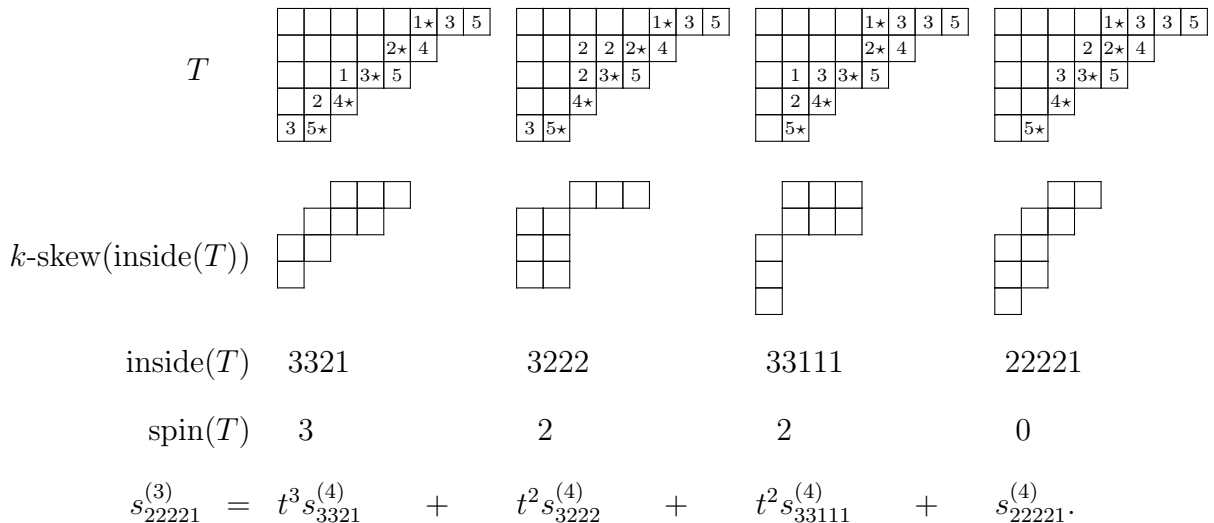
Example 2.11. Let $k = 4$. For $\tau = 663331111$ and $\kappa = 665443221$, $\mathbf{p}(\tau) = 332221111$ and $\mathbf{p}(\kappa) = 222222221$. Thus $\tau \Rightarrow \kappa$ is a strong cover and it has two distinct markings:



Remark 2.12. Although strong marked covers are typically marked by the content of the northeastmost box of a connected component of κ/τ , it is equivalent (and more natural for us) to use row indices.

Given a $k + 1$ -core κ and $\lambda = \mathbf{p}(\kappa)$, the k -skew diagram of λ denoted $k\text{-skew}(\lambda)$, is the subdiagram of κ consisting of boxes with hook length $\leq k$. Hence the row lengths of $k\text{-skew}(\lambda)$ are given by λ itself.

Example 2.13. According to Theorem 2.6, the expansion of $s_{22221}^{(3)}$ into 4-Schur functions is obtained by summing $t^{\text{spin}(T)} s_{\text{inside}(T)}^{(4)}$ over the set $\text{VSMT}_{(5)}^4(33332)$ of vertical strong marked tableaux given below. Note that $86532 = \mathbf{p}^{-1}(33332)$ is the outer shape of each diagram on the first line.



Spin

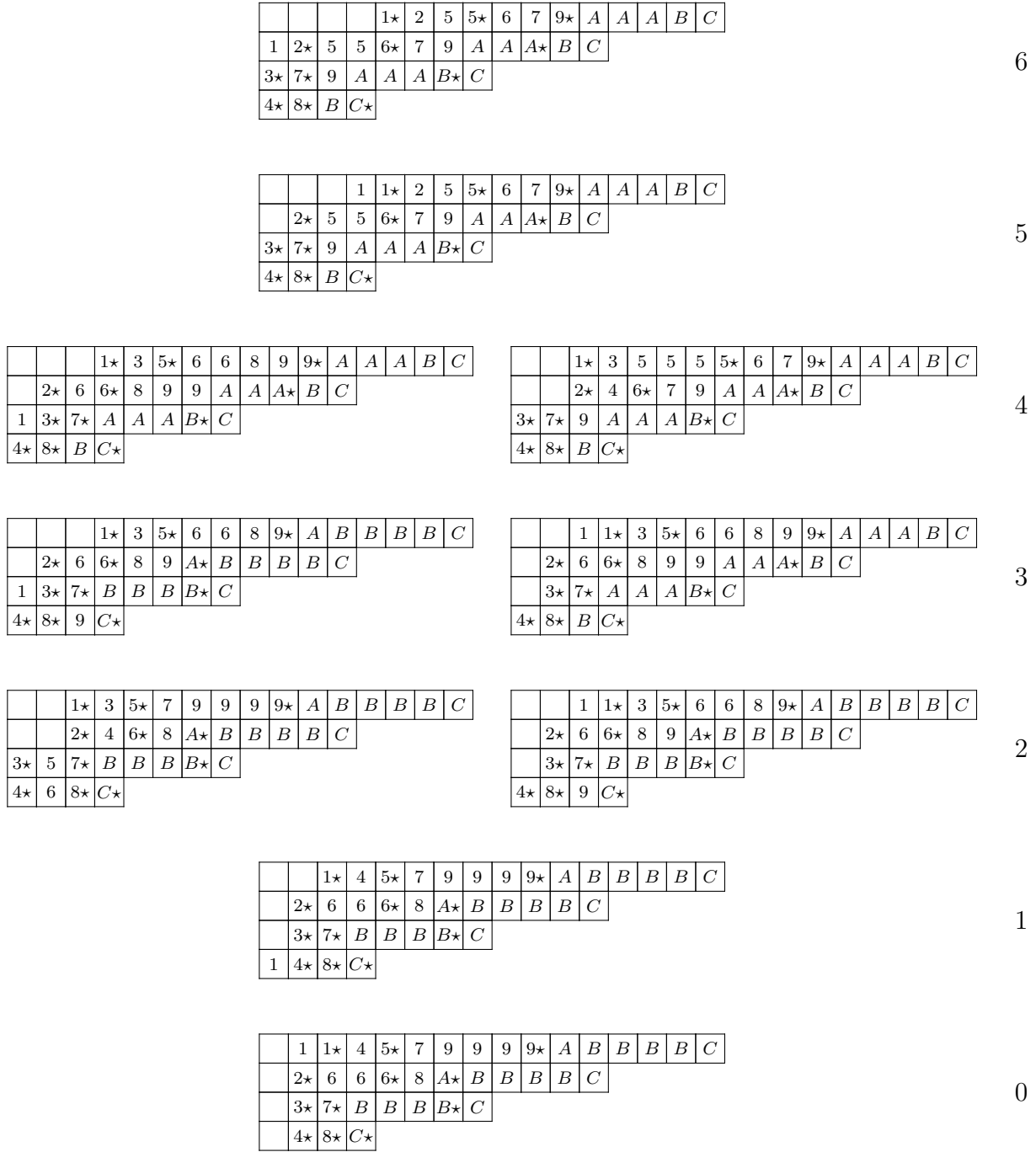


Figure 1: According to Theorem 2.7 with $k = 1, \ell = 4$, the Schur expansion of the 1-Schur function $s_{1111}^{(1)}$ (also equal to the modified Hall-Littlewood polynomial H_{1111}) is obtained by summing $t^{\text{spin}(T)} s_{\text{inside}(T)}$ over the set $\text{VSMT}_{(4,4,4)}^4(4, 4, 4, 4)$ of vertical strong marked tableaux T given above. We have written A, B, C in place 10, 11, 12. Note that T having weight $\ell^m = (4, 4, 4)$ means that the skew shapes (strong covers) labeled by $1, 2, \dots, \ell$ have stars in rows $1, 2, \dots, \ell$, as do the next ℓ skew shapes, and so on.

Example 2.14. We compute the Schur expansion of $s_{3321}^{(4)}$ using the (proof of) Theorem 2.7: it is given by the sum $t^{\text{spin}(T)} s_{\text{inside}(T)}$ over the set $\text{VSMT}_{(4,4,4)}^7(6654)$ of vertical strong marked tableaux given below. Note that the m in Theorem 2.7 is a convenient choice, but often a smaller m suffices; the 7-Schur expansion of $s_{3321}^{(4)}$ is already the Schur expansion, so $m = 3$ (hence $k + m = 7$) suffices.

	Spin																																												
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td></td><td></td><td></td><td></td><td></td><td>1*</td><td>3</td><td>5*</td><td>7</td><td>9*</td><td>B</td></tr> <tr><td></td><td></td><td></td><td></td><td></td><td>2*</td><td>6</td><td>6*</td><td>8</td><td>A*</td><td>C</td></tr> <tr><td></td><td>3*</td><td>5</td><td>7*</td><td>9</td><td>B*</td><td></td><td></td><td></td><td></td><td></td></tr> <tr><td></td><td>4*</td><td>8*</td><td>A</td><td>C*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>						1*	3	5*	7	9*	B						2*	6	6*	8	A*	C		3*	5	7*	9	B*							4*	8*	A	C*							3
					1*	3	5*	7	9*	B																																			
					2*	6	6*	8	A*	C																																			
	3*	5	7*	9	B*																																								
	4*	8*	A	C*																																									
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td></td><td></td><td></td><td></td><td>1*</td><td>4</td><td>5</td><td>5*</td><td>7</td><td>9*</td><td>B</td></tr> <tr><td></td><td></td><td></td><td></td><td>2*</td><td>6</td><td>6*</td><td>8</td><td>A*</td><td>C</td><td></td></tr> <tr><td></td><td>3*</td><td>7*</td><td>9</td><td>B*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> <tr><td></td><td>4*</td><td>8*</td><td>A</td><td>C*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>					1*	4	5	5*	7	9*	B					2*	6	6*	8	A*	C			3*	7*	9	B*								4*	8*	A	C*							2
				1*	4	5	5*	7	9*	B																																			
				2*	6	6*	8	A*	C																																				
	3*	7*	9	B*																																									
	4*	8*	A	C*																																									
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td></td><td></td><td></td><td></td><td>1*</td><td>4</td><td>5*</td><td>8</td><td>9</td><td>9*</td><td>B</td></tr> <tr><td></td><td></td><td></td><td></td><td>2*</td><td>6*</td><td>A</td><td>A</td><td>A</td><td>A*</td><td>C</td></tr> <tr><td></td><td></td><td>3*</td><td>7*</td><td>B*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> <tr><td></td><td>4*</td><td>5</td><td>8*</td><td>C*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>					1*	4	5*	8	9	9*	B					2*	6*	A	A	A	A*	C			3*	7*	B*								4*	5	8*	C*							1
				1*	4	5*	8	9	9*	B																																			
				2*	6*	A	A	A	A*	C																																			
		3*	7*	B*																																									
	4*	5	8*	C*																																									
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td></td><td></td><td></td><td></td><td>1</td><td>1*</td><td>4</td><td>5*</td><td>7</td><td>9*</td><td>B</td></tr> <tr><td></td><td></td><td></td><td></td><td>2*</td><td>6*</td><td>A</td><td>A</td><td>A</td><td>A*</td><td>C</td></tr> <tr><td></td><td></td><td>3*</td><td>7*</td><td>9</td><td>B*</td><td></td><td></td><td></td><td></td><td></td></tr> <tr><td></td><td>4*</td><td>8*</td><td>C*</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>					1	1*	4	5*	7	9*	B					2*	6*	A	A	A	A*	C			3*	7*	9	B*							4*	8*	C*								0
				1	1*	4	5*	7	9*	B																																			
				2*	6*	A	A	A	A*	C																																			
		3*	7*	9	B*																																								
	4*	8*	C*																																										
<table border="1" style="border-collapse: collapse; margin: auto;"> <tr><td></td><td></td><td></td><td>1*</td><td>5</td><td>5</td><td>5*</td><td>8</td><td>9</td><td>9*</td><td>B</td></tr> <tr><td></td><td></td><td></td><td>2*</td><td>6*</td><td>A</td><td>A</td><td>A</td><td>A*</td><td>C</td><td></td></tr> <tr><td></td><td></td><td>3*</td><td>7*</td><td>B*</td><td></td><td></td><td></td><td></td><td></td><td></td></tr> <tr><td></td><td>4*</td><td>8*</td><td>C*</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr> </table>				1*	5	5	5*	8	9	9*	B				2*	6*	A	A	A	A*	C				3*	7*	B*								4*	8*	C*								0
			1*	5	5	5*	8	9	9*	B																																			
			2*	6*	A	A	A	A*	C																																				
		3*	7*	B*																																									
	4*	8*	C*																																										

$$s_{3321}^{(4)} = t^3 s_{54} + t^2 (s_{441} + s_{531}) + t (s_{432} + s_{4311}) + s_{3321}.$$

3. CATALAN FUNCTIONS AS G -EQUIVARIANT EULER CHARACTERISTICS

We first review the geometric description of Catalan functions from [7, 30], and then summarize prior work on these polynomials. This serves to provide context for our results but is not necessary for the remainder of the paper.

Let $G = GL_\ell(\mathbb{C})$, $B \subset G$ the standard lower triangular Borel subgroup, and $H \subset B$ the subgroup of diagonal matrices. The character group of H (integral weights) is identified with \mathbb{Z}^ℓ via the correspondence sending $\gamma \in \mathbb{Z}^\ell$ to the character $H \rightarrow \mathbb{C}^\times$ given by $\text{diag}(z_1, \dots, z_\ell) \mapsto z_1^{\gamma_1} \cdots z_\ell^{\gamma_\ell}$.

The *character* of a G -module M is defined by

$$\text{ch}(M) = \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_\ell) \\ \lambda_1 \geq \dots \geq \lambda_\ell \geq 0}} \dim \text{Hom}(V_\lambda, M) s_\lambda \in \Lambda, \quad (3.1)$$

where V_λ denotes the irreducible G -module of highest weight λ .

Given a B -module N , let $G \times_B N$ denote the homogeneous G -vector bundle on G/B with fiber N above $B \in G/B$, and let $\mathcal{L}_{G/B}(N)$ denote the locally free $\mathcal{O}_{G/B}$ -module of its sections. For $\gamma \in \mathbb{Z}^\ell$, let \mathbb{C}_γ denote the one-dimensional B -module of weight γ . Consider

Broer proved that in the case $\Psi = \Delta^+$, the cohomology in (3.2) vanishes for $i > 0$ if and only if $\gamma_i - \gamma_j \geq -1$ for $i < j$ (see [4, Theorem 2.4] and [3, Proposition 2(iii)]). Broer posed Conjecture 3.3 in the parabolic case $\Psi = \Delta(\eta)$ (see, e.g., [34, Conjecture 5]) and proved it in the subcase γ is a *dominant sequence of rectangles*, meaning that $\gamma = (a_1^{\eta_1}, a_2^{\eta_2}, \dots, a_r^{\eta_r})$ for some $a_1 \geq a_2 \geq \dots \geq a_r$ [5, Theorem 2.2].

Panyushev proved that the cohomology in (3.2) vanishes for $i > 0$ when the weight $\gamma - \rho + \sum_{(i,j) \in \Delta^+ \setminus \Psi} \epsilon_i - \epsilon_j$ is weakly decreasing, where $\rho = (\ell - 1, \ell - 2, \dots, 0)$ (see [30, Theorem 3.2] for the full statement of which this is a special case); this includes the case where γ is a partition with distinct parts and Ψ is any root ideal as well as many instances where γ is not a partition.

A natural combinatorial problem arising here is to find a positive combinatorial formula for the Schur expansion of $H(\Psi; \gamma)$ when γ is a partition. Shimozono-Weyman posed a (still open) conjecture that the Schur expansion of $H(\Delta(\eta); \gamma)$ when γ is a partition can be described using an intricate combinatorial procedure called katabolism [34]. Progress has been made in the case γ is a dominant sequence of rectangles: the Schur expansion was described by Schilling-Warnaar [31] and Shimozono [32] (independently) using a cocyclage poset on Littlewood-Richardson tableaux; by Shimozono [33] using affine Demazure crystals; and by A. N. Kirillov-Schilling-Shimozono [18] using rigged configurations. Chen-Haiman conjectured a generalization of the Shimozono-Weyman katabolism formula to any root ideal Ψ and partition γ [7, Conjecture 5.4.3].

4. CATALAN FUNCTIONS

We elaborate on the definition of the Catalan functions $H(\Psi; \gamma)$ and give another description of these polynomials using Hall-Littlewood vertex operators. We then prove (2.7), (2.8), and Theorem 2.8, along the way establishing some basic facts about Catalan functions which may have further applications beyond this paper.

4.1. Notation. Throughout the paper we use the following notation/conventions: We write $[a, b]$ for the interval $\{i \in \mathbb{Z} \mid a \leq i \leq b\}$ and $[n] := [1, n]$. For $i \in [\ell]$, we write $\epsilon_i \in \mathbb{Z}^\ell$ for the weight with a 1 in position i and 0's elsewhere, and for a set $S \subset [\ell]$, denote $\epsilon_S = \sum_{i \in S} \epsilon_i$. We often omit set braces on singleton sets to avoid clutter.

4.2. Indexed root ideals. We regard the set $\Delta_\ell^+ = \Delta^+ := \{(i, j) \mid 1 \leq i < j \leq \ell\}$ as labels for the set of positive roots of the root system of type $A_{\ell-1}$ (though for brevity we refer to elements of Δ^+ as roots as well). For $\alpha = (i, j) \in \Delta_\ell^+$, we write $\epsilon_\alpha = \epsilon_i - \epsilon_j \in \mathbb{Z}^\ell$ for the corresponding positive root (not to be confused with $\epsilon_{\{i,j\}} = \epsilon_i + \epsilon_j$). As previously mentioned, a *root ideal* is an upper order ideal of the poset Δ^+ with partial order given by $(a, b) \leq (c, d)$ when $a \geq c$ and $b \leq d$. We also work with the complement $\Delta^+ \setminus \Psi$, a lower order ideal of Δ^+ .

An *indexed root ideal of length ℓ* is a pair (Ψ, γ) consisting of a root ideal $\Psi \subset \Delta_\ell^+$ and a weight $\gamma \in \mathbb{Z}^\ell$.

Given an indexed root ideal (Ψ, γ) of length ℓ , we represent the Catalan function $H(\Psi; \gamma)$ by the $\ell \times \ell$ grid of boxes (labeled by matrix-style coordinates as in Figure 2),



$$\Psi = \{(1, 3), (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\} \quad \Delta^+ \setminus \Psi = \{(1, 2), (2, 3), (4, 5)\}$$

Figure 2: For $\ell = 5$, a root ideal Ψ and its complement $\Delta^+ \setminus \Psi$:

with the boxes of Ψ shaded and the entries of γ written along the diagonal. For example, with Ψ as in Figure 2 and $\gamma = 33411$,

$$H(\Psi; \gamma) = \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & 3 & & & \\ \hline & & 4 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array}.$$

This is the most convenient way we have found to compute with these polynomials.

4.3. Catalan functions. Recall from (2.1) that for any $\gamma \in \mathbb{Z}^\ell$, the Schur function s_γ is defined to be $\det(h_{\gamma_i+j-i})$. The s_γ are expressed in the basis of Schur functions indexed by partitions by the following *straightening rule*:

Proposition 4.1 (Schur function straightening). *For any $\gamma \in \mathbb{Z}^\ell$,*

$$s_\gamma(\mathbf{x}) = \begin{cases} \operatorname{sgn}(\gamma + \rho) s_{\operatorname{sort}(\gamma+\rho)-\rho}(\mathbf{x}) & \text{if } \gamma + \rho \text{ has distinct nonnegative parts,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\rho = (\ell - 1, \ell - 2, \dots, 0)$, $\operatorname{sort}(\beta)$ denotes the weakly decreasing sequence obtained by sorting β , and $\operatorname{sgn}(\beta)$ denotes sign of the shortest permutation taking β to $\operatorname{sort}(\beta)$.

Proof. This is a consequence of the Jacobi-Trudi formula $s_\lambda = \det(h_{\lambda_i+j-i})$ for partitions λ and the row interchange property of the determinant. \square

Corollary 4.2. *For a fixed degree $d \in \mathbb{Z}$, the Schur function $s_\gamma(\mathbf{x})$ is nonzero for only finitely many of the weights $\{\gamma \in \mathbb{Z}^\ell : |\gamma| = d\}$.*

Example 4.3. For $\ell = 4$ and $\gamma = (3, 1, 2, 5)$, $s_\gamma = 0$ since $\gamma + \rho = (6, 3, 3, 5)$ does not have distinct parts. For $\gamma = (4, 7, 1, 6)$, $s_{4716} = s_{6552}$ since $\gamma + \rho = (7, 9, 2, 6)$, $\operatorname{sgn}(\gamma + \rho) = +1$, and $\operatorname{sort}(\gamma + \rho) - \rho = (9, 7, 6, 2) - (3, 2, 1, 0) = (6, 5, 5, 2)$.

Let (Ψ, γ) be an indexed root ideal. In (2.2), we defined the Catalan function $H(\Psi; \gamma)$ using raising operators R_{ij} . Raising operators do not give rise to well-defined operators on Λ (for example $R_{12} s_{12} = s_{21} \neq 0$, but $s_{12} = 0$), so they should be thought of as acting on the subscripts γ rather than the s_γ themselves. A precise way to interpret (2.2) is

$$H(\Psi; \gamma)(\mathbf{x}; t) = \tilde{\pi} \left(\prod_{(i,j) \in \Psi} (1 - tz_i/z_j)^{-1} \mathbf{z}^\gamma \right), \tag{4.1}$$

where the map $\tilde{\pi}$ is defined by first letting $\pi : \mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm] \rightarrow \mathbb{Q}[h_1, h_2, \dots]$ be the linear map determined by $\mathbf{z}^\gamma \mapsto s_\gamma$, and then $\tilde{\pi} : (\mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm])[t] \rightarrow (\mathbb{Q}[h_1, h_2, \dots])[t]$ is its natural extension, given by $\sum_{i \geq 0} f_i t^i \mapsto \sum_{i \geq 0} \pi(f_i) t^i$ for any $f_i \in \mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm]$. It follows from Corollary 4.2 that the right side of (4.1) actually lies in the degree $|\gamma|$ part of the ring of symmetric functions $\Lambda = \mathbb{Q}(t)[h_1, h_2, \dots]$.

Remark 4.4. The map π is essentially the Demazure operator corresponding to the longest element of \mathcal{S}_ℓ , though the Demazure operator is typically defined as a map from $\mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm] \rightarrow \mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm]^{S_\ell}$ (see, e.g., [34, §2.1]).

Example 4.5. With $\ell = 4$, $\mu = 3321$, and $\Psi = \{(1, 3), (2, 4), (1, 4)\}$, we have

$$\begin{aligned} H(\Psi; \mu) &= (1 - tR_{13})^{-1}(1 - tR_{24})^{-1}(1 - tR_{14})^{-1}s_{3321} \\ &= s_{3321} + t(s_{3420} + s_{4311} + s_{4320}) + t^2(s_{4410} + s_{5301} + s_{5310}) \\ &\quad + t^3(s_{63-11} + s_{5400} + s_{6300}) + t^4(s_{64-10} + s_{73-10}) \\ &= s_{3321} + t(s_{4320} + s_{4311}) + t^2(s_{4410} + s_{5310}) + t^3s_{5400}. \end{aligned}$$

Proposition 4.1 is used to truncate the series to terms s_α with $\alpha + \rho \in \mathbb{Z}_{\geq 0}^\ell$ for the second equality and it is used again for the third to give $s_{3420} = s_{5301} = s_{64-10} = s_{73-10} = 0$ and $s_{63-11} = -s_{6300}$.

This demonstrates the direct way to obtain the Schur expansion from the definition of Catalan functions. In contrast, Example 2.14 illustrates the Schur expansion of $H(\Psi; \mu) = \mathfrak{s}_{3321}^{(4)}$ obtained via Theorem 2.7; the former involves cancellation whereas the latter is manifestly positive.

4.4. Compositional Hall-Littlewood polynomials. A useful alternative description of the Catalan functions involves Garsia's version [10] of Jing's Hall-Littlewood vertex operators [16]. These are the symmetric function operators defined for any $m \in \mathbb{Z}$ by

$$\mathbf{H}_m = \sum_{i,j \geq 0} (-1)^{i+j} t^j h_{m+i+j}(\mathbf{x}) e_i^\perp h_j^\perp \in \text{End}(\Lambda) \quad (4.2)$$

(see [37, Definition 2.5.2] for this formula for \mathbf{H}_m). Now for $\gamma \in \mathbb{Z}^\ell$, define

$$\mathbf{H}_\gamma = \mathbf{H}_{\gamma_1} \mathbf{H}_{\gamma_2} \cdots \mathbf{H}_{\gamma_\ell} \quad \text{and} \quad H_\gamma(\mathbf{x}; t) = \mathbf{H}_\gamma \cdot 1. \quad (4.3)$$

When μ is a partition, $H_\mu(\mathbf{x}; t)$ is the modified Hall-Littlewood polynomial [10, Theorem 2.1]. For general γ , the $H_\gamma(\mathbf{x}; t)$ are known as *compositional Hall-Littlewood polynomials*. They played a key role in the recent proof [6] of the Shuffle Conjecture [12]. As we now show, any Catalan function can be conveniently expressed in terms of compositional Hall-Littlewood polynomials, making them central to our work as well.

Proposition 4.6. *The compositional Hall-Littlewood polynomials are Catalan functions for the root ideal Δ^+ . That is, for any $\gamma \in \mathbb{Z}^\ell$,*

$$H(\Delta^+; \gamma)(\mathbf{x}; t) = \prod_{(i,j) \in \Delta^+} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x}) = \mathbf{H}_\gamma \cdot 1 = H_\gamma(\mathbf{x}; t).$$

Proof. This follows from Proposition 7 and Remark 2 of [35]. \square

Proposition 4.7. *For any indexed root ideal (Ψ, γ) ,*

$$H(\Psi; \gamma)(\mathbf{x}; t) = \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - t\mathbf{R}_{ij})H_\gamma(\mathbf{x}; t), \quad (4.4)$$

where the raising operator \mathbf{R}_{ij} acts on the subscripts of the H_γ by $\mathbf{R}_{ij}H_\gamma = H_{\gamma + \epsilon_i - \epsilon_j}$.

The right side of (4.4) can be expressed more formally as

$$\Phi \left(\prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - tz_i/z_j) \mathbf{z}^\gamma \right), \quad (4.5)$$

where Φ is the linear map $\mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm][t] \rightarrow \Lambda$ determined by $t^i \mathbf{z}^\gamma \mapsto t^i H_\gamma$.

Proof. It follows from Proposition 4.6 that Φ is equal to the composition $\tilde{\pi} \circ D$, where $D : \mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm][t] \rightarrow \mathbb{Q}[z_1^\pm, \dots, z_\ell^\pm][[t]]$ is the linear map given by left multiplication by $\prod_{(i,j) \in \Delta^+} (1 - tz_i/z_j)^{-1}$. Using the description (4.1) of the Catalan function $H(\Psi; \gamma)$, we have

$$\begin{aligned} H(\Psi; \gamma) &= \tilde{\pi} \left(\prod_{(i,j) \in \Psi} (1 - tz_i/z_j)^{-1} \mathbf{z}^\gamma \right) \\ &= \tilde{\pi} \left(\prod_{(i,j) \in \Delta^+} (1 - tz_i/z_j)^{-1} \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - tz_i/z_j) \mathbf{z}^\gamma \right) \\ &= \tilde{\pi} \circ D \left(\prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - tz_i/z_j) \mathbf{z}^\gamma \right), \end{aligned}$$

which agrees with (4.5). □

The Garsia-Jing operator \mathbf{H}_m at $t = 1$ reduces to multiplication by $h_m(\mathbf{x})$ (this is equivalent to the identity (9.14) arising in a proof later on). Hence for any $\gamma \in \mathbb{Z}^\ell$,

$$H_\gamma(\mathbf{x}; 1) = h_\gamma(\mathbf{x}) = h_{\gamma_1}(\mathbf{x})h_{\gamma_2}(\mathbf{x}) \cdots h_{\gamma_\ell}(\mathbf{x}),$$

where $h_m(\mathbf{x}) = 0$ if $m < 0$. This yields the following explicit description of the Catalan functions at $t = 1$ in terms of homogeneous symmetric functions.

Corollary 4.8. *For any indexed root ideal (Ψ, γ) ,*

$$H(\Psi; \gamma)(\mathbf{x}; 1) = \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - \mathbf{R}_{ij})h_\gamma(\mathbf{x}).$$

Example 4.9. The Catalan function from Example 4.5 at $t = 1$ is computed using Corollary 4.8 as follows: for $\mu = 3321$ and $\Delta^+ \setminus \Psi = \{(1, 2), (2, 3), (3, 4)\}$,

$$\begin{aligned} H(\Psi; \mu)(\mathbf{x}; 1) &= (1 - \mathbf{R}_{12})(1 - \mathbf{R}_{23})(1 - \mathbf{R}_{34})h_{3321} \\ &= h_{3321} - h_{3330} - h_{3411} - h_{4221} + h_{3420} + h_{4230} + h_{4311} - h_{4320} \\ &= h_{3321} - h_{3330} \quad - h_{4221} \quad + h_{4230} \\ &= s_{3321} + s_{4320} + s_{4311} + s_{4410} + s_{5310} + s_{5400}. \end{aligned}$$

4.5. Proof of Property (2.7). We first give relations satisfied by the Garsia-Jing and e_d^\perp operators and then deduce a general result about the action of e_d^\perp on Catalan functions.

Lemma 4.10. *Let $d \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}$, and $\gamma \in \mathbb{Z}^\ell$. Then*

$$e_d^\perp \mathbf{H}_m = \mathbf{H}_m e_d^\perp + \mathbf{H}_{m-1} e_{d-1}^\perp, \quad (4.6)$$

$$e_d^\perp \mathbf{H}_\gamma = \sum_{S \subset [\ell], |S| \leq d} \mathbf{H}_{\gamma - \epsilon_S} e_{d-|S|}^\perp. \quad (4.7)$$

Proof. The first relation is [11, Equation 5.37] and the second follows from the first by a straightforward induction on ℓ . \square

Lemma 4.11. *For $d \in \mathbb{Z}_{\geq 0}$ and (Ψ, γ) any indexed root ideal of length ℓ ,*

$$e_d^\perp (H(\Psi; \gamma)) = \sum_{S \subset [\ell], |S|=d} H(\Psi; \gamma - \epsilon_S). \quad (4.8)$$

Proof. Proposition 4.7 allows us to express any Catalan function in terms of compositional Hall-Littlewood polynomials. The result then follows from Lemma 4.10, (4.3), and the fact that $e_i^\perp(1) = 0$ for $i > 0$. \square

In particular, letting $d = \ell$ in the lemma gives $e_\ell^\perp H(\Psi; \gamma) = H(\Psi; \gamma - 1^\ell)$. In turn, we have $e_\ell^\perp \mathfrak{s}_{\mu+1^\ell}^{(k+1)} = \mathfrak{s}_\mu^{(k)}$ since the root ideals $\Delta^k(\mu)$ and $\Delta^{k+1}(\mu + 1^\ell)$ are equal.

4.6. Proof of Property (2.8). The Catalan functions of length ℓ contain those of length $\ell - 1$ in a natural way.

Proposition 4.12. *For an indexed root ideal (Ψ, γ) of length ℓ with $\gamma_\ell = 0$,*

$$H(\Psi; \gamma) = H(\hat{\Psi}; \hat{\gamma}), \quad \text{where } \hat{\Psi} = \{(i, j) \in \Psi \mid j < \ell\} \text{ and } \hat{\gamma} = (\gamma_1, \dots, \gamma_{\ell-1}).$$

Proof. Using the description of the Catalan functions from Proposition 4.7, we have

$$H(\Psi; \gamma) = \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - t\mathbf{R}_{ij}) H_\gamma = \prod_{(h,\ell) \in \Delta_\ell^+ \setminus \Psi} (1 - t\mathbf{R}_{h\ell}) \prod_{(i,j) \in \Delta_{\ell-1}^+ \setminus \hat{\Psi}} (1 - t\mathbf{R}_{ij}) H_\gamma = H(\hat{\Psi}; \hat{\gamma}).$$

The last equality uses that $\mathbf{H}_0 \cdot 1 = 1$ and $\mathbf{H}_m \cdot 1 = 0$ for $m < 0$ to conclude that for any $\alpha \in \mathbb{Z}^\ell$ with $\alpha_\ell = 0$, we have $\prod_{(h,\ell) \in \Delta_\ell^+ \setminus \Psi} (1 - t\mathbf{R}_{h\ell}) H_\alpha = H_{(\alpha_1, \dots, \alpha_{\ell-1})}$. \square

Property (2.8) now follows easily: let $\mu \in \text{Par}_\ell^k$ with $|\mu| \leq k$. We need to show $\mathfrak{s}_\mu^{(k)} = s_\mu$. Proposition 4.12 allows us to reduce to the case $\mu_\ell > 0$. Then we have $k \geq |\mu| \geq \mu_i + \ell - i$ for all $i \in [\ell]$. Hence the root ideal $\Delta^k(\mu) = \{(i, j) \in \Delta^+ \mid k - \mu_i + i < j\}$ is empty and $\mathfrak{s}_\mu^{(k)} = H(\Delta^k(\mu); \mu) = s_\mu$ follows.

4.7. Proof of Theorem 2.8. Define the *dominance partial order* \supseteq on \mathbb{Z}^ℓ by $\gamma \supseteq \delta$ if $\gamma_1 + \dots + \gamma_i \geq \delta_1 + \dots + \delta_i$ for all $i \in [\ell]$.

Lemma 4.13. *For $\gamma \in \mathbb{Z}^\ell$ and $k = \max(\gamma)$, $H_\gamma \in \text{span}_{\mathbb{Q}(t)} \{H_\lambda \mid \lambda \in \text{Par}_\ell^k \text{ and } \lambda \supseteq \gamma\}$.*

Proof. Since $\mathbf{H}_m \cdot 1 = 0$ for $m < 0$, the result is a consequence of the stronger claim:

$$\mathbf{H}_\gamma \in \text{span}_{\mathbb{Q}(t)} \{ \mathbf{H}_\lambda \mid \lambda \in \mathbb{Z}_{\leq k}^\ell \text{ is weakly decreasing and } \lambda \supseteq \gamma \}.$$

This follows from repeated application of the identity [10, Theorem 2.2]

$$\mathbf{H}_m \mathbf{H}_n = t \mathbf{H}_{m+1} \mathbf{H}_{n-1} + t \mathbf{H}_n \mathbf{H}_m - \mathbf{H}_{n-1} \mathbf{H}_{m+1}. \quad (4.9)$$

with $m < n$, noting that for $n = m + 1$, we must rearrange to obtain $\mathbf{H}_m \mathbf{H}_{m+1} = t \mathbf{H}_{m+1} \mathbf{H}_m$, rather than apply (4.9) directly. \square

The proof of Theorem 2.8 now goes as follows: by Proposition 4.7, for any $\mu \in \text{Par}_\ell^k$,

$$\mathfrak{s}_\mu^{(k)} = \prod_{i=1}^{\ell} \prod_{j=i+1}^{k-\mu_i+i} (1 - t \mathbf{R}_{ij}) H_\mu. \quad (4.10)$$

Consider H_γ arising from such a successive application of raising operators to H_μ . Then γ_i is obtained by adding some amount not exceeding $k - \mu_i$ to μ_i . Hence $\gamma \in \mathbb{Z}_{\leq k}^\ell$, implying $H_\gamma \in \text{span}_{\mathbb{Q}(t)} \{ H_\lambda \mid \lambda \in \text{Par}_\ell^k \text{ and } \lambda \supseteq \gamma \}$ by Lemma 4.13. Since each application of a raising operator strictly increases dominance order, $\gamma \supseteq \mu$. It follows that $\mathfrak{s}_\mu^{(k)} \in \Lambda_\ell^k$ and the transition matrix expressing $\{ \mathfrak{s}_\mu^{(k)} \}_{\mu \in \text{Par}_\ell^k}$ in terms of $\{ H_\lambda \}_{\lambda \in \text{Par}_\ell^k}$ is upper unitriangular with respect to dominance order, implying that the former is a basis for Λ_ℓ^k .

5. RECURRENCES FOR THE CATALAN FUNCTIONS

Computations with Catalan functions are facilitated by recurrences which express a Catalan function as the sum of two Catalan functions with similar indexed root ideals. The bounce graph of a root ideal, defined below, is the natural combinatorial object arising in these computations.

5.1. Bounce graphs. We say that $\alpha \in \Psi$ is a *removable root* of Ψ if $\Psi \setminus \alpha$ is a root ideal and a root $\beta \in \Delta^+ \setminus \Psi$ is *addable to* Ψ if $\Psi \cup \beta$ is a root ideal.

Definition 5.1. Fix a root ideal $\Psi \subset \Delta_\ell^+$ and $x \in [\ell]$. If there is a removable root (x, j) of Ψ , then define $\text{down}_\Psi(x) = j$; otherwise, $\text{down}_\Psi(x)$ is undefined. Similarly, if there is a removable root (i, x) of Ψ , then define $\text{up}_\Psi(x) = i$; otherwise, $\text{up}_\Psi(x)$ is undefined.

Definition 5.2. The *bounce graph* of a root ideal $\Psi \subset \Delta_\ell^+$ is the graph on the vertex set $[\ell]$ with edges $(r, \text{down}_\Psi(r))$ for each $r \in [\ell]$ such that $\text{down}_\Psi(r)$ is defined. The bounce graph of Ψ is a disjoint union of paths called *bounce paths* of Ψ .

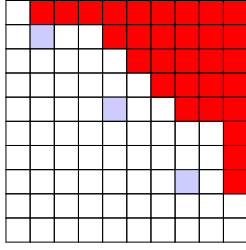
For each vertex $r \in [\ell]$, distinguish $\text{bot}_\Psi(r)$ (resp. $\text{top}_\Psi(r)$) to be the maximum (resp. minimum) element of the bounce path of Ψ containing r . For $a, b \in [\ell]$ in the same bounce path of Ψ with $a \leq b$, we define

$$\text{path}_\Psi(a, b) = (a, \text{down}_\Psi(a), \text{down}_\Psi^2(a), \dots, b),$$

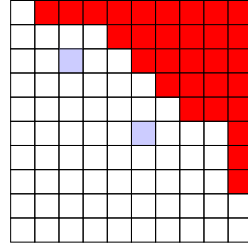
i.e., the list of indices in this path lying between a and b . We also set $\text{downpath}_\Psi(r) = \text{path}(r, \text{bot}_\Psi(r))$ and $\text{uppath}_\Psi(r)$ to be the reverse of $\text{path}(\text{top}_\Psi(r), r)$ for any $r \in [\ell]$. By a slight abuse of notation, we also write $\text{path}_\Psi(a, b)$, $\text{downpath}_\Psi(r)$, and $\text{uppath}_\Psi(r)$ for the corresponding sets of indices. For $b = \text{down}_\Psi^m(a)$, the *bounce* from a to b is

$$B_\Psi(a, b) := |\text{path}_\Psi(a, b)| - 1 = m.$$

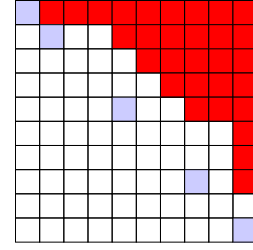
Example 5.3. Examples of downpath, uppath, and bounce for the root ideal Ψ below:



$$\text{path}_{\Psi}(2, 8) = 2, 5, 8$$



$$\text{downpath}_{\Psi}(3) = 3, 6$$



$$\text{uppath}_{\Psi}(10) = 10, 8, 5, 2, 1$$

$$B_{\Psi}(2, 8) = 2, B_{\Psi}(1, 10) = 4, B_{\Psi}(3, 6) = 1, \text{ and } B_{\Psi}(3, 3) = 0.$$

Definition 5.4. A root ideal Ψ is said to have

- a wall in rows $r, r + 1$ if rows r and $r + 1$ of Ψ have the same length,*
- a ceiling in columns $c, c + 1$ if columns c and $c + 1$ of Ψ have the same length, and*
- a mirror in rows $r, r + 1$ if Ψ has removable roots $(r, c), (r + 1, c + 1)$ for some $c > r + 1$.*

Example 5.5. The root ideal Ψ in the previous example has a ceiling in columns 2, 3, in columns 3, 4, and in columns 8, 9, a wall in rows 6, 7, in rows 7, 8, and in rows 9, 10, and a mirror in rows 2, 3, in rows 3, 4, and in rows 4, 5.

5.2. Recurrences for the Catalan functions.

Proposition 5.6. Let (Ψ, μ) be an indexed root ideal. For any root β addable to Ψ ,

$$H(\Psi; \mu) = H(\Psi \cup \beta; \mu) - tH(\Psi \cup \beta; \mu + \varepsilon_{\beta}). \quad (5.1)$$

For any removable root α of Ψ ,

$$H(\Psi; \mu) = H(\Psi \setminus \alpha; \mu) + tH(\Psi; \mu + \varepsilon_{\alpha}). \quad (5.2)$$

Proof. The first identity (5.1) follows directly from Proposition 4.7 :

$$\begin{aligned} H(\Psi; \mu) &= \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - t\mathbf{R}_{ij}) H_{\mu} \\ &= (1 - t\mathbf{R}_{\beta}) \prod_{(i,j) \in \Delta^+ \setminus (\Psi \cup \beta)} (1 - t\mathbf{R}_{ij}) H_{\mu} \\ &= \prod_{(i,j) \in \Delta^+ \setminus (\Psi \cup \beta)} (1 - t\mathbf{R}_{ij}) H_{\mu} - t \prod_{(i,j) \in \Delta^+ \setminus (\Psi \cup \beta)} (1 - t\mathbf{R}_{ij}) H_{\mu + \varepsilon_{\beta}}. \end{aligned}$$

The second identity (5.2) is then obtained by applying (5.1) with $\Psi = \Psi \setminus \alpha$ and $\beta = \alpha$. \square

We also record a convenient application of the recurrence (5.2) obtained by iterating it along a downpath.

Corollary 5.7. Let (Ψ, μ) be an indexed root ideal of length ℓ and $m \in [\ell]$. Then

$$H(\Psi; \mu) = \sum_{z \in \text{downpath}_{\Psi}(m)} t^{B_{\Psi}(m,z)} H(\Psi^z; \mu + \varepsilon_m - \varepsilon_z), \quad (5.3)$$

where $\Psi^z := \Psi \setminus \{(z, \text{down}_{\Psi}(z))\}$ for $z \neq \text{bot}_{\Psi}(m)$ and $\Psi^{\text{bot}_{\Psi}(m)} := \Psi$.

Proof. The proof is by induction on $|\text{downpath}_\Psi(m)|$. The base case $|\text{downpath}_\Psi(m)| = 1$ is clear. Now suppose $|\text{downpath}_\Psi(m)| > 1$ and set $m' = \text{down}_\Psi(m)$. The desired result is obtained by expanding $H(\Psi; \mu)$ on the root (m, m') using the recurrence (5.2) and then applying the inductive hypothesis:

$$\begin{aligned} H(\Psi; \mu) &= H(\Psi^m; \mu) + t H(\Psi; \mu + \epsilon_m - \epsilon_{m'}) \\ &= H(\Psi^m; \mu) + t \sum_{z \in \text{downpath}_\Psi(m')} t^{B_\Psi(m', z)} H(\Psi^z; \mu + \epsilon_m - \epsilon_{m'} + \epsilon_{m'} - \epsilon_z) \\ &= \sum_{z \in \text{downpath}_\Psi(m)} t^{B_\Psi(m, z)} H(\Psi^z; \mu + \epsilon_m - \epsilon_z). \quad \square \end{aligned}$$

6. MIRROR LEMMAS

We first give a natural generalization of Schur function straightening to Catalan functions (Lemma 6.1) and then deduce two *Mirror Lemmas*. The first gives sufficient conditions for a Catalan function to be zero and the second gives sufficient conditions for two Catalan functions to be equal. We further show that these lemmas often “commute” with certain generalizations of the operator e_d^\pm called subset lowering operators.

The symmetric group \mathcal{S}_ℓ acts on the ring $\mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_\ell^{\pm 1}]$ by permuting variables. The simple reflections $\tau_1, \dots, \tau_{\ell-1} \in \mathcal{S}_\ell$ act on the basis of Laurent monomials $\{\mathbf{z}^\gamma\}_{\gamma \in \mathbb{Z}^\ell}$ by $\tau_i \mathbf{z}^\gamma = \mathbf{z}^{\tau_i \gamma}$, where $\tau_i \gamma = (\gamma_1, \dots, \gamma_{i-1}, \gamma_{i+1}, \gamma_i, \gamma_{i+2}, \dots)$. This action extends in the natural way to an action on $\mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_\ell^{\pm 1}][[t]]$. We also consider the action of \mathcal{S}_ℓ on subsets $\Psi \subset [\ell] \times [\ell]$ given by $\tau_i \Psi = \{(\tau_i(a), \tau_i(b)) \mid (a, b) \in \Psi\}$.

Lemma 6.1. *Let $\Psi \subset \Delta_\ell^+$ be a root ideal such that $\tau_i \Psi = \Psi$. Then for any $\gamma \in \mathbb{Z}^\ell$,*

$$H(\Psi; \gamma) + H(\Psi; \epsilon_{i+1} - \epsilon_i + \tau_i \gamma) = 0.$$

Proof. Set $f^\Psi = \prod_{(i,j) \in \Psi} (1 - t z_i / z_j)^{-1}$. Recall from (4.1) that the Catalan functions may be defined in terms of the linear map $\tilde{\pi}$ by $H(\Psi; \gamma) = \tilde{\pi}(f^\Psi \mathbf{z}^\gamma)$. Thus

$$H(\Psi; \gamma) + H(\Psi; \epsilon_{i+1} - \epsilon_i + \tau_i \gamma) = \tilde{\pi}(f^\Psi \mathbf{z}^\gamma + f^\Psi \mathbf{z}^{\epsilon_{i+1} - \epsilon_i + \tau_i \gamma}).$$

We have $\tau_i \Psi = \Psi$ implies $\tau_i f^\Psi = f^\Psi$ (note that $\Psi \subset \Delta^+$ and $\tau_i \Psi = \Psi$ imply $(i, i+1) \notin \Psi$). Hence we obtain

$$\tilde{\pi}(f^\Psi \mathbf{z}^\gamma + f^\Psi \mathbf{z}^{\epsilon_{i+1} - \epsilon_i + \tau_i \gamma}) = \tilde{\pi} \circ (1 + \frac{z_{i+1}}{z_i} \tau_i)(f^\Psi \mathbf{z}^\gamma),$$

where $1 + \frac{z_{i+1}}{z_i} \tau_i$ is regarded as an operator on $\mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_\ell^{\pm 1}][[t]]$. We now claim that the operator $\tilde{\pi} \circ (1 + \frac{z_{i+1}}{z_i} \tau_i) : \mathbb{Q}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_\ell^{\pm 1}][[t]] \rightarrow \mathbb{Q}[h_1, h_2, \dots][[t]]$ is identically 0, which will complete the proof. It suffices to show that $\pi \circ (1 + \frac{z_{i+1}}{z_i} \tau_i)(\mathbf{z}^\delta) = 0$ for any $\delta \in \mathbb{Z}^\ell$, where π is the map used to define $\tilde{\pi}$ (see (4.1)). We have

$$\pi \circ (1 + \frac{z_{i+1}}{z_i} \tau_i)(\mathbf{z}^\delta) = \pi(\mathbf{z}^\delta + \mathbf{z}^{\epsilon_{i+1} - \epsilon_i + \tau_i \delta}) = s_\delta + s_{\epsilon_{i+1} - \epsilon_i + \tau_i \delta} = 0,$$

where the last equality is by the Schur function straightening rule (Proposition 4.1). \square

Lemma 6.2. *Let (Ψ, μ) be an indexed root ideal of length ℓ and $z \in [\ell - 1]$, and suppose*

$$\Psi \text{ has a ceiling in columns } z, z + 1; \quad (6.1)$$

$$\Psi \text{ has a wall in rows } z, z + 1; \quad (6.2)$$

$$\mu_z = \mu_{z+1} - 1. \quad (6.3)$$

Then $H(\Psi; \mu) = 0$.

Proof. Conditions (6.1)–(6.2) are just another way of saying $\tau_z \Psi = \Psi$. By (6.3), $\epsilon_{z+1} - \epsilon_z + \tau_z \mu = \mu$. Hence the result follows from Lemma 6.1. \square

Example 6.3. By Lemma 6.2 with $z = 2$, the following Catalan function is zero:

$$\begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & 1 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} = 0.$$

Lemma 6.4 (Mirror Lemma I). *Let (Ψ, μ) be an indexed root ideal of length ℓ , and let y, z, w be indices in the same bounce path of Ψ with $1 \leq y \leq z \leq w < \ell$, satisfying*

$$\Psi \text{ has a ceiling in columns } y, y + 1; \quad (6.4)$$

$$\Psi \text{ has a mirror in rows } x, x + 1 \text{ for all } x \in \text{path}_\Psi(y, \text{up}_\Psi(w)); \quad (6.5)$$

$$\Psi \text{ has a wall in rows } w, w + 1; \quad (6.6)$$

$$\mu_x = \mu_{x+1} \text{ for all } x \in \text{path}_\Psi(y, w) \setminus \{z\}; \quad (6.7)$$

$$\mu_z = \mu_{z+1} - 1. \quad (6.8)$$

Then $H(\Psi; \mu) = 0$.

Proof. The proof is by induction on $w - y$. The base case $y = w$ is Lemma 6.2. Now assume $y < w$. By (6.5), the root $\beta = (\text{up}_\Psi(w + 1), w)$ is addable to Ψ . So we can expand $H(\Psi; \mu)$ using (5.1) to obtain

$$H(\Psi; \mu) = H(\Psi \cup \beta; \mu) - tH(\Psi \cup \beta; \mu + \epsilon_\beta).$$

The root ideal $\Psi \cup \beta$ has a wall in rows $\text{up}_\Psi(w), \text{up}_\Psi(w) + 1$ and a ceiling in columns $w, w + 1$. Hence, if $z \neq w$, we have $H(\Psi \cup \beta; \mu) = 0$ by the inductive hypothesis and $H(\Psi \cup \beta; \mu + \epsilon_\beta) = 0$ by Lemma 6.2 ((6.3) holds by $(\mu + \epsilon_\beta)_w = (\mu + \epsilon_\beta)_{w+1} - 1$). If $z = w$, then we have $H(\Psi \cup \beta; \mu) = 0$ by Lemma 6.2 and $H(\Psi \cup \beta; \mu + \epsilon_\beta) = 0$ by the inductive hypothesis ((6.8) holds with $\mu + \epsilon_\beta$ in place of μ and $\text{up}_\Psi(w)$ in place of z). \square

Here is another useful variant:

Lemma 6.5 (Mirror Lemma II). *Let (Ψ, μ) be an indexed root ideal of length ℓ , and let y, w be indices in the same bounce path of Ψ with $1 \leq y \leq w < \ell$, satisfying (6.4)–(6.6) and*

$$\mu_x = \mu_{x+1} \text{ for all } x \in \text{path}_\Psi(y, w). \quad (6.9)$$

If Ψ has a removable root α in column y , then $H(\Psi; \mu) = H(\Psi \setminus \alpha; \mu)$. Similarly, if Ψ has a removable root β in row $w + 1$, then $H(\Psi; \mu) = H(\Psi \setminus \beta; \mu)$.

Proof. Apply (5.2) with the removable root α to obtain

$$H(\Psi; \mu) = H(\Psi \setminus \alpha; \mu) + tH(\Psi; \mu + \varepsilon_\alpha) = H(\Psi \setminus \alpha; \mu),$$

where the second equality is by Lemma 6.4 applied with indexed root ideal $(\Psi, \mu + \varepsilon_\alpha)$ and $z = y$ ((6.8) holds since $(\mu + \varepsilon_\alpha)_y = (\mu + \varepsilon_\alpha)_{y+1} - 1$). A similar argument with β in place of α gives $H(\Psi; \mu) = H(\Psi \setminus \beta; \mu)$. \square

Example 6.6. By Lemma 6.5 with $y = 2$, $w = 4$, we have

$$\begin{array}{|c|c|c|c|c|} \hline 3 & \alpha & & & \\ \hline & 2 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 3 & \alpha & & & \\ \hline & 2 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 3 & \alpha & & & \\ \hline & 2 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array}.$$

The vertical dual Pieri rule gives a combinatorial description of $e_d^\perp \mathfrak{s}_\mu^{(k)}$. The proof of this rule requires extending it to a combinatorial description of a more general operator on $\mathfrak{s}_\mu^{(k)}$, which we now define.

Definition 6.7. For $d \in \mathbb{Z}_{\geq 0}$ and $V \subset [\ell]$, the *subset lowering operator* $L_{d,V}$ is given by

$$L_{d,V}H(\Psi; \mu) = \sum_{S \subset V, |S|=d} H(\Psi; \mu - \varepsilon_S),$$

where (Ψ, μ) is any indexed root ideal of length ℓ . We use this notation for the functions $\mathfrak{s}_\mu^{(k)}$ as well: $L_{d,V} \mathfrak{s}_\mu^{(k)}$ means $L_{d,V}H(\Delta^k(\mu); \mu) = \sum_{S \subset V, |S|=d} H(\Delta^k(\mu); \mu - \varepsilon_S)$.

With this notation, Lemma 4.11 says that $L_{d,[\ell]}H(\Psi; \mu) = e_d^\perp H(\Psi; \mu)$ for any $d \geq 0$.

Remark 6.8. Just as for raising operators, the subset lowering operators should be thought of as acting on the input μ in $H(\Psi; \mu)$ rather than on the polynomials themselves. They are not in general well-defined operators on symmetric functions: for instance, $H(\emptyset; 12) = 0$ but $L_{1,\{1\}}H(\emptyset; 12) := H(\emptyset; 02) = -s_{11} \neq 0$. Also see Example 6.11.

Despite the fact that $L_{d,V}$ is not a well-defined operator on symmetric functions, it commutes with raising operators and the recurrences of Proposition 5.6. Moreover, it commutes with the Mirror Lemmas under some mild assumptions. This means that if we have any computation involving Catalan functions that only uses the recurrences and Mirror Lemmas in a controlled way, then we can commute $L_{d,V}$ through this entire computation. This is a powerful technique and is crucial to the proof of the vertical dual Pieri rule.

Proposition 6.9. *Let (Ψ, μ) be an indexed root ideal. For any root β addable to Ψ ,*

$$L_{d,V}H(\Psi; \mu) = L_{d,V}H(\Psi \cup \beta; \mu) - tL_{d,V}H(\Psi \cup \beta; \mu + \varepsilon_\beta). \quad (6.10)$$

For any removable root α of Ψ ,

$$L_{d,V}H(\Psi; \mu) = L_{d,V}H(\Psi \setminus \alpha; \mu) + tL_{d,V}H(\Psi; \mu + \varepsilon_\alpha). \quad (6.11)$$

Proof. This is immediate from the definition of $L_{d,V}$ and Proposition 5.6. \square

Lemma 6.10. *Let (Ψ, μ) be an indexed root ideal of length ℓ , let $z \in [\ell - 1]$, and $V \subset [\ell]$. Suppose this data satisfies (6.1)–(6.3) together with*

$$V \text{ contains both or neither of } z, z + 1. \quad (6.12)$$

Then $L_{d,V}H(\Psi; \mu) = 0$ for any $d \geq 0$.

Proof. By definition,

$$L_{d,V}H(\Psi; \mu) = \sum_{S \subset V, |S|=d} H(\Psi; \mu - \epsilon_S).$$

The terms in the sum such that S contains both or neither of $z, z+1$ are zero by Lemma 6.2. If $V \cap \{z, z+1\} = \emptyset$, then this accounts for all the terms, and we are done. If $\{z, z+1\} \subset V$, then the remaining terms come in pairs:

$$\sum_{S \subset V, |S|=d} H(\Psi; \mu - \epsilon_S) = \sum_{\substack{S' \subset V \setminus \{z, z+1\} \\ |S'|=d-1}} \left(H(\Psi; \mu - \epsilon_{S'} - \epsilon_z) + H(\Psi; \mu - \epsilon_{S'} - \epsilon_{z+1}) \right). \quad (6.13)$$

By Lemma 6.1 with $\gamma = \mu - \epsilon_{S'} - \epsilon_{z+1}$ (using that $t_z \gamma = \gamma$ and $\mu - \epsilon_{S'} - \epsilon_z = \epsilon_{z+1} - \epsilon_z + t_z \gamma$), $H(\Psi; \mu - \epsilon_{S'} - \epsilon_z) + H(\Psi; \mu - \epsilon_{S'} - \epsilon_{z+1}) = 0$. Hence the right side of (6.13) is zero. \square

Example 6.11. By Lemma 6.10 with $z = 2$,

$$L_{1,\{2,3\}} \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 1 & \\ \hline & & 2 \\ \hline \end{array} := \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 0 & \\ \hline & & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} = 0.$$

For comparison here is an example in which (6.12) is not satisfied:

$$L_{1,\{1,2\}} \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 1 & \\ \hline & & 2 \\ \hline \end{array} := \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & 1 & \\ \hline & & 2 \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 0 & \\ \hline & & 2 \\ \hline \end{array} = - \begin{array}{|c|c|c|} \hline 3 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline \end{array} = -s_{311} - ts_{410} \neq 0.$$

With Lemma 6.10 in hand, we easily obtain generalizations of Lemmas 6.4 and 6.5 to the $L_{d,V}H(\Psi; \mu)$.

Lemma 6.12. *Let (Ψ, μ) be an indexed root ideal of length ℓ , let y, z, w be indices in the same bounce path of Ψ with $1 \leq y \leq z \leq w < \ell$, and let $V \subset [\ell]$. Suppose that this data satisfies (6.4)–(6.8) together with*

$$V \text{ contains both or neither of } x, x + 1 \text{ for all } x \in \text{path}_\Psi(y, w). \quad (6.14)$$

Then $L_{d,V}H(\Psi; \mu) = 0$ for any $d \geq 0$.

Proof. Repeat the proof of Lemma 6.4 using Lemma 6.10 in place of Lemma 6.2 and (6.10) in place of (5.1). \square

Lemma 6.13. *Let (Ψ, μ) be an indexed root ideal of length ℓ , let y, w be indices in the same bounce path of Ψ with $1 \leq y \leq w < \ell$, let $d \geq 0$, and let $V \subset [\ell]$ satisfy (6.4)–(6.6), (6.9), and (6.14). If Ψ has a removable root α in column y , then $L_{d,V}H(\Psi; \mu) = L_{d,V}H(\Psi \setminus \alpha; \mu)$. Similarly, if Ψ has a removable root β in row $w + 1$, then $L_{d,V}H(\Psi; \mu) = L_{d,V}H(\Psi \setminus \beta; \mu)$.*

Proof. Repeat the proof of Lemma 6.5 using (6.11) in place of (5.2) and Lemma 6.12 in place of Lemma 6.4. \square

Example 6.14. By Lemma 6.13 with $y = 2$, $w = 4$, for any $d \geq 0$ and $V \subset \{1, 2, 3, 4, 5, 6\}$ such that $V \cap \{2, 3\}$ has size 0 or 2 and $V \cap \{4, 5\}$ has size 0 or 2, there holds

$$L_{d,V} \begin{array}{|c|c|c|c|c|c|} \hline 3 & \alpha & & & & \\ \hline & 2 & & & & \\ \hline & & 2 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array} = L_{d,V} \begin{array}{|c|c|c|c|c|c|} \hline 3 & \alpha & & & & \\ \hline & 2 & & & & \\ \hline & & 2 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array} = L_{d,V} \begin{array}{|c|c|c|c|c|c|} \hline 3 & \alpha & & & & \\ \hline & 2 & & & & \\ \hline & & 2 & & & \\ \hline & & & 1 & & \\ \hline & & & & 1 & \beta \\ \hline & & & & & 1 \\ \hline \end{array}.$$

7. k -SCHUR STRAIGHTENING

One beautiful consequence of identifying k -Schur functions as a subclass of Catalan functions is that we obtain a natural generalization of k -Schur functions to a class of Catalan functions indexed by the set of weights

$$\widetilde{\text{Par}}_\ell^k := \{ \mu \in \mathbb{Z}_{\leq k}^\ell \mid \mu_1 + \ell - 1 \geq \mu_2 + \ell - 2 \geq \dots \geq \mu_\ell \}, \tag{7.1}$$

which contains Par_ℓ^k but many nonpartitions as well. Here we show that these Catalan functions obey a k -Schur straightening rule similar to that of ordinary Schur functions, and this plays a crucial role in the proof of the dual Pieri rules.

7.1. The k -Schur root ideal. The definition of $\mathfrak{s}_\mu^{(k)}$ from (2.4) carries over unchanged to this more general setting, which we record for convenience:

Definition 7.1. For $\mu \in \widetilde{\text{Par}}_\ell^k$, define the root ideal

$$\Delta^k(\mu) = \{ (i, j) \in \Delta_\ell^+ \mid k - \mu_i + i < j \}, \tag{7.2}$$

and the associated Catalan function

$$\mathfrak{s}_\mu^{(k)} = H(\Delta^k(\mu); \mu) = \prod_{i=1}^\ell \prod_{j=k+1-\mu_i+i}^\ell (1 - tR_{ij})^{-1} s_\mu. \tag{7.3}$$

Example 7.2. Here are some examples of the Catalan functions $\mathfrak{s}_\mu^{(k)}$ for $k = 4$:

$$\mathfrak{s}_{3321}^{(4)} = \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline & 3 & & \\ \hline & & 2 & \\ \hline & & & 1 \\ \hline \end{array}, \quad \mathfrak{s}_{4432320}^{(4)} = \begin{array}{|c|c|c|c|c|c|} \hline 4 & & & & & \\ \hline & 4 & & & & \\ \hline & & 3 & & & \\ \hline & & & 2 & & \\ \hline & & & & 3 & \\ \hline & & & & & 2 \\ \hline & & & & & & 0 \\ \hline \end{array}, \quad \mathfrak{s}_{-1012211}^{(4)} = \begin{array}{|c|c|c|c|c|c|} \hline -1 & & & & & \\ \hline & 0 & & & & \\ \hline & & 1 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 1 \\ \hline \end{array}.$$

The first was computed explicitly in Examples 2.14 and 4.5.

The following is essentially a restatement of the definition of Δ^k , which we will reference frequently.

Proposition 7.3. For $\mu \in \widetilde{\text{Par}}_\ell^k$, the western border of the root ideal $\Delta^k(\mu)$ consists of the roots $(i, k + 1 - \mu_i + i)$ for i such that $k + 1 - \mu_i + i \leq \ell$. Moreover, $(i, k + 1 - \mu_i + i)$ is a removable root of $\Delta^k(\mu)$ if and only if $\mu_i \geq \mu_{i+1}$ and $k + 1 - \mu_i + i \leq \ell$.

7.2. k -Schur straightening.

Lemma 7.4 (k -Schur straightening I). *Let $\mu \in \widetilde{\text{Par}}_\ell^k$, $\Psi = \Delta^k(\mu)$, and $z \in [\ell - 1]$. Suppose*

$$y + 1 = \text{up}_\Psi(z + 1) \text{ is defined}; \quad (7.4)$$

$$\mu_z = \mu_{z+1} - 1; \quad (7.5)$$

$$\mu_{z+1} \geq \mu_{z+2} \text{ and } \mu_y \geq \mu_{y+1}. \quad (7.6)$$

Then

$$\mathfrak{s}_\mu^{(k)} = t \mathfrak{s}_{\mu + \epsilon_{y+1} - \epsilon_{z+1}}^{(k)}. \quad (7.7)$$

Proof. Expand $H(\Psi; \mu)$ using (5.2) with the removable root $\delta = (y + 1, z + 1)$ to obtain

$$H(\Psi; \mu) = H(\Psi \setminus \delta; \mu) + t H(\Psi; \mu + \epsilon_\delta). \quad (7.8)$$

Using that $\mu_z = \mu_{z+1} - 1$ and $\mu_y \geq \mu_{y+1}$ with the definition of Δ^k shows that $\Psi \setminus \delta$ has a wall in rows $z, z + 1$ and a ceiling in columns $z, z + 1$. Hence $H(\Psi \setminus \delta; \mu) = 0$ by Lemma 6.2.

The root $\alpha = (y + 1, \text{down}_\Psi(y + 1) - 1) = (y + 1, z)$ is addable to Ψ by the assumption $\mu_y \geq \mu_{y+1}$; also set $B = \{\beta\}$ with $\beta = (z + 1, \text{down}_\Psi(z + 1))$ if this is defined and $B = \emptyset$ otherwise. Lemma 6.5, applied to the indexed root ideal $(\Psi \cup \alpha, \mu + \epsilon_\delta)$, yields

$$H(\Psi; \mu + \epsilon_\delta) = H(\Psi \cup \alpha; \mu + \epsilon_\delta) = H(\Psi \cup \alpha \setminus B; \mu + \epsilon_\delta) = \mathfrak{s}_{\mu + \epsilon_\delta}^{(k)}.$$

For the last equality we are using $\mu_{z+1} \geq \mu_{z+2}$ to conclude that $\text{down}_\Psi(z + 1)$ is defined if and only if the $(z + 1)$ -st row of Ψ is nonempty. \square

Remark 7.5. For conditions such as (7.6) arising in this section and the next, corner cases $z = \ell - 1$ and $y = 0$ are conveniently handled by defining $\mu_0 = k$ and $\mu_{\ell+1} = 0$ for $\mu \in \text{Par}_\ell^k$. The latter is a standard convention, however, we must take care here since the definitions of $\Delta^k(\mu)$ and $\mathfrak{s}_\mu^{(k)} = H(\Delta^k(\mu); \mu)$ depend on ℓ ; we still regard $\Delta^k(\mu)$ as a subset of $[\ell] \times [\ell]$ not $[\ell + 1] \times [\ell + 1]$.

Example 7.6. By Lemma 7.4 with $z = 2, y = 0$, we have

$$\mathfrak{s}_{32321}^{(4)} = \begin{array}{|c|c|c|c|} \hline 3 & & & \\ \hline & 2 & & \\ \hline & & 3 & \\ \hline & & & 2 \\ \hline & & & 1 \\ \hline \end{array} = t \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline & 2 & & \\ \hline & & 2 & \\ \hline & & & 2 \\ \hline & & & 1 \\ \hline \end{array} = t \mathfrak{s}_{42221}^{(4)}.$$

Remark 7.7. It is not difficult to show using Lemma 6.2 that for $\mu \in \widetilde{\text{Par}}_\ell^k$, $\mathfrak{s}_\mu^{(k)} = 0$ if (7.5) holds and $\text{up}_{\Delta^k(\mu)}(z + 1)$ is undefined. Using this in combination with Lemma 7.4, one can show that for any $\mu \in \widetilde{\text{Par}}_\ell^k$, $\mathfrak{s}_\mu^{(k)}$ is equal to 0 or a power of t times $\mathfrak{s}_\nu^{(k)}$ for $\nu \in \text{Par}_\ell^k$. However, we will not need this general statement. Instead, we focus on the special case $\mu = \lambda - \epsilon_z$ for $\lambda \in \text{Par}_\ell^k$, where we can give an explicit combinatorial description of when $\mathfrak{s}_\mu^{(k)}$ is zero and the power of t when it is nonzero.

Lemma 7.8. Let $\mu \in \widetilde{\text{Par}}_\ell^k$ and $\Psi = \Delta^k(\mu)$. Let $h \in \mathbb{Z}_{\geq 1}$ and $z \in [\ell - h]$. Suppose that

$$y + 1 = \text{up}_\Psi(z + 1) \text{ is defined}; \quad (7.9)$$

$$\mu_z = \mu_{z+1} - 1; \quad (7.10)$$

$$\mu_{z+1} = \cdots = \mu_{z+h} \text{ and } \mu_{y+1} = \cdots = \mu_{y+h}; \quad (7.11)$$

$$\mu_{z+h} \geq \mu_{z+h+1} \text{ and } \mu_y \geq \mu_{y+1}. \quad (7.12)$$

Then

$$\mathfrak{s}_\mu^{(k)} = t^h \mathfrak{s}_{\mu + \epsilon_{[y+1, y+h]} - \epsilon_{[z+1, z+h]}}^{(k)}. \quad (7.13)$$

Proof. Set $\mu^i = \mu + \epsilon_{[y+1, y+i]} - \epsilon_{[z+1, z+i]}$ for $i \in [0, h]$. By h applications of Lemma 7.4, we obtain

$$\mathfrak{s}_\mu^{(k)} = t \mathfrak{s}_{\mu^1}^{(k)} = \cdots = t^h \mathfrak{s}_{\mu^h}^{(k)}.$$

We verify that the hypotheses of Lemma 7.4 are satisfied at each step: it follows from Proposition 7.3 that $(y + i + 1, k - \mu_{y+i+1} + y + i + 2) = (y + i + 1, z + i + 1)$ (using (7.11) and $k + 1 - \mu_{y+1} + y + 1 = z + 1$ for the equality) is a removable root of $\Delta^k(\mu^i)$ for each $i \in [0, h - 1]$; this ensures that (7.4) holds for each application of the lemma. The assumptions (7.11)–(7.12) ensure that (7.5)–(7.6) hold for each application of the lemma. \square

The following definition is forced on us by the combinatorics arising in k -Schur straightening. Examples of this definition and Lemma 7.8 are given at the end of the subsection.

Definition 7.9. Let $\lambda \in \text{Par}_\ell^k$ and $z \in [\ell]$. Set $\mu = \lambda - \epsilon_z$ and $\Psi = \Delta^k(\mu)$. Let $c = |\text{uppath}_\Psi(z)|$. If $z = \ell$ or $\lambda_z > \lambda_{z+1}$ or $\text{up}_\Psi^c(z + 1)$ is undefined, then set $h = 0$; otherwise, set $y + 1 = \text{up}_\Psi^c(z + 1)$ and let $h \in [\ell - z]$ be as large as possible such that

$$\begin{aligned} \mu \text{ is constant on each of the intervals } [z+1, z+h], [\text{up}_\Psi(z), \text{up}_\Psi(z)+h], \\ [\text{up}_\Psi^2(z), \text{up}_\Psi^2(z)+h], \dots, [\text{top}_\Psi(z), \text{top}_\Psi(z)+h], \text{ and } [y+1, y+h]. \end{aligned} \quad (7.14)$$

Define

$$\text{cover}_z(\lambda) = \lambda + \epsilon_{[y+1, y+h]} - \epsilon_{[z, z+h]}.$$

If y is undefined or, equivalently, $h = 0$ then $\text{cover}_z(\lambda) = \mu$ (we interpret $[y + 1, y + h] = \emptyset$ in this case). We also define the *bounce* from λ to $\text{cover}_z(\lambda)$ by

$$\text{bounce}(\text{cover}_z(\lambda), \lambda) = h \cdot c. \quad (7.15)$$

For the proof of k -Schur straightening II below and the results of the next subsection, we also define integers c' and h_x as follows: if $\lambda_{z+h} > \lambda_{z+h+1}$, then $c' := -1$; otherwise,

$$c' := \max \{i \in [0, c - 1] \mid \mu_{x+h} = \mu_{x+h+1} \text{ for all } x \in \text{path}_\Psi(\text{up}_\Psi^i(z), \text{up}_\Psi(z))\}. \quad (7.16)$$

And for $x \in \text{uppath}_\Psi(z)$, let

$$h_x = \begin{cases} h + 1 & \text{if } x = \text{up}_\Psi^s(z) \text{ with } s \leq c', \\ h & \text{otherwise.} \end{cases} \quad (7.17)$$

Lemma 7.10. *The following facts clarify Definition 7.9.*

- (i) *If y is defined (equivalently, $h > 0$), then $\mu_y > \mu_{y+1}$.*

- (ii) $\text{cover}_z(\lambda) \in \text{Par}_\ell^k$ if and only if $\lambda_{z+h} > \lambda_{z+h+1}$.
- (iii) If $\text{up}_\Psi^c(z+1)$ is defined and $z < \ell$, then (7.14) holds with $h = 1$ (so there does exist a largest $h \in [\ell - z]$ satisfying (7.14)).

Note that we have used Remark 7.5 to handle corner cases in (i) and (ii).

Proof. Statements (i) and (iii) are straightforward consequences of Proposition 7.3 and (ii) is immediate from (i). \square

Lemma 7.11. *The intervals in (7.14) are pairwise disjoint.*

Proof. It suffices to show that $x + h_x < \text{down}_\Psi(x)$ for all $x \in \text{uppath}_\Psi(z) \setminus \{z\}$ and $y + h < \text{top}_\Psi(z)$. We begin by proving the former (this is stronger than what we need, but we will use this version later). Suppose for a contradiction that this fails; then there is a largest $x \in \text{uppath}_\Psi(z) \setminus \{z\}$ such that $x + h_x \geq \text{down}_\Psi(x)$. By definition of h_x , we have $\mu_x = \cdots = \mu_{\text{down}_\Psi(x)} = \cdots = \mu_{x+h_x}$. We thus cannot have $x = \text{up}_\Psi(z)$ since this would contradict $\mu_{z-1} > \mu_z$. So $x = \text{up}_\Psi^a(z)$ for some $a \geq 2$. We then have

$$h_{\text{down}_\Psi(x)} \geq h_x \geq \text{down}_\Psi(x) - x = k + 1 - \mu_x = k + 1 - \mu_{\text{down}_\Psi(x)} = \text{down}_\Psi^2(x) - \text{down}_\Psi(x),$$

where we have used Proposition 7.3 for the first and third equalities. This contradicts our choice of x .

Now to prove $y + h < \text{top}_\Psi(z)$, suppose for a contradiction that $y + h \geq \text{top}_\Psi(z)$. Then by definition of h , we have $\mu_{y+1} = \cdots = \mu_{\text{top}_\Psi(z)} = \cdots = \mu_{y+h}$. If $z = \text{top}_\Psi(z)$, this contradicts $\mu_{z-1} > \mu_z$ and we are done, so assume $\text{top}_\Psi(z) < z$. We have

$$\begin{aligned} h &\geq \text{top}_\Psi(z) - y = \text{down}_\Psi(y+1) - (y+1) \\ &= k + 1 - \mu_{y+1} = k + 1 - \mu_{\text{top}_\Psi(z)} = \text{down}_\Psi(\text{top}_\Psi(z)) - \text{top}_\Psi(z), \end{aligned}$$

where the first, second, and fourth equalities follow from Proposition 7.3. This contradicts the result of the previous paragraph $x + h_x < \text{down}_\Psi(x)$ for $x = \text{top}_\Psi(z)$. \square

Theorem 7.12 (*k*-Schur straightening II). *Maintain the notation of Definition 7.9. Then*

$$\mathfrak{s}_\mu^{(k)} = t^{hc} \mathfrak{s}_{\text{cover}_z(\lambda)}^{(k)} = t^{\text{bounce}(\text{cover}_z(\lambda), \lambda)} \mathfrak{s}_{\text{cover}_z(\lambda)}^{(k)} \quad (7.18)$$

and this is equal to 0 if $\text{cover}_z(\lambda) \notin \text{Par}_\ell^k$.

Proof. We first prove (7.18). This is trivial if $h = 0$, so assume $h > 0$ for the remainder of this paragraph. Let $\text{path}_\Psi(y+1, z+1) = (b_c + 1, b_{c-1} + 1, \dots, b_0 + 1)$ (thus $b_0 = z$ and $b_c = y$). Set $\mu^i = \mu + \epsilon_{[b_i+1, b_i+h]} - \epsilon_{[z+1, z+h]}$ for $i \in [c]$ (thus $\mu^c = \text{cover}_z(\lambda)$). By c applications of Lemma 7.8, we obtain

$$\mathfrak{s}_\mu^{(k)} = t^h \mathfrak{s}_{\mu^1}^{(k)} = \cdots = t^{hc} \mathfrak{s}_{\mu^c}^{(k)}.$$

The hypotheses of the lemma are satisfied at each step: (7.9)–(7.10) are clear from the definition of h , (7.11) follows from the definition of h together with Lemma 7.11, and (7.12) follows from $\mu + \epsilon_z \in \text{Par}_\ell^k$.

Now suppose $\nu := \text{cover}_z(\lambda) \notin \text{Par}_\ell^k$; this can only happen if $\lambda_{z+h} = \lambda_{z+h+1}$ by Lemma 7.10 (ii). If $z+h = \ell$, then $\lambda_z = \lambda_{z+h} = 0$, and we have $\mathfrak{s}_\nu^{(k)} = 0$ by Proposition 4.7 and the fact that $H_\gamma = 0$ for $\gamma_\ell < 0$. So now assume $z+h < \ell$. Set $\tilde{\Psi} = \Delta^k(\nu)$. We will apply Lemma 6.4 to show $\mathfrak{s}_\nu^{(k)} = 0$.

First consider the case $h > 0$ and define b_i as above. Let c' be as in (7.16). Note that $\tilde{\Psi}$ and Ψ are identical in rows $[y+h+1, z-1]$ and have removable roots in these rows. It then follows from Proposition 7.3 and the definitions of h and c' that

$$\begin{aligned} \text{down}_{\tilde{\Psi}}(b_i+h) &= \text{down}_{\Psi}(b_i+h) = b_{i-1}+h \quad \text{and} \\ \text{down}_{\tilde{\Psi}}(b_i+h+1) &= \text{down}_{\Psi}(b_i+h+1) = b_{i-1}+h+1 \quad \text{for each } i \in [c']. \end{aligned} \quad (7.19)$$

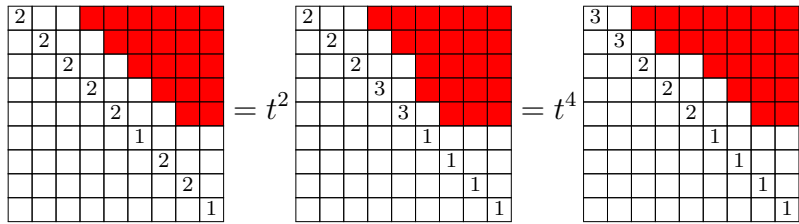
(We need that $b_i+h+1 \leq z-1$, which holds since $\mu_{b_i} = \dots = \mu_{b_i+h+1}$ and $\mu_{z-1} > \mu_z$.) Also by the definitions of h and c' , we have $\mu_{b_{c'+1}+h} > \mu_{b_{c'+1}+h+1}$ (we need to check that $\mu_{b_{c'+1}+h} < \mu_{b_{c'+1}+h+1}$ cannot occur, but this follows from $\mu_{z-1} > \mu_z$ and $\mu_{b_{c'+1}} = \dots = \mu_{b_{c'+1}+h}$). This implies $\nu_{b_{c'+1}+h} > \nu_{b_{c'+1}+h+1}$. It then follows from Proposition 7.3 that

$$\text{down}_{\tilde{\Psi}}(b_{c'+1}+h) \leq \text{down}_{\Psi}(b_{c'+1}+h) = b_{c'}+h \quad \text{and} \quad \text{down}_{\tilde{\Psi}}(b_{c'+1}+h+1) > b_{c'}+h+1.$$

This means that $\tilde{\Psi}$ has a ceiling in columns $b_{c'}+h, b_{c'}+h+1$. Together with (7.19), this shows that the hypotheses of Lemma 6.4 are satisfied for the indexed root ideal $(\tilde{\Psi}, \nu)$ (with y, z, w of the lemma equal to $b_{c'}+h, z+h, z+h$, respectively) and hence $\mathfrak{s}_\nu^{(k)} = 0$.

The case $h = 0$ is similar but easier. Here we define the b_i by $\text{uppath}_\Psi(z) = (b_0, b_1, \dots, b_{c-1})$. The proof in the previous paragraph still works (though some arguments simplify since $\nu = \mu$ and $\tilde{\Psi} = \Psi$) except for the proof that $\tilde{\Psi}$ has a ceiling in columns $b_{c'}+h, b_{c'}+h+1$ in the case $c' = c-1$; but this follows directly from the fact that $\text{up}_\Psi(b_{c'}+h+1)$ is undefined (by Definition 7.9, $\text{up}_\Psi^c(z+1)$ is undefined since $h = 0, \lambda_z = \lambda_{z+1}$, and $z < \ell$). \square

Example 7.13. We illustrate Definition 7.9 and different cases of the proof of Theorem 7.12 with several examples, all with $k = 4$. First, let $\lambda = 222222221, \mu = 222221221, z = 6$. Then $\text{uppath}_{\Delta^k(\mu)}(z) = (6, 3), \text{uppath}_{\Delta^k(\mu)}(z+1) = (7, 4, 1), y+1 = 1, c = 2, h = 2$, and $\text{cover}_z(\lambda) = 332221111$. We illustrate the two applications of Lemma 7.8 used in the proof of Theorem 7.12 to obtain $\mathfrak{s}_{222221221}^{(4)} = t^4 \mathfrak{s}_{332221111}^{(4)}$:



For comparison, here is a similar example where $\mathfrak{s}_\mu^{(k)} = 0$: $\lambda = 222222222, \mu = 222221222, z = 6$. Then $c = 2, h = 2, c' = 0$, and $\text{cover}_z(\lambda) = 332221112$. The

If $\text{cover}_z(\lambda) \in \text{Par}_\ell^k$ we are done. Otherwise, repeat the last three paragraphs of the proof of Theorem 7.12 with Lemma 6.12 in place of Lemma 6.4; Lemma 6.12 requires that V contains all or none of $[x+h, x+h+1]$ for all $x \in \text{path}_{\tilde{\Psi}}(\text{up}_{\tilde{\Psi}}^{\mathcal{C}'}(z), z) = \text{path}_{\Psi}(\text{up}_{\Psi}^{\mathcal{C}'}(z), z)$, where the equality is by (7.19) and \mathcal{C}' and $\tilde{\Psi}$ are as in the proof of Theorem 7.12; this condition on V combined with that of the previous paragraph is equivalent to the assumption on V in the statement. \square

Example 7.17. We continue Example 7.13. For the first example, $\mu = 222221221$, we have $h_x = h = 2$ for all $x \in \text{uppath}_{\Delta^k(\mu)}(z) = (6, 3)$. Hence the hypothesis on V in Theorem 7.16 is that V must contain all or none of the intervals $\{3, 4, 5\}$ and $\{6, 7, 8\}$. For the second example, $\mu = 222221222$, we have $\text{uppath}_{\Delta^k(\mu)}(z) = (6, 3)$, $h = 2$, $h_6 = 3$, $h_3 = 2$. Thus the intervals $[x, x+h_x]$ are $\{3, 4, 5\}$ and $\{6, 7, 8, 9\}$. So, for instance, $V = [8]$ would work for the previous example, but not this one. For the third example, $\mu = 432221222$, we have $\text{uppath}_{\Delta^k(\mu)}(z) = (6, 3)$, $h = 1$, $h_6 = 2$, $h_3 = 2$. Thus the intervals $[x, x+h_x]$ are $\{3, 4, 5\}$ and $\{6, 7, 8\}$.

Theorem 7.16 gives the most general conditions on V for which k -Schur straightening is possible, however we will only need it for $V = [m-1]$ for each $m \in \text{uppath}_{\Psi}(z)$. Theorem 7.16 does indeed apply in this case, as the following result shows.

Lemma 7.18. *Let μ , $\Psi = \Delta^k(\mu)$, and z be as in Definition 7.9, and h_x be as in (7.17). Then for any $m \in \text{uppath}_{\Psi}(z)$, the set $[m-1]$ contains all or none of the interval $[x, x+h_x]$ for all $x \in \text{uppath}_{\Psi}(z)$.*

Proof. This is immediate from the first paragraph of the proof of Lemma 7.11. \square

8. THE BOUNCE GRAPH TO CORE DICTIONARY

Here we connect the combinatorics associated to bounce graphs and k -Schur straightening to strong (marked) covers. In particular, we give an explicit description of strong (marked) covers in terms of the corresponding k -bounded partitions (Propositions 8.10 and 8.12). To make this connection, we use a description of strong covers in terms of edge sequences and offset sequences; we follow [21] for this background.

Throughout this section, fix a positive integer k and let $n = k + 1$.

The *edge sequence* of a partition κ is the bi-infinite binary word $p(\kappa) = p = \dots p_{-1}p_0p_1\dots$ obtained by tracing the border of the diagram of κ from southwest to northeast, such that every letter 1 (resp. 0) represents a north (resp. east) step. We adopt the convention¹ that the meeting point of the edges p_i and p_{i+1} is the southeast corner of a box in diagonal i , where the *diagonal* of a box (r, c) of a partition diagram is $c - r$.

A partition κ is an n -core if and only if each subsequence $\dots p_{i-2n}p_{i-n}p_i p_{i+n}p_{i+2n}\dots$ has the form $\dots 111000\dots$. Thus an n -core κ is specified by recording, for each $i \in \mathbb{Z}$, the integer d_i such that $p_{i+n(d_i-1)} = 1$ and $p_{i+nd_i} = 0$. The sequence $(d_i)_{i \in \mathbb{Z}}$ is the *extended offset sequence* of κ . Note that

$$d_{i-n} = d_i + 1 \quad \text{for all } i \in \mathbb{Z}. \tag{8.1}$$

¹This is off by 1 from the convention in [21], but our extended offset sequences agree.

The affine symmetric group $\widehat{\mathcal{S}}_n$ can be identified with the set of permutations w of \mathbb{Z} such that $w(i+n) = w(i)+n$ for all $i \in \mathbb{Z}$ and $\sum_{i=1}^n (w(i)-i) = 0$ (see [28]). For $r < s$ with $r \not\equiv s \pmod n$, the reflection $t_{r,s} \in \widehat{\mathcal{S}}_n$ is defined by $t_{r,s}(r+jn) = s+jn$, $t_{r,s}(s+jn) = r+jn$ for all $j \in \mathbb{Z}$ and $t_{r,s}(i) = i$ for all $i \in \mathbb{Z}$ such that $r-i, s-i \notin n\mathbb{Z}$. There is a natural action of $\widehat{\mathcal{S}}_n$ on edge sequences and extended offset sequences: the reflection $t_{r,s}$ acts on an edge sequence p by exchanging the bits p_{r+in} and p_{s+in} for all $i \in \mathbb{Z}$, and $t_{r,s}d$ is obtained from d in the same way. This gives an $\widehat{\mathcal{S}}_n$ action on n -cores.

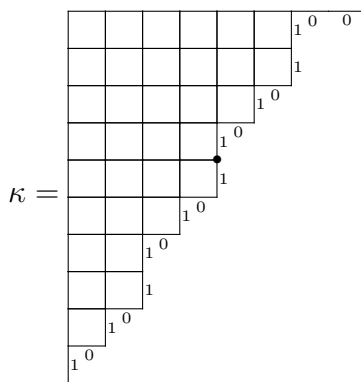
It is worth pointing out (though we will only make use of this implicitly through citations) that there is a bijection between minimal coset representatives $\widehat{\mathcal{S}}_n/\mathcal{S}_n$ and n -cores, compatible with the $\widehat{\mathcal{S}}_n$ actions, given by $w\mathcal{S}_n \mapsto w \cdot \emptyset$, where \emptyset denotes the empty partition. Moreover, this bijection matches strong Bruhat order with the inclusion partial order on n -cores (see Propositions 8.7, 8.8, 8.9, and 9.3 of [21]).

We also need one new definition, not given in the reference [21]. For an n -core κ , the *row map* of κ is the function

$$f : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}, \text{ given by } f(z) = \kappa_z - z + 1. \quad (8.2)$$

With this definition, $f(z) - 1$ is the diagonal of the box (z, κ_z) on the eastern border of κ . By our convention above for the sequence p , this means that $p_{f(z)}$ corresponds to the north step in row z ; in other words, $f(z)$ is equal to the index i such that $p_i = 1$ and $p_i p_{i+1} \cdots$ contains z 1's.

Example 8.1. Let $k = 4$, $n = 5$. Below is the diagram of the n -core $\kappa = 665443221$ with its edge sequence p labeled, where the \bullet separates p_0 and p_1 :



i	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
p_i	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	0
d_i	4	0	3	0	3	3	-1	2	-1	2	2	-2	1	-2	1	1	-3	0

It is convenient to depict the edge sequence in an $\infty \times n$ array with the r -th row equal to the sequence $p_{1+nr} p_{2+nr} \cdots p_{n+nr}$, where the horizontal line divides rows 0 and -1 . Then

the entries $d_1 d_2 \cdots d_n$ record the heights the 1's attain in columns $1, 2, \dots, n$.

$$\begin{array}{cccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \\
 0 & 0 & 0 & 0 & 0 & \\
 1 & 0 & 0 & 0 & 0 & \\
 1 & 0 & 1 & 0 & 1 & \\
 \hline
 1 & 0 & 1 & 0 & 1 & \\
 1 & 0 & 1 & 0 & 1 & \\
 1 & 1 & 1 & 1 & 1 & \\
 \vdots & \vdots & \vdots & \vdots & \vdots &
 \end{array}$$

The row map is given by $f(1), f(2), \dots = 6, 5, 3, 1, 0, -2, -4, -5, -7, -9, -10, -11, \dots$

Our first task in this section is to give the translation between bounce paths and extended offset sequences. We write \mathbf{c} for the inverse of the bijection \mathbf{p} , which is a map from Par^k to $k+1$ -cores.

Proposition 8.2. *Let $\lambda \in \text{Par}_\ell^k$ and $\Phi = \Delta^k(\lambda)$. Let $\kappa = \mathbf{c}(\lambda)$ have edge sequence p , extended offset sequence d , and row map f . Let $r \in \mathbb{Z}_{\geq 1}$ and $z \in [\ell]$. Then*

- (a) $f(r) = f(r + n - \lambda_r) + n$;
- (b) $f(\text{up}_\Phi(z)) = f(z) + n$ whenever $\text{up}_\Phi(z)$ is defined;
- (c) $d_{f(z)} = d_{f(\text{up}_\Phi(z))} + 1$ whenever $\text{up}_\Phi(z)$ is defined;
- (d) $d_{f(z)} > 1 \iff \text{up}_\Phi(z)$ is defined;
- (e) $d_{f(z)} = |\text{uppath}_\Phi(z)|$;
- (f) $n - \lambda_z = p_{f(z)-n+1} + p_{f(z)-n+2} + \cdots + p_{f(z)}$;
- (g) $\lambda_z = \lambda_{z+1} \iff$ there are no 0's in $d_{f(z+1)} d_{f(z+1)+1} \cdots d_{f(z)}$.

Proof. We first prove (a). First suppose $\kappa_r > \lambda_r$ and let $(r, c) = (r, \kappa_r - \lambda_r)$ be the easternmost box in the r -th row of κ having hook length $> n$. For this box to have hook length $> n$, the box $(r + n - \lambda_r, c)$ must lie in κ . By definition of the map \mathbf{p} (see Section 2), the box $(r, c + 1)$ has hook length $< n$ in κ , and thus $(r + n - \lambda_r, c + 1) \notin \kappa$. Hence $(r + n - \lambda_r, c)$ lies on the eastern border of κ and we have

$$f(r) = \kappa_r - r + 1 = c + \lambda_r - r + 1 = c - (r + n - \lambda_r) + 1 + n = f(r + n - \lambda_r) + n.$$

This argument also works in the case $\kappa_r = \lambda_r$ if we consider the column $\{(i, 0) \mid i \in \mathbb{Z}_{\geq 1}\}$ to be part of κ .

Statement (b) follows from (a) by setting $r = \text{up}_\Phi(z)$ and using that $z = \text{down}_\Phi(r) = r + n - \lambda_r$. Statement (c) follows from (b) and (8.1), (e) follows from (c) and (d), (f) follows from (a), and (g) follows from (f).

It remains to prove (d). The \Leftarrow direction is immediate from (c). For the \Rightarrow direction, suppose $d_{f(z)} > 1$. Then $p_{f(z)+n} = 1$ and $f(r) = f(z) + n$ for some $r < z$; hence $z = r + n - \lambda_r$ by (a). Note that $\text{down}_\Phi(r)$ is defined exactly when $r + n - \lambda_r \leq \ell$ (by Proposition 7.3); since $z \leq \ell$ we then have $z = \text{down}_\Phi(r)$ and $\text{up}_\Phi(z) = r$. \square

Recall that a strong cover $\tau \Rightarrow \kappa$ is a pair of n -cores such that $\tau \subset \kappa$ and $|\mathbf{p}(\tau)| + 1 = |\mathbf{p}(\kappa)|$.

Lemma 8.3. *Let κ be an n -core with extended offset sequence d and $t_{r,s} \in \widehat{\mathcal{S}}_n$ a reflection.*

- (i) $\kappa \Rightarrow t_{r,s}\kappa$ if and only if $d_r > d_s$ and for all $r < i < s$, $d_i \notin [d_s, d_r]$.
- (ii) $t_{r,s}\kappa \Rightarrow \kappa$ if and only if $s - n < r$, $d_r < d_s$, and for all $r < i < s$, $d_i \notin [d_r, d_s]$.
- (iii) If $t_{r,s}\kappa \Rightarrow \kappa$, then the skew shape $\kappa/(t_{r,s}\kappa)$ has components $R_{d_r}, \dots, R_{d_s-1}$, where R_j is the ribbon with $s - r$ boxes in diagonals $r + nj, r + 1 + nj, \dots, s - 1 + nj$.
- (iv) For each $j \in [d_r, d_s - 1]$, let z_j be the smallest row index of the ribbon R_j from (iii). Then $f(z_j) = s + nj$ and $\text{uppath}_{\Delta^k(\mathfrak{p}(\kappa))}(z_{d_r}) = (z_{d_r}, z_{d_r+1}, \dots, z_{d_s-1})$.

Proof. Statement (i) is [21, Lemma 9.4 (2)]. Statement (ii) follows from (i) and (8.1). Statement (iii) is essentially [21, Proposition 9.5]; it follows from the interpretation of $t_{r,s}\kappa \Rightarrow \kappa$ in terms of edge sequences. We now prove (iv). By (iii), the northeastmost box of the ribbon R_j is the box in diagonal $s - 1 + nj$ on the eastern border of κ . The row z_j containing this box satisfies $f(z_j) = s + nj$ by the discussion following (8.2). The statement about the uppath then follows from Proposition 8.2 (b) and (e). \square

The next result gives a description of strong covers with the focus on the row indices of the shape κ rather than diagonals. This result as well as Lemmas 8.7 and 8.9 prepare us to prove the main results of this section, which connect strong covers to $\text{cover}_z(\lambda)$ from k -Schur straightening.

Lemma 8.4. *Let κ be an n -core with extended offset sequence d and row map f . Let $z \in [\ell(\kappa)]$ and set $s = f(z)$. Let $r < s$ be as small as possible such that*

$$(d_i > d_s \text{ or } d_i < 0) \text{ for all } i \in [r + 1, s - 1]. \quad (8.3)$$

There exists a strong cover $\tau \Rightarrow \kappa$ such that the southwestmost component of κ/τ has smallest row index z if and only if $(d_r = 0 \text{ and } s - n < r)$. Moreover, this cover is unique if it exists and $\tau = t_{r,s}\kappa$.

Definition 8.5. We define $\text{COVER}_z(\kappa) = \tau$ if the strong cover in Lemma 8.4 exists, and otherwise we say that $\text{COVER}_z(\kappa)$ does not exist.

Proof. For the “only if” direction, suppose $\tau \Rightarrow \kappa$ is a strong cover such that the southwestmost component of κ/τ has smallest row index z . We can write $\tau = t_{r',s'}\kappa$ with $r' < s'$ determined uniquely by requiring $d_{r'} = 0$. Then $s = f(z) = s' + nd_{r'} = s'$ by Lemma 8.3 (iii)–(iv). By Lemma 8.3 (ii), $r = r'$, and hence $d_r = 0$ and $s - n < r$. This also establishes uniqueness.

For the “if” direction, suppose $d_r = 0$ and $s - n < r$. Note that $d_s > 0$ since $s \in \text{Image}(f)$. Then by Lemma 8.3 (ii), $t_{r,s}\kappa \Rightarrow \kappa$. By Lemma 8.3 (iii)–(iv), the southwestmost component of κ/τ has smallest row index z . \square

Example 8.6. For $\kappa = 665443221$ and $z = 6$, we have $\text{COVER}_z(\kappa) = 663331111$. The strong cover $\text{COVER}_z(\kappa) \Rightarrow \kappa$ is the same as that of Example 2.11. The key relevant quantities from Lemma 8.4 are $s = -2$, $r = -6$, and $d_r d_{r+1} \cdots d_s = 0 \ 3 \ 3 \ -1 \ 2$.

Lemma 8.7. *Let $\lambda \in \text{Par}_\ell^k$ and set $\Phi = \Delta^k(\lambda)$. Let $j \in [\ell]$ and $i \in \text{uppath}_\Phi(j)$. Then $|\text{path}_\Phi(i, j)| \leq |\text{uppath}_\Phi(j + 1)|$ if and only if $\lambda_x = \lambda_{x+1}$ for all $x \in \text{path}_\Phi(i, j) \setminus \{j\}$; if these conditions hold, then $\text{path}_\Phi(i + 1, j + 1) = \{x + 1 \mid x \in \text{path}_\Phi(i, j)\}$.*

Proof. Suppose $\lambda_x = \lambda_{x+1}$ for all $x \in \text{path}_\Phi(i, j) \setminus \{j\}$. Then by Proposition 7.3, $\text{down}_\Phi(x) + 1 = \text{down}_\Phi(x + 1)$ for all $x \in \text{path}_\Phi(i, j) \setminus \{j\}$. Hence $\text{downpath}_\Phi(i + 1)$

contains $j + 1$ and $\text{path}_\Phi(i + 1, j + 1) = \{x + 1 \mid x \in \text{path}_\Phi(i, j)\}$ as desired. Since $\text{path}_\Phi(i + 1, j + 1)$ is contained in $\text{uppath}_\Phi(j + 1)$, we have $|\text{path}_\Phi(i, j)| \leq |\text{uppath}_\Phi(j + 1)|$.

For the “only if” direction, suppose $\lambda_x \neq \lambda_{x+1}$ for some $x \in \text{path}_\Phi(i, j) \setminus \{j\}$; let y be the maximum such. By the previous paragraph and Proposition 7.3, $\text{uppath}_\Phi(j + 1) = \{x + 1 \mid x \in \text{path}_\Phi(i, j) \text{ and } x > y\}$. Hence $|\text{path}_\Phi(i, j)| > |\text{uppath}_\Phi(j + 1)|$. \square

Remark 8.8. Let $\lambda \in \text{Par}_\ell^k$ and $\mu = \lambda - \epsilon_z$ ($z \in [\ell]$), and set $\Phi = \Delta^k(\lambda)$ and $\Psi = \Delta^k(\mu)$. We have $\text{uppath}_\Phi(z) = \text{uppath}_\Psi(z)$ and if $|\text{uppath}_\Psi(z + 1)| > 1$, then $\text{uppath}_\Phi(z + 1) = \text{uppath}_\Psi(z + 1)$.

Lemma 8.9. *Maintain the notation of Definition 7.9 and set $\Phi = \Delta^k(\lambda)$. Then $h = \min(k - \lambda_z, h')$, where h' is the largest element of $[0, \ell - z]$ such that*

$$|\text{uppath}_\Phi(z)| < |\text{uppath}_\Phi(z + i)| \text{ and } \lambda_z = \lambda_{z+i} \text{ for all } i \in [h']. \quad (8.4)$$

Proof. One checks directly that $h = 0$ if and only if $h' = 0$. So now assume $h > 0$ and $h' > 0$. One checks using Remark 8.8 and Lemma 8.7 with $j = z + 1$, then $j = z + 2, \dots, j = z + h' - 1$ and $i = \text{up}_\Phi^c(j)$ that h' is the largest element of $[z - \ell]$ such that

$$\begin{aligned} &\lambda \text{ is constant on each of the intervals } [z + 1, z + h'], [\text{up}_\Psi(z), \text{up}_\Psi(z) + h'], \\ &[\text{up}_\Psi^2(z), \text{up}_\Psi^2(z) + h'], \dots, [\text{top}_\Psi(z), \text{top}_\Psi(z) + h'], \text{ and } [y + 1, y + h']. \end{aligned}$$

This is the same as the definition of h except with λ in place of $\mu = \lambda - \epsilon_z$. Hence showing $h = \min(k - \lambda_z, h')$ amounts to checking

$$\min(k - \lambda_z, g(\lambda)) = \min(z - \text{up}_\Psi(z + 1), g(\lambda)) = g(\mu), \quad (8.5)$$

where $g(\nu) := \max\{i \mid \nu \text{ is constant on } [\text{up}_\Psi(z + 1), \text{up}_\Psi(z + 1) + i - 1]\}$ for $\nu \in \mathbb{Z}^\ell$

(the second equality of (8.5) holds by $\mu_{z-1} > \mu_z \implies g(\mu) \leq z - \text{up}_\Psi(z + 1)$). We have

$$k - \lambda_z \geq k - \lambda_{\text{up}_\Psi(z+1)} = k - \mu_{\text{up}_\Psi(z+1)} = z - \text{up}_\Psi(z + 1),$$

with equality if and only if $g(\lambda) > z - \text{up}_\Psi(z + 1)$ (the last equality is by Proposition 7.3). This proves (8.5). \square

The next three results establish a dictionary between constructions on the core side (strong covers, spin) and constructions on the bounce graph side ($\text{cover}_z(\lambda)$, bounce).

Proposition 8.10. *Let $\lambda \in \text{Par}_\ell^k$, $\kappa = \mathbf{c}(\lambda)$, and $z \in [\ell]$. Then*

- (i) $\text{COVER}_z(\kappa)$ exists if and only if $\text{cover}_z(\lambda) \in \text{Par}_\ell^k$;
- (ii) if $\text{cover}_z(\lambda) \in \text{Par}_\ell^k$, then $\text{cover}_z(\lambda) = \mathbf{p}(\text{COVER}_z(\kappa))$.

Proof. If $\lambda_z = 0$, then both conditions in (i) fail. So now assume $\lambda_z > 0$. Maintain the notation of Lemma 8.4 so that $s = f(z)$ and r is defined by (8.3). The edge sequence, extended offset sequence, and row map of κ are denoted p , d , and f , respectively.

We first prove (i). For an index $r' < s$, define $\tilde{h}(r') = p_{r'+1} + p_{r'+2} + \dots + p_{s-1} = |\{i \in [r' + 1, s - 1] : d_i > 0\}|$. One checks using Proposition 8.2 (e) and (g) that $\tilde{h}(r)$ is equal to the h' defined in Lemma 8.9 and

$$d_r = 0 \iff \lambda_{z+h'} > \lambda_{z+h'+1}. \quad (8.6)$$

By Lemma 8.4 and Lemma 7.10 (ii), statement (i) amounts to showing

$$(d_r = 0 \text{ and } s - n < r) \iff \lambda_{z+h} > \lambda_{z+h+1}, \quad (8.7)$$

where h is as in Definition 7.9. We treat the cases $s - n < r$ and $s - n \geq r$ separately. By Proposition 8.2 (f), $k - \lambda_z = \tilde{h}(s - n)$; this together with Lemma 8.9 yields $s - n < r \implies h = h'$, and $s - n \geq r \implies h = k - \lambda_z$. Hence in the case $s - n < r$, (8.6) implies (8.7). In the case $s - n \geq r$, using that $d_{s-n} = d_s + 1 > 0$ and $h = k - \lambda_z = \tilde{h}(s - n)$ we have $f(z + h + 1) = s - n$; by definition of r , there are no 0's in $d_{f(z+h+1)} \cdots d_{f(z+h)}$, and thus $\lambda_{z+h} = \lambda_{z+h+1}$ by Proposition 8.2 (g).

We now prove (ii). By (i), $\tau := \text{COVER}_z(\kappa)$ exists. The border sequence of τ is $q := t_{r,s}p$. Define the sequence \tilde{p} by $\tilde{p}_i = n - \sum_{j \in [i-n+1, i]} p_j$, and define \tilde{q} similarly in terms of q . Since $q = p + \sum_{i \in [0, d_s - 1]} (\epsilon_{r+in} - \epsilon_{s+in})$, we have

$$\tilde{q} = \tilde{p} + \epsilon_{[r+d_s n, s+d_s n - 1]} - \epsilon_{[r, s - 1]}. \quad (8.8)$$

By Proposition 8.2 (f), λ (resp. $\mathbf{p}(\tau)$) is the subsequence of \tilde{p} (resp. \tilde{q}) over those indices j such that $p_j = 1$ (resp. $q_j = 1$). By the definition of r , the sequence \tilde{p} is constant on each interval $[r + in, s + in]$ for $i \in [0, d_s - 1]$. Using this, (8.8), and $q = t_{r,s}p$, one checks that $\mathbf{p}(\tau) = \lambda + \epsilon_I - \epsilon_J$, where $I = f^{-1}([r + d_s n, s + d_s n - 1])$ and $J = f^{-1}([r, s])$. Let y be as in Definition 7.9. We have $I = [y + 1, y + h]$ and $J = [z, z + h]$ by $h = \tilde{h}(r)$ (from the proof of (i)) and Proposition 8.2 (b) and (e). Hence $\mathbf{p}(\tau) = \text{cover}_z(\lambda)$ as desired. \square

Example 8.11. Let $\lambda = 222222221$ and $\kappa = 665443221$. By Examples 7.13 and 8.6,

$$\text{cover}_z(\lambda) = 332221111 = \mathbf{p}(663331111) = \mathbf{p}(\text{COVER}_z(\kappa)),$$

in agreement with Proposition 8.10 (ii).

Proposition 8.12. Fix $\lambda \in \text{Par}_\ell^k$ and set $\Phi = \Delta^k(\lambda)$ and $\kappa = \mathbf{c}(\lambda)$. There is a bijection

$$\{(\nu, z, m) \in \text{Par}_\ell^k \times [\ell] \times [\ell] \mid z \in \text{downpath}_\Phi(m), \nu = \text{cover}_z(\lambda)\} \xrightarrow{\cong} \text{VSMT}_{(1)}^k(\lambda)$$

given by $(\nu, z, m) \mapsto (\mathbf{c}(\nu) \xrightarrow{m} \kappa)$.

Proof. The set $\text{VSMT}_{(1)}^k(\lambda)$ is just another notation for the set of all strong marked covers $\tau \xrightarrow{m} \kappa$, which, by Lemma 8.4, can be written as the union of $\{(\text{COVER}_z(\kappa) \xrightarrow{m} \kappa)\}$ over all $z, m \in [\ell]$ such that $\text{COVER}_z(\kappa)$ exists and m is a possible marking of $\text{COVER}_z(\kappa) \Rightarrow \kappa$. By Lemma 8.3 (iii)–(iv), the possible markings of $\text{COVER}_z(\kappa) \Rightarrow \kappa$ are exactly the elements of $\text{uppath}_\Phi(z)$. The result then follows from Proposition 8.10. \square

Recall the definitions of spin from Section 2 and bounce from Definition 7.9.

Proposition 8.13. The bijection of Proposition 8.12 takes bounce to spin: for $(\text{cover}_z(\lambda), z, m) \mapsto T$, we have

$$\text{bounce}(\text{cover}_z(\lambda), \lambda) + B_\Phi(m, z) = \text{spin}(T).$$

Proof. We have $T = (\mathbf{c}(\text{cover}_z(\lambda)) \xrightarrow{m} \kappa) = (\text{COVER}_z(\kappa) \xrightarrow{m} \kappa)$ (by Proposition 8.10).

Let c and h be as in Definition 7.9. By Lemma 8.3 (iv), the number of components of $\kappa / \text{COVER}_z(\kappa)$ is $|\text{uppath}_\Phi(z)| = c$. By Lemma 8.3 (iii), the height of each component

is the number of 1's (north steps) in $p_r p_{r+1} \cdots p_{s-1} p_s$, where p is the edge sequence of κ and $\text{COVER}_z(\kappa) = t_{r,s} \kappa$ as in Lemma 8.4. This is equal to $\tilde{h}(r) + 1 = h + 1$ by the proof of Proposition 8.10 (i). So $\text{bounce}(\text{cover}_z(\lambda), \lambda)$ is equal to the number of components of $\kappa / \text{COVER}_z(\kappa)$ times one less than the height of each component. Finally, by Lemma 8.3 (iv), the number of components of $\kappa / \text{COVER}_z(\kappa)$ entirely contained in rows $> m$ is equal to $|\text{path}_\Phi(\text{down}_\Phi(m), z)| = B_\Phi(m, z)$. \square

Example 8.14. In Example 2.11 we computed the spins of the possible markings of the strong cover $663331111 = \tau \Rightarrow \kappa = 665443221$ (reproduced on the right in (8.9)–(8.10)). We have $\tau = \text{COVER}_z(\kappa)$ for $z = 6$ (see Example 8.6) and $\lambda = \mathbf{p}(\kappa) = 222222221$. We verify Proposition 8.13 for each $m \in \text{uppath}_{\Delta^k(\lambda)}(z) = \{6, 3\}$:

$$\text{bounce}(\text{cover}_z(\lambda), \lambda) + B_{\Delta^k(\lambda)}(6, z) = 4 + 0 = 4 = \text{spin}(\tau \xrightarrow{6} \kappa), \quad (8.9)$$

$$\text{bounce}(\text{cover}_z(\lambda), \lambda) + B_{\Delta^k(\lambda)}(3, z) = 4 + 1 = 5 = \text{spin}(\tau \xrightarrow{3} \kappa), \quad (8.10)$$

where $\text{bounce}(\text{cover}_z(\lambda), \lambda) = h \cdot c = 2 \cdot 2 = 4$ (by Example 7.13).

9. THE DUAL PIERI RULES

To better understand the dual Pieri rules for k -Schur functions, let us first consider the dual Pieri rule for ordinary Schur functions for $d = 1$. This rule can be stated in the following elegant way:

$$e_1^\perp s_\lambda = \sum_{z=1}^{\ell} s_{\lambda - \epsilon_z}.$$

This is the same as the sum of Schur functions indexed by partitions obtained by removing a corner from λ since $s_{\lambda - \epsilon_z}$ is nonzero if and only if (z, λ_z) is a removable corner. We have the following analogous formula for the $\mathfrak{S}_\lambda^{(k)}$ (by summing (9.3) below over $m \in [\ell]$):

$$e_1^\perp \mathfrak{S}_\lambda^{(k)} = \sum_{m \in [\ell]} \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m, z)} \mathfrak{S}_{\lambda - \epsilon_z}^{(k)}; \quad (9.1)$$

the (vertical) dual Pieri rule for $d = 1$ is then obtained by evaluating the right side using k -Schur straightening II.

To prove the vertical dual Pieri rule for general d , we need a more refined version to induct on d . This refined statement involves the subset lowering operators, recalled below. Somewhat miraculously, the combinatorics of bounce graphs and k -Schur straightening matches exactly the combinatorics of strong covers—we obtain an algebraic meaning for the sum over strong covers with a fixed marking; see Theorem 9.2. This is the refined statement we need for $d = 1$. The general case is then handled in Theorem 9.3. (The former is a special case of the latter so is unnecessary but we include it as an instructive warmup.)

We will need the following variants of the notation from Definition 6.7 for the subset lowering operators: for $d \geq 0$, $m \in [\ell]$, and an indexed root ideal (Φ, λ) of length ℓ , define

$$\begin{aligned} L_{d,m}H(\Phi; \lambda) &:= L_{d,[m]}H(\Phi; \lambda) = \sum_{S \subset [m], |S|=d} H(\Phi; \lambda - \epsilon_S); \\ \tilde{L}_{d,m}H(\Phi; \lambda) &:= L_{d,m}H(\Phi; \lambda) - L_{d,m-1}H(\Phi; \lambda) = \sum_{S \subset [\ell], |S|=d, \max(S)=m} H(\Phi; \lambda - \epsilon_S). \end{aligned} \quad (9.2)$$

By Lemma 4.11, $L_{d,\ell}H(\Phi; \lambda) = e_d^\perp H(\Phi; \lambda)$ for any $d \geq 0$, and $e_d^\perp H(\Phi; \lambda) = 0$ when $d > \ell$. Note that $L_{d,m}H(\Phi; \lambda) = \tilde{L}_{d,m}H(\Phi; \lambda) = 0$ when $d > m$, the natural generalization of this latter fact.

Proposition 9.1. *Let $\lambda \in \text{Par}_\ell^k$, $\Phi = \Delta^k(\lambda)$, and $m \in [\ell]$. Then*

$$\tilde{L}_{1,m}\mathfrak{S}_\lambda^{(k)} = H(\Phi; \lambda - \epsilon_m) = \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} \mathfrak{S}_{\lambda - \epsilon_z}^{(k)}. \quad (9.3)$$

Proof. We have

$$H(\Phi; \lambda - \epsilon_m) = \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} H(\Psi^z; \lambda - \epsilon_z) = \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} \mathfrak{S}_{\lambda - \epsilon_z}^{(k)},$$

where $\Psi^z := \Phi \setminus \{(z, \text{down}_\Phi(z))\}$ for $z \neq \text{bot}_\Phi(m)$ and $\Psi^z := \Phi$ for $z = \text{bot}_\Phi(m)$. The first equality is by Corollary 5.7 and the second is by $\Delta^k(\lambda - \epsilon_z) = \Psi^z$. \square

For a (vertical) strong marked tableau $T = (\kappa^{(0)} \xrightarrow{r_1} \kappa^{(1)} \xrightarrow{r_2} \dots \xrightarrow{r_m} \kappa^{(m)})$, define $\text{marks}(T)$ to be the set $\{r_1, \dots, r_m\}$ of markings that appear in T .

Theorem 9.2. *For any $\lambda \in \text{Par}_\ell^k$ and $m \in [\ell]$,*

$$\tilde{L}_{1,m}\mathfrak{S}_\lambda^{(k)} = \sum_{T \in \text{VSMT}_{(1)}^k(\lambda), \text{marks}(T)=\{m\}} t^{\text{spin}(T)} \mathfrak{S}_{\text{inside}(T)}^{(k)}. \quad (9.4)$$

Proof. Set $\Phi = \Delta^k(\lambda)$. Beginning with Proposition 9.1, we obtain

$$\tilde{L}_{1,m}\mathfrak{S}_\lambda^{(k)} = \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} \mathfrak{S}_{\lambda - \epsilon_z}^{(k)} \quad (9.5)$$

$$= \sum_{\{z \in \text{downpath}_\Phi(m) \mid \text{cover}_z(\lambda) \in \text{Par}_\ell^k\}} t^{B_\Phi(m,z) + \text{bounce}(\text{cover}_z(\lambda), \lambda)} \mathfrak{S}_{\text{cover}_z(\lambda)}^{(k)} \quad (9.6)$$

$$= \sum_{T \in \text{VSMT}_{(1)}^k(\lambda), \text{marks}(T)=\{m\}} t^{\text{spin}(T)} \mathfrak{S}_{\text{inside}(T)}^{(k)}, \quad (9.7)$$

where the second equality is by Theorem 7.12 and the third equality is by Propositions 8.12 and 8.13. \square

9.1. The vertical dual Pieri rule.

Theorem 9.3. *For any $\lambda \in \text{Par}_\ell^k$ and integers $d \geq 0$, $m \in [\ell]$, we have*

$$L_{d,m}\mathfrak{S}_\lambda^{(k)} = \sum_{T \in \text{VSMT}_{(d)}^k(\lambda), \text{marks}(T) \subset [m]} t^{\text{spin}(T)} \mathfrak{S}_{\text{inside}(T)}^{(k)}; \quad (9.8)$$

$$\tilde{L}_{d,m}\mathfrak{S}_\lambda^{(k)} = \sum_{T \in \text{VSMT}_{(d)}^k(\lambda), \max(\text{marks}(T))=m} t^{\text{spin}(T)} \mathfrak{S}_{\text{inside}(T)}^{(k)}. \quad (9.9)$$

In the special case $m = \ell$, the condition $\text{marks}(T) \subset [m]$ in (9.8) is no restriction at all. Hence this proves Property (2.6).

Proof. Since $\tilde{L}_{d,m}\mathfrak{S}_\lambda^{(k)} = L_{d,m}\mathfrak{S}_\lambda^{(k)} - L_{d,m-1}\mathfrak{S}_\lambda^{(k)}$, statements (9.8) and (9.9) are equivalent. We will prove them simultaneously by induction on d . Specifically, we will prove (9.9) using (9.8) for $d-1, m-1$. The base cases $d=0$ and $m < d$ are trivial. So now assume $0 < d \leq m$.

Set $\Phi = \Delta^k(\lambda)$. By definition (9.2), $\tilde{L}_{d,m}\mathfrak{S}_\lambda^{(k)} = \sum_{S \subset [\ell], |S|=d, \max(S)=m} H(\Phi; \lambda - \epsilon_S)$. For each term $H(\Phi; \lambda - \epsilon_S)$ in this sum, expand on $\text{downpath}_\Phi(m)$ using Corollary 5.7 to obtain

$$H(\Phi; \lambda - \epsilon_S) = \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} H(\Psi^z; \lambda - \epsilon_z - \epsilon_{S'}), \quad (9.10)$$

where $\Psi^z := \Phi \setminus \{(z, \text{down}_\Phi(z))\}$ for $z \neq \text{bot}_\Phi(m)$ and $\Psi^z := \Phi$ for $z = \text{bot}_\Phi(m)$, and $S' := S \setminus \{m\}$.

Summing (9.10) over all $S \subset [\ell]$ such that $|S|=d$ and $\max(S)=m$, we obtain

$$\begin{aligned} \tilde{L}_{d,m}\mathfrak{S}_\lambda^{(k)} &= \sum_{S' \subset [m-1], |S'|=d-1} \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} H(\Psi^z; \lambda - \epsilon_z - \epsilon_{S'}) \\ &= \sum_{z \in \text{downpath}_\Phi(m)} t^{B_\Phi(m,z)} L_{d-1,m-1} H(\Psi^z; \lambda - \epsilon_z) \\ &= \sum_{\{z \in \text{downpath}_\Phi(m) \mid \text{cover}_z(\lambda) \in \text{Par}_\ell^k\}} t^{B_\Phi(m,z) + \text{bounce}(\text{cover}_z(\lambda), \lambda)} L_{d-1,m-1} \mathfrak{S}_{\text{cover}_z(\lambda)}^{(k)} \\ &= \sum_{V \in \text{VSMT}_{(1)}^k(\lambda), \text{marks}(V)=\{m\}} t^{\text{spin}(V)} L_{d-1,m-1} \mathfrak{S}_{\text{inside}(V)}^{(k)} \\ &= \sum_{\substack{V \in \text{VSMT}_{(1)}^k(\lambda) \\ \text{marks}(V)=\{m\}}} \sum_{\substack{U \in \text{VSMT}_{(d-1)}^k(\text{inside}(V)) \\ \text{marks}(U) \subset [m-1]}} t^{\text{spin}(V) + \text{spin}(U)} \mathfrak{S}_{\text{inside}(U)}^{(k)} \\ &= \sum_{T \in \text{VSMT}_{(d)}^k(\lambda), \max(\text{marks}(T))=m} t^{\text{spin}(T)} \mathfrak{S}_{\text{inside}(T)}^{(k)}. \end{aligned}$$

We need to justify the last four equalities. The third equality is the delicate step where we apply Theorem 7.16 to each term $L_{d-1,m-1} H(\Psi^z; \lambda - \epsilon_z) = L_{d-1,m-1} \mathfrak{S}_{\lambda - \epsilon_z}^{(k)}$; the hypotheses of the theorem are satisfied by Lemma 7.18 and Remark 8.8. The fourth equality is by Propositions 8.12 and 8.13 (just as in the proof of Theorem 9.2). The fifth equality is by

the inductive hypothesis, and the last equality holds since a vertical strong marked tableau T of weight (d) with $\max(\text{marks}(T)) = m$ is the same as a vertical strong marked tableau V of weight (1) with $\text{marks}(V) = \{m\}$ followed by a vertical strong marked tableau U of weight $(d-1)$ such that $\text{marks}(U) \subset [m-1]$ and $\text{inside}(V) = \text{outside}(U)$. \square

Example 9.4. Let $k = 3$, $d = 2$, and $m = 4$. According to Theorem 9.3, $L_{d,m}\mathfrak{s}_{222222}^{(k)}$ equals the sum of $t^{\text{spin}(T)}\mathfrak{s}_{\text{inside}(T)}^{(k)}$ over strong marked tableaux $T \in \text{VSMT}_{(d)}^k(222222)$ such that $\text{marks}(T) \subset [m]$, as shown:

T					
$k\text{-skew}(\text{inside}(T))$					
$\text{spin}(T)$	2	3	4	4	3
$\text{inside}(T)$	222211	222211	222211	322111	222220
	$L_{2,4}\mathfrak{s}_{222222}^{(3)} = t^2\mathfrak{s}_{222211}^{(3)} + t^3\mathfrak{s}_{222211}^{(3)} + t^4\mathfrak{s}_{222211}^{(3)} + t^4\mathfrak{s}_{322111}^{(3)} + t^3\mathfrak{s}_{222220}^{(3)}$				

Remark 9.5. Theorem 9.3 is somewhat delicate. By definition,

$$L_{d,m}\mathfrak{s}_{\lambda}^{(k)} = \sum_{S \subset V, |S|=d} H(\Delta^k(\lambda); \lambda - \epsilon_S).$$

However, each Catalan function $H(\Delta^k(\lambda); \lambda - \epsilon_S)$ in the sum is not necessarily equal to $\sum_{T \in \text{VSMT}_{(d)}^k(\lambda), \text{marks}(T)=S} t^{\text{spin}(T)}\mathfrak{s}_{\text{inside}(T)}^{(k)}$. In fact, the Catalan functions $H(\Delta^k(\lambda); \lambda - \epsilon_S)$ are not always Schur positive, as can be seen from a more detailed study of the previous example. Setting $\Phi = \Delta^3(222222)$, we have

$$\begin{aligned} L_{2,4}\mathfrak{s}_{222222}^{(3)} &= H(\Phi; 112222) + H(\Phi; 121222) + H(\Phi; 122122) \\ &\quad + H(\Phi; 211222) + H(\Phi; 212122) + H(\Phi; 221122), \end{aligned}$$

and these terms have the following expansions into 3-Schur functions:

$$\begin{aligned} H(\Phi; 112222) &= t^4\mathfrak{s}_{222211}^{(3)}, & H(\Phi; 121222) &= -t^3\mathfrak{s}_{222211}^{(3)}, & H(\Phi; 122122) &= t^3\mathfrak{s}_{222211}^{(3)}, \\ H(\Phi; 211222) &= t^3\mathfrak{s}_{222211}^{(3)}, & H(\Phi; 212122) &= t^4\mathfrak{s}_{322111}^{(3)} + t^3\mathfrak{s}_{222220}^{(3)}, & H(\Phi; 221122) &= t^2\mathfrak{s}_{222211}^{(3)}. \end{aligned}$$

Summing these, we recover the positive expression in Example 9.4. The proof of Theorem 9.3 implicitly handles cancellation of this form to produce the positive sum in (9.8).

9.2. The horizontal dual Pieri rule. We prove this rule (Property (2.5)) using the vertical dual Pieri rule and a trick involving symmetric functions in noncommuting variables, inspired by [9, 2].

Fix positive integers ℓ and k . Recall that by Theorem 2.8, $\{\mathfrak{s}_\mu^{(k)} \mid \mu \in \text{Par}_\ell^k\}$ is a basis for $\Lambda_\ell^k \subset \Lambda$. We define the *strong marked cover operators* $u_1, \dots, u_\ell \in \text{End}(\Lambda_\ell^k)$ by

$$\mathfrak{s}_\mu^{(k)} \cdot u_r = \sum_{T \in \text{SMT}_{(1)}^k(\mu), \text{marks}(T)=\{r\}} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(k)}. \quad (9.11)$$

We have found it more natural here to work with right operators, so for this subsection we consider all operators to act on the right. Note that the set of T in the sum is just another notation for the set of strong marked covers $\tau \xrightarrow{r} \kappa$ with $\mathfrak{p}(\kappa) = \mu$.

For $d \geq 0$, define the following versions of the elementary and homogeneous symmetric functions in the (noncommuting) operators u_i :

$$\tilde{e}_d = \sum_{\ell \geq i_1 > i_2 > \dots > i_d \geq 1} u_{i_1} u_{i_2} \cdots u_{i_d}, \quad (9.12)$$

$$\tilde{h}_d = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_d \leq \ell} u_{i_1} u_{i_2} \cdots u_{i_d}; \quad (9.13)$$

by convention, $\tilde{e}_0 = \tilde{h}_0 = 1$ and $\tilde{e}_d = 0$ for $d > \ell$.

Proof of Property (2.5). First, we have the basic identity

$$h_m^\perp - h_{m-1}^\perp e_1^\perp + h_{m-2}^\perp e_2^\perp - \dots + (-1)^m e_m^\perp = 0 \quad \text{for } m > 0. \quad (9.14)$$

This follows directly from the well-known identity $\sum_{i=0}^m (-1)^i e_i h_{m-i} = 0$ [36, Equation 7.13]. Hence the operators $h_1^\perp, h_2^\perp, \dots$ can be recursively expressed in terms of $e_1^\perp, e_2^\perp, \dots$.

The operators \tilde{h}_d and \tilde{e}_d were cooked up so that $\mathfrak{s}_\mu^{(k)} \cdot \tilde{h}_d$ (resp. $\mathfrak{s}_\mu^{(k)} \cdot \tilde{e}_d$) equals the right side of (2.5) (resp. (2.6)). Hence by Property (2.6), e_d^\perp restricts to an operator on Λ_ℓ^k which agrees with \tilde{e}_d . Then by the previous paragraph, h_d^\perp also restricts to an operator on Λ_ℓ^k and Property (2.5) is equivalent to the identity $h_d^\perp = \tilde{h}_d$ of operators on Λ_ℓ^k . It now suffices to show that (9.14) holds with \tilde{e}_d and \tilde{h}_d in place of e_d^\perp and h_d^\perp .

Let y be a formal variable that commutes with u_1, \dots, u_ℓ . The elements \tilde{e}_d and \tilde{h}_d can be packaged into the generating functions

$$\begin{aligned} \tilde{E}(y) &= \sum_{d=0}^{\ell} y^d \tilde{e}_d = (1 + yu_\ell) \cdots (1 + yu_1) \in \text{End}(\Lambda_\ell^k)[y], \\ \tilde{H}(y) &= \sum_{d=0}^{\infty} y^d \tilde{h}_d = (1 - yu_1)^{-1} \cdots (1 - yu_\ell)^{-1} \in \text{End}(\Lambda_\ell^k)[[y]]. \end{aligned}$$

The identity $\tilde{H}(y)\tilde{E}(-y) = 1$ implies that for any $m > 0$,

$$\tilde{h}_m - \tilde{h}_{m-1} \tilde{e}_1 + \tilde{h}_{m-2} \tilde{e}_2 - \dots + (-1)^m \tilde{e}_m = 0, \quad (9.15)$$

as desired. \square

10. THE CHEN-HAIMAN ROOT IDEAL FOR SKEW-LINKING DIAGRAMS

Consider a skew partition κ/η and denote its rows by $\lambda = (\kappa_1 - \eta_1, \dots, \kappa_\ell - \eta_\ell)$ and the rows of its transpose $(\kappa/\eta)'$ by μ . If λ and μ are both partitions, we say κ/η is a *skew-linking diagram* and write $\lambda \xrightarrow{\kappa/\eta} \mu$. Any k -skew diagram (defined in §2.2) is a skew-linking diagram; this follows from the fact that the transpose of a $k+1$ -core is again a $k+1$ -core and the existence of the bijection \mathfrak{p} .

Definition 10.1 ([7, §5.3]). The root ideal associated to a skew-linking diagram $\lambda \xrightarrow{\kappa/\eta} \mu$ is given by

$$\Phi(\kappa/\eta) = \{(i, j) \in \Delta_{\ell(\kappa)}^+ \mid i \leq \ell(\eta) \text{ and } j \geq \mu_{\eta_i} + i\}.$$

Equivalently, $\Phi(\kappa/\eta)$ is the root ideal with removable roots $\{(i, \mu_{\eta_i} + i) \mid i \leq \ell(\eta)\}$.

See Example 10.5.

Chen and Haiman conjectured that the Catalan functions include the k -Schur functions as a subclass but used the root ideal $\Phi(k\text{-skew}(\lambda))$ rather than our $\Delta^k(\lambda)$. We reconcile this difference in Theorem 10.4 below. This conjecture fits in the broader context of a study of Catalan functions associated to skew-linking diagrams. We briefly mention two conjectures of Chen-Haiman arising in this study: for any skew-linking diagram $\lambda \xrightarrow{\kappa/\eta} \mu$, the Catalan function $H(\Phi(\kappa, \eta); \lambda)$ (1) is the graded character of an explicitly constructed module for the symmetric group (see [7, Corollary 5.4.6]), and (2) is equal to the Schur positive sum $\sum t^{\text{charge}(T)} s_{\text{shape}(T)}$ over semistandard tableaux T satisfying certain katabolizability conditions [7, Conjecture 5.4.3].

Lemma 10.2. *Let $\lambda \xrightarrow{\kappa/\eta} \mu$ be a skew-linking diagram. If $\kappa_y = \kappa_{y+1}$, then $\eta_y = \eta_{y+1}$ and $\lambda_y = \lambda_{y+1}$. If $\eta_y = \eta_{y+1} > 0$, then $\kappa_{\mu_{\eta_y} + y} = \kappa_{\mu_{\eta_y} + y + 1}$.*

Proof. These facts are direct consequences of the definition of a skew-linking diagram. The first uses that λ is a partition and the second that μ is. \square

Lemma 10.3. *Let $\lambda \xrightarrow{\kappa/\eta} \mu$ be a skew-linking diagram and Ψ a root ideal which agrees with $\Phi(\kappa/\eta)$ in rows $\geq y$. Suppose that Ψ has a removable root of the form $\alpha = (a, y)$, Ψ has a ceiling in columns $y, y+1$, and $\kappa_y = \kappa_{y+1}$. Then $H(\Psi; \lambda) = H(\Psi \setminus \alpha; \lambda)$.*

Proof. Apply Lemma 6.5 with indexed root ideal (Ψ, λ) and $w = \text{bot}(y)$; we need to verify the hypotheses (6.4)–(6.6), (6.9), and $w < \ell(\kappa)$. The first, (6.4), holds by assumption. Next, by Lemma 10.2, $\kappa_y = \kappa_{y+1}$ implies $\lambda_y = \lambda_{y+1}$ and $\eta_y = \eta_{y+1}$. By definition of $\Phi(\kappa/\eta)$, if $\eta_y > 0$, then $\text{down}_\Psi(y) = \mu_{\eta_y} + y$ and thus $\kappa_{\text{down}_\Psi(y)} = \kappa_{\text{down}_\Psi(y)+1}$ by Lemma 10.2. Iterating this argument gives $\kappa_x = \kappa_{x+1}$, $\lambda_x = \lambda_{x+1}$, and $\eta_x = \eta_{x+1}$ for all $x \in \text{downpath}_\Psi(y)$, which verifies (6.5), (6.9), and also $w < \ell(\kappa)$ (since $\kappa_w = \kappa_{w+1} > 0$). Lastly, since Ψ has a removable root in row i for $z \leq i \leq \ell(\eta)$ and no roots in rows $> \ell(\eta)$, we have $w > \ell(\eta)$ and Ψ has a wall in rows $w, w+1$, i.e., (6.6) holds. \square

Theorem 10.4. *Let $\lambda \in \text{Par}^k$ and κ/η be the k -skew diagram of λ . Then*

$$\mathfrak{s}_\lambda^{(k)} = H(\Delta^k(\lambda); \lambda) = H(\Phi(\kappa/\eta); \lambda).$$

It is most natural to regard $(\Delta^k(\lambda), \lambda)$ as an indexed root ideal of length $\ell(\lambda) = \ell(\kappa)$, and we do this in the proof below, however this is not strictly necessary by Proposition 4.12.

Proof. Let μ denote the rows of $(\kappa/\eta)'$. Let

$$\Psi^s = \{(i, j) \in \Delta^k(\lambda) \mid i < s\} \sqcup \{(i, j) \in \Phi(\kappa/\eta) \mid i \geq s\}.$$

We have $\Psi^1 = \Phi(\kappa/\eta)$ and $\Psi^{\ell(\eta)+1} = \Delta^k(\lambda)$ since both Ψ and $\Delta^k(\lambda)$ have no roots (i, j) for $i > \ell(\eta)$; namely, when $i > \ell(\eta)$, κ/η is a k -skew diagram implies that $k + 1 - \lambda_i + i \geq \mu_1 + i > \ell(\eta) + \mu_1 = \ell(\kappa)$. It thus suffices to show $H(\Psi^{s+1}; \lambda) = H(\Psi^s; \lambda)$ for all $s \in [\ell(\eta)]$.

Let $s \in [\ell(\eta)]$. Since κ/η is a k -skew diagram, $\lambda_s + \mu_{\eta_s} > k$ and thus

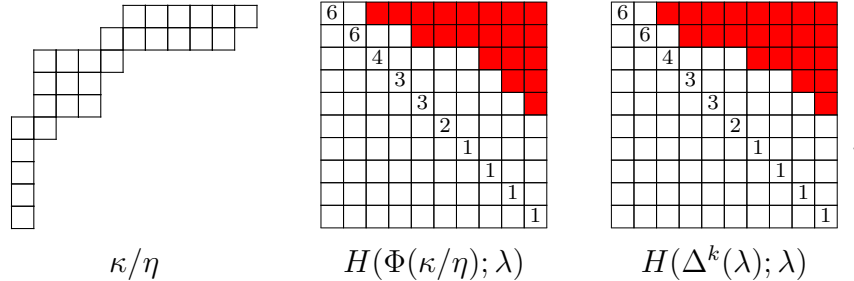
$$\Psi^{s+1} = \Psi^s \sqcup \{(s, j) \mid j \in J\}, \text{ where } J = [k + 1 - \lambda_s + s, \mu_{\eta_s} + s - 1]. \quad (10.1)$$

We have

$$H(\Psi^{s+1}; \lambda) = H(\Psi^{s+1} \setminus (s, k + 1 - \lambda_s + s); \lambda) = \dots = H(\Psi^s; \lambda)$$

by repeated application of Lemma 10.3. The hypotheses hold at each step by (10.1) and the facts (1) Ψ^{s+1} has a ceiling in columns $j, j + 1$ for $j \in J$, and (2) $\kappa_j = \kappa_{j+1}$ for $j \in J$. Fact (1) holds by (10.1) and $\text{down}_{\Psi^{s+1}}(s + 1) = \mu_{\eta_{s+1}} + s + 1 > \mu_{\eta_s} + s$. For (2), note that the box $(s, \eta_s + 1)$ has hook length $\leq k$ in κ , and hence the box $(k + 1 - \lambda_s + s, \eta_s + 1) \notin \kappa$. Together with $(\mu_{\eta_s} + s, \eta_s) \in \kappa$, this implies $\kappa_{k+1-\lambda_s+s} = \eta_s = \kappa_{\mu_{\eta_s}+s}$. \square

Example 10.5. For $k = 7$ and $\lambda = 6643321111$, here are the k -skew diagram κ/η of λ and the two associated Catalan functions from Theorem 10.4:



Acknowledgments. We thank Elaine So for help typing and typesetting figures.

REFERENCES

- [1] Sami H. Assaf and Sara C. Billey. Affine dual equivalence and k -Schur functions. *J. Comb.*, 3(3):343–399, 2012.
- [2] Jonah Blasiak and Sergey Fomin. Noncommutative Schur functions, switchboards, and Schur positivity. *Selecta Math. (N.S.)*, 23(1):727–766, 2017.
- [3] Abraham Broer. A vanishing theorem for Dolbeault cohomology of homogeneous vector bundles. *J. Reine Angew. Math.*, 493:153–169, 1997.
- [4] Bram Broer. Line bundles on the cotangent bundle of the flag variety. *Invent. Math.*, 113(1):1–20, 1993.
- [5] Bram Broer. Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 1–19. Birkhäuser Boston, Boston, MA, 1994.
- [6] Erik Carlsson and Anton Mellit. A proof of the shuffle conjecture. *J. Amer. Math. Soc.*, 2017.

- [7] Li-Chung Chen. *Skew-Linked Partitions and a Representation-Theoretic Model for k -Schur Functions*. PhD thesis, UC Berkeley, 2010.
- [8] J. Désarménien, B. Leclerc, and J.-Y. Thibon. Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory. *Sém. Lothar. Combin.*, 32:Art. B32c, approx. 38, 1994.
- [9] Sergey Fomin and Curtis Greene. Noncommutative Schur functions and their applications. *Discrete Math.*, 193(1-3):179–200, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).
- [10] Adriano M. Garsia. Orthogonality of Milne’s polynomials and raising operators. *Discrete Math.*, 99(1-3):247–264, 1992.
- [11] Adriano M. Garsia and Claudio Procesi. On certain graded S_n -modules and the q -Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992.
- [12] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke Math. J.*, 126(2):195–232, 2005.
- [13] Mark Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001.
- [14] Mark Haiman. Combinatorics, symmetric functions, and Hilbert schemes. In *Current developments in mathematics, 2002*, pages 39–111. Int. Press, Somerville, MA, 2003.
- [15] Wim H. Hesselink. Cohomology and the resolution of the nilpotent variety. *Math. Ann.*, 223(3):249–252, 1976.
- [16] Nai Huan Jing. Vertex operators and Hall-Littlewood symmetric functions. *Adv. Math.*, 87(2):226–248, 1991.
- [17] Joel Kamnitzer. Geometric constructions of the irreducible representations of GL_n . In *Geometric representation theory and extended affine Lie algebras*, volume 59 of *Fields Inst. Commun.*, pages 1–18. Amer. Math. Soc., Providence, RI, 2011.
- [18] Anatol N. Kirillov, Anne Schilling, and Mark Shimozono. A bijection between Littlewood-Richardson tableaux and rigged configurations. *Selecta Math. (N.S.)*, 8(1):67–135, 2002.
- [19] Thomas Lam. Schubert polynomials for the affine Grassmannian. *J. Amer. Math. Soc.*, 21(1):259–281, 2008.
- [20] Thomas Lam. Affine Schubert classes, Schur positivity, and combinatorial Hopf algebras. *Bull. Lond. Math. Soc.*, 43(2):328–334, 2011.
- [21] Thomas Lam, Luc Lapointe, Jennifer Morse, and Mark Shimozono. Affine insertion and Pieri rules for the affine Grassmannian. *Mem. Amer. Math. Soc.*, 208(977):xii+82, 2010.
- [22] Thomas Lam, Luc Lapointe, Jennifer Morse, and Mark Shimozono. The poset of k -shapes and branching rules for k -Schur functions. *Mem. Amer. Math. Soc.*, 223(1050):vi+101, 2013.
- [23] L. Lapointe, A. Lascoux, and J. Morse. Tableau atoms and a new Macdonald positivity conjecture. *Duke Math. J.*, 116(1):103–146, 2003.
- [24] L. Lapointe and J. Morse. Schur function analogs for a filtration of the symmetric function space. *J. Combin. Theory Ser. A*, 101(2):191–224, 2003.
- [25] Luc Lapointe and Jennifer Morse. Tableaux on $k + 1$ -cores, reduced words for affine permutations, and k -Schur expansions. *J. Combin. Theory Ser. A*, 112(1):44–81, 2005.
- [26] Luc Lapointe and Jennifer Morse. A k -tableau characterization of k -Schur functions. *Adv. Math.*, 213(1):183–204, 2007.
- [27] Luc Lapointe and Jennifer Morse. Quantum cohomology and the k -Schur basis. *Trans. Amer. Math. Soc.*, 360(4):2021–2040, 2008.
- [28] George Lusztig. Some examples of square integrable representations of semisimple p -adic groups. *Trans. Amer. Math. Soc.*, 277(2):623–653, 1983.
- [29] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995.
- [30] Dmitri I. Panyushev. Generalised Kostka-Foulkes polynomials and cohomology of line bundles on homogeneous vector bundles. *Selecta Math. (N.S.)*, 16(2):315–342, 2010.
- [31] Anne Schilling and S. Ole Warnaar. Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials. *Comm. Math. Phys.*, 202(2):359–401, 1999.

- [32] Mark Shimozono. A cyclage poset structure for Littlewood-Richardson tableaux. *European J. Combin.*, 22(3):365–393, 2001.
- [33] Mark Shimozono. Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. *J. Algebraic Combin.*, 15(2):151–187, 2002.
- [34] Mark Shimozono and Jerzy Weyman. Graded characters of modules supported in the closure of a nilpotent conjugacy class. *European J. Combin.*, 21(2):257–288, 2000.
- [35] Mark Shimozono and Mike Zabrocki. Hall-Littlewood vertex operators and generalized Kostka polynomials. *Adv. Math.*, 158(1):66–85, 2001.
- [36] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
- [37] Michael Alan Zabrocki. *On the action of the Hall-Littlewood vertex operator*. ProQuest LLC, Ann Arbor, MI, 1998. Thesis (Ph.D.)—University of California, San Diego.

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104
Email address: jblasiak@gmail.com

DEPARTMENT OF MATH, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VA 22904
Email address: morsej@virginia.edu

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104
Email address: annapunying@gmail.com

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA 19104
Email address: danielsummers72@gmail.com