

k -SCHUR EXPANSIONS OF CATALAN FUNCTIONS

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ABSTRACT. We make a broad conjecture about the k -Schur positivity of Catalan functions, symmetric functions which generalize the (parabolic) Hall-Littlewood polynomials. We resolve the conjecture with positive combinatorial formulas in cases which address the k -Schur expansion of (1) Hall-Littlewood polynomials, proving the $q = 0$ case of the strengthened Macdonald positivity conjecture from [24]; (2) the product of a Schur function and a k -Schur function when the indexing partitions concatenate to a partition, describing a class of Gromov-Witten invariants for the quantum cohomology of complete flag varieties; (3) k -split polynomials, solving a substantial special case of a problem of Broer and Shimozono-Weyman on parabolic Hall-Littlewood polynomials [37]. In addition, we prove the conjecture that the k -Schur functions defined via k -split polynomials [25] agree with those defined in terms of strong tableaux [21].

1. INTRODUCTION

Catalan functions are elements of the ring $\Lambda = \mathbb{Z}[t][h_1, h_2, \dots]$ of symmetric functions in infinitely many variables $\mathbf{x} = (x_1, x_2, \dots)$, where $h_d = h_d(\mathbf{x}) = \sum_{i_1 \leq \dots \leq i_d} x_{i_1} \cdots x_{i_d}$. Studied in full generality first by Chen-Haiman [8] and Panyushev [33], Catalan functions are GL_ℓ -equivariant Euler characteristics of vector bundles on the flag variety.

They can be defined by the following Demazure-operator formula. Fix $\ell \in \mathbb{Z}_{\geq 0}$. A *root ideal* is an upper order ideal of the poset $\Delta^+ = \Delta_\ell^+ := \{(i, j) \mid 1 \leq i < j \leq \ell\}$ with partial order given by $(a, b) \leq (c, d)$ when $a \geq c$ and $b \leq d$. An *indexed root ideal of length ℓ* is a pair (Ψ, γ) consisting of a root ideal $\Psi \subset \Delta_\ell^+$ and a weight $\gamma \in \mathbb{Z}^\ell$. Schur functions can be defined for any $\gamma \in \mathbb{Z}^\ell$ by

$$s_\gamma = s_\gamma(\mathbf{x}) = \det(h_{\gamma_{i_i+j-i}}(\mathbf{x}))_{1 \leq i, j \leq \ell} \in \Lambda, \tag{1.1}$$

where by convention $h_0(\mathbf{x}) = 1$ and $h_d(\mathbf{x}) = 0$ for $d < 0$.

Definition 1.1. The *Catalan function* associated to an indexed root ideal (Ψ, γ) of length ℓ is

$$H(\Psi; \gamma)(\mathbf{x}; t) := \prod_{(i, j) \in \Psi} (1 - tR_{ij})^{-1} s_\gamma(\mathbf{x}) \in \Lambda, \tag{1.2}$$

where the raising operator R_{ij} acts on the subscripts of the s_γ by $R_{ij}s_\gamma = s_{\gamma + \epsilon_i - \epsilon_j}$ and $\epsilon_i \in \mathbb{Z}^\ell$ denotes the weight with a 1 in position i and 0's elsewhere; for a discussion of raising operators, including a more precise version of (1.2), see [5, §4.3]. By convention, $H(\emptyset; \emptyset) := 1$ when $\ell = 0$, where \emptyset denotes the empty set and \emptyset denotes the weight/partition of length 0.

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Twenty years ago, symmetric functions known as k -Schur functions were discovered in the study of the Macdonald positivity conjecture [24] and were subsequently connected to affine Schubert calculus [28, 29, 19]. Many conjecturally equivalent candidates for k -Schur functions have been proposed in the years since their discovery. Among them is the following subclass of Catalan functions, which we recently connected to other k -Schur candidates [5], proving a conjecture of Chen-Haiman [8] (see §1.2).

Fix $k \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}_{\geq 0}$. Write $\text{Par}_\ell^k = \{(\mu_1, \dots, \mu_\ell) \in \mathbb{Z}^\ell : k \geq \mu_1 \geq \dots \geq \mu_\ell \geq 0\}$ for the set of partitions contained in the $\ell \times k$ -rectangle and $\text{Par}^k = \bigsqcup_{m \geq 0} \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}^m \mid k \geq \mu_1 \geq \mu_2 \geq \dots \geq \mu_m > 0\}$ for the set of k -bounded partitions. For any partition μ (elements of Par_ℓ^k and Par^k included), the *length* $\ell(\mu)$ is the number of nonzero parts of μ .

Definition 1.2. For $\mu \in \text{Par}_\ell^k$, define the root ideal

$$\Delta^k(\mu) = \{(i, j) \in \Delta_\ell^+ \mid k - \mu_i + i < j\}, \quad (1.3)$$

and the k -Schur Catalan function

$$\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) := H(\Delta^k(\mu); \mu)(\mathbf{x}; t) = \prod_{i=1}^{\ell} \prod_{j=k+1-\mu_i+i}^{\ell} (1 - tR_{ij})^{-1} s_\mu(\mathbf{x}). \quad (1.4)$$

1.1. A k -Schur positivity conjecture. A central theme in the investigations of Catalan functions [37, 8, 33] can be articulated as Schur positivity conjectures. We have discovered that a conjecture addressing k -Schur positivity encompasses the earlier conjectures as well as questions concerning Gromov-Witten invariants and Macdonald polynomials.

Let $H_\mu(\mathbf{x}; t)$ denote the modified Hall-Littlewood polynomial indexed by the partition μ , which is equal to the Catalan function $H(\Delta^+; \mu)(\mathbf{x}; t)$.

Proposition 1.3. *The k -Schur Catalan functions $\{\mathfrak{s}_\mu^{(k)}\}_{\mu \in \text{Par}^k}$ form a $\mathbb{Z}[t]$ -basis for*

$$\Lambda^k := \text{span}_{\mathbb{Z}[t]} \{H_\mu(\mathbf{x}; t) \mid \mu \in \text{Par}^k\} \subset \Lambda. \quad (1.5)$$

Proof. Note that Λ is a free $\mathbb{Z}[t]$ -module with bases $\{h_\mu\}$, $\{s_\mu\}$, $\{H_\mu\}$ (μ ranges over partitions) since $\{h_\mu\}$ and $\{s_\mu\}$ are related by a unitriangular change of basis, as are $\{H_\mu\}$ and $\{s_\mu\}$. Thus Λ^k is a free $\mathbb{Z}[t]$ -module. Since $\{\mathfrak{s}_\mu^{(k)}\}_{\mu \in \text{Par}^k}$ and $\{H_\mu\}_{\mu \in \text{Par}^k}$ are related by a unitriangular change of basis by the proof of [5, Theorem 2.8], the result follows. \square

Given an indexed root ideal (Ψ, μ) of length ℓ , for each $i \in [\ell]$ define

$$\bar{\mathfrak{r}}(\Psi)_i := |\{(i, j) \in \Delta^+ \setminus \Psi : j \in [\ell]\}|, \quad (1.6)$$

$$\text{band}(\Psi, \mu)_i := \bar{\mathfrak{r}}(\Psi)_i + \mu_i. \quad (1.7)$$

Proposition 1.4. *If (Ψ, μ) is an indexed root ideal with $\text{band}(\Psi, \mu)_i \leq k$ for all i , then $H(\Psi; \mu) \in \Lambda^k$.*

Proof. It was established in the proof of Theorem 2.8 on page 17 of [5] that $H(\Psi; \mu) \in \Lambda^k$ when $\mu \in \text{Par}_\ell^k$ and $\Psi = \Delta^k(\mu)$, but all that was used about (Ψ, μ) is $\text{band}(\Psi, \mu)_i \leq k$ for all i , so the proof applies in the present setting unchanged. \square

Conjecture 1.5. *If (Ψ, μ) is an indexed root ideal with $\mu \in \text{Par}^k$ and $\text{band}(\Psi, \mu)_i \leq k$ for all i , then the Catalan function $H(\Psi; \mu)$ is k -Schur positive. That is,*

$$H(\Psi; \mu)(\mathbf{x}; t) = \sum_{\lambda \in \text{Par}^k} K_{\lambda, \mu}^{\Psi, k}(t) \mathfrak{s}_{\lambda}^{(k)}(\mathbf{x}; t) \quad \text{with } K_{\lambda, \mu}^{\Psi, k}(t) \in \mathbb{N}[t]. \quad (1.8)$$

We call the $K_{\lambda, \mu}^{\Psi, k}(t)$ k -Catalan-Kostka coefficients.

Remark 1.6. The branching property [5, Theorem 2.6] — $\mathfrak{s}_{\nu}^{(k)}$ is $k+1$ -Schur positive for all $\nu \in \text{Par}^k$ — tells us that for fixed (Ψ, μ) the conjectured k -Schur positivity of $H(\Psi; \mu)$ is strongest for $k = \max_i \{ \text{band}(\Psi, \mu)_i \}$. Also, k -Schur positivity implies Schur positivity since a k -Schur function reduces to a Schur function for large k .

Conjecture 1.5 is a broad conjecture which encompasses several open positivity problems in algebraic combinatorics. We elaborate below and preview our main results which resolve three natural special cases of this conjecture.

Strengthened Macdonald positivity. Lapointe, Lascoux, and Morse [24] constructed a family of symmetric functions—now one of many conjecturally equivalent k -Schur candidates—and conjectured (i) they form a basis for Λ^k , (ii) they are Schur positive, and (iii) for $\mu \in \text{Par}^k$, the expansion of the modified Macdonald polynomial $H_{\mu}(\mathbf{x}; q, t)$ in this basis has coefficients in $\mathbb{N}[q, t]$. This factors the problem of Schur expanding Macdonald polynomials into the two positivity problems (ii) and (iii). In [5], we established that $\{ \mathfrak{s}_{\mu}^{(k)} \}_{\mu \in \text{Par}^k}$ is a Schur positive basis for Λ^k , i.e., it satisfies (i)–(ii). Here we resolve the $q = 0$ specialization of (iii) in the strongest possible terms by giving a positive combinatorial formula for the k -Schur expansion of $H_{\mu}(\mathbf{x}; 0, t) = H_{\mu}(\mathbf{x}; t)$ for all $\mu \in \text{Par}^k$ (Theorem 4.1); since $H_{\mu}(\mathbf{x}; 0, t) = H(\Delta^+; \mu)$, this resolves Conjecture 1.5 in the case $\Psi = \Delta^+$.

Gromov-Witten invariants. The k -Schur expansion coefficients in products of two k -Schur functions at $t = 1$ agree (see Section 7) with the 3-point Gromov-Witten invariants of genus 0 which, informally, count equivalence classes of certain rational curves in the flag variety Fl_{k+1} . They are structure constants of the quantum cohomology ring of Fl_{k+1} and specialize to Schubert polynomial structure constants. It turns out that the product of k -Schur functions is realizable as a single Catalan function, as we now explain.

Define a binary operation \uplus on root ideals as follows: for $\Psi \subset \Delta_r^+$ and $\Phi \subset \Delta_m^+$, the root ideal $\Psi \uplus \Phi \subset \Delta_{r+m}^+$ is the result of placing Ψ and Φ catty-corner and including the full $r \times m$ rectangle of roots in between; equivalently, $\Psi \uplus \Phi$ is determined by

$$\Delta_{r+m}^+ \setminus (\Psi \uplus \Phi) = (\Delta_r^+ \setminus \Psi) \sqcup \{ (i+r, j+r) : (i, j) \in \Delta_m^+ \setminus \Phi \}. \quad (1.9)$$

Let $\mu \in \text{Par}_r^k$, $\nu \in \text{Par}_m^k$ and $\mu\nu = (\mu, \nu)$ denote the concatenation of μ and ν . We have $H(\Delta^k(\mu) \uplus \Delta^k(\nu); \mu\nu)(\mathbf{x}; 1) = \mathfrak{s}_{\mu}^{(k)}(\mathbf{x}; 1) \mathfrak{s}_{\nu}^{(k)}(\mathbf{x}; 1)$ and thus its k -Schur expansion coefficients are the (nonnegative) Gromov-Witten invariants. Conjecture 1.5 predicts a stronger result in the more restricted setting when $\mu\nu$ is a partition: the k -Schur expansion coefficients of $H(\Delta^k(\mu) \uplus \Delta^k(\nu); \mu\nu)(\mathbf{x}; t)$ lie in $\mathbb{N}[t]$. We resolve this with a positive combinatorial formula in the case $\mathfrak{s}_{\mu}^{(k)}$ is a Schur function (Theorem 5.1) and deduce a tableau enumeration formula for a new class of Gromov-Witten invariants (Theorem 7.6).

Positive formulas for Gromov-Witten invariants have been obtained in many special cases [2, 3, 4, 7, 9, 10, 12, 31, 32, 35, 39]. Our formula is one of the few for which all three input permutations are allowed to vary among $\Omega(2^k)$ many possibilities, in contrast to, say, the quantum Monk or Pieri formula in which one of the permutations has a very special form.

Schur positivity and parabolic Hall-Littlewood polynomials. Conjecture 1.5 strengthens the following conjecture of Chen-Haiman [8] (by Remark 1.6):

Conjecture 1.7. *The Catalan function $H(\Psi; \mu)$ is Schur positive for any root ideal Ψ and partition μ .*

Broer and Shimozono-Weyman [6, 37] had earlier studied Conjecture 1.7 for *parabolic Hall-Littlewood polynomials*, those Catalan functions of the form

$$H(\emptyset_{r_1} \uplus \cdots \uplus \emptyset_{r_d}; \mu) \quad (1.10)$$

with $r_1, \dots, r_d \in \mathbb{Z}_{\geq 1}$ and partition μ , where $\emptyset_r \subset \Delta_r^+$ denotes the empty root ideal of length r . Broer posed Conjecture 1.7 for this class of Catalan functions, or rather the stronger conjecture that the higher cohomology of an associated vector bundle on the flag variety vanishes (see [37, Conjecture 5]). Shimozono-Weyman conjectured that the Schur expansion coefficients of parabolic Hall-Littlewood polynomials can be described by a combinatorial procedure called katabolism [37]; this conjecture was later generalized by Chen-Haiman to address any Catalan function with partition weight [8, Conjecture 5.4.3].

Conjecture 1.5 predicts that the parabolic Hall-Littlewood polynomials are not only Schur positive but in fact k -Schur positive for $k = \max\{(\mu^i)_1 + r_i - 1 \mid i \in [d]\}$, where $\mu = (\mu^1, \dots, \mu^d)$ is the decomposition of μ into sequences of lengths r_1, \dots, r_d . We settle this conjecture for a subclass of parabolic Hall-Littlewood polynomials studied in [25, 26] called k -split polynomials (Theorem 6.2).

1.2. Unifying the k -Schur candidates. The desired properties (i)–(iii) of a k -Schur basis were never simultaneously satisfied by any one proposed candidate. For this reason, the unification of the various definitions has been an important open problem. Table 1 summarizes the current state of the art in this regard.

Circled entries follow from our recent work in [5], Theorem 1.8, and Theorem 4.1. The k -Schur candidate $\{\tilde{A}_\lambda^{(k)}\}$ was defined via k -split polynomials in [25, 26] (recalled in §6.2); they were shown to form a basis for Λ^k in [25] and satisfy the k -rectangle property in [26]. The candidate $\{s_\lambda^{(k)}\}$ was proposed in [21, §9.3] (see also [1, 22]) and is defined as a sum of monomials over strong tableaux, certain chains in the strong (Bruhat) order of $\widehat{\mathcal{S}}_{k+1}$. The candidates on the last two rows are only defined at $t = 1$ (and the checks indicate properties at $t = 1$). For these candidates, Schur positivity was proven geometrically in [20] and combinatorially in [21, 22]; the k -rectangle property was proven in [28]. The remaining checkmarks follow directly from the definitions of the respective candidates and the equivalence of the last two candidates [19].

Theorem 1.8. *The k -Schur functions defined via k -split polynomials [25, 26], k -Schur Catalan functions, and strong tableau k -Schur functions [21, §9.3] coincide:*

$$\tilde{A}_\mu^{(k)}(\mathbf{x}; t) = \mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) = s_\mu^{(k)}(\mathbf{x}; t) \quad \text{for all } \mu \in \text{Par}^k. \quad (1.11)$$

k -Schur candidate	basis of Λ^k	Schur positive	the $H_\mu(\mathbf{x}; q, t)$ are k -Schur positive	k -rectangle property
(1998) Young tableaux and katabolism [24]		✓		
(2001) k -split polynomials [25, 26]	$\tilde{A}_\lambda^{(k)}$	✓	⊙ ($q = 0$)	✓
(2006) Strong tableaux [21]	$s_\lambda^{(k)}$	⊙	⊙ ($q = 0$)	⊙
(2008) Catalan functions [8]	$\mathfrak{s}_\lambda^{(k)}$	⊙	⊙ ($q = 0$)	⊙
(2014) Affine Kostka matrix [11, 30]		✓	✓ ($q = 0$)	
(2004) Weak tableaux [28] ($t = 1$)	$\bar{s}_\lambda^{(k)}$	✓	✓ ($q = 0$)	✓
(2005) Schubert classes in $H_*(\text{Gr})$ [19] ($t = 1$)		✓	✓ ($q = 0$)	✓

TABLE 1. Conjecturally equivalent definitions of k -Schur functions and known properties.

Moreover, their $t = 1$ specializations $\{s_\mu^{(k)}(\mathbf{x}; 1)\}$ match a definition $\{\bar{s}_\mu^{(k)}(\mathbf{x})\}$ using weak tableaux from [28], and represent Schubert classes in the homology of the affine Grassmannian Gr_G of $G = SL_{k+1}$.

Proof. The first equality of (1.11) is proved in Section 10 as a consequence of the above-mentioned Theorem 6.2, while the second was established in [5, Theorem 2.4]. The $\{s_\mu^{(k)}(\mathbf{x}; 1)\}$ agree with the weak tableau k -Schur functions by [21, Theorem 4.11] and with affine Grassmannian Schubert classes by [19, Theorem 7.1]. \square

2. STRONG PIERI OPERATORS

The strong Pieri operators were introduced in [5] and played a peripheral role there. We have since discovered they are key to establishing elegant formulas for k -Schur expansions of Catalan functions. The operators are defined combinatorially using strong marked covers and have a simple description in terms of Catalan functions.

For most of the background below, we follow [5]. Strong marked covers were introduced in [21] and our version below differs only in that markings are by rows rather than diagonals; this difference, though seemingly minor, has turned out to be quite important. For examples of the subsequent definitions, see [5, §2.2] and Examples 4.2 and 5.3.

2.1. Strong tableaux. The *diagram* of a partition λ is the subset of boxes $\{(r, c) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \mid c \leq \lambda_r\}$ in the plane, drawn in English (matrix-style) notation so that rows (resp. columns) are increasing from north to south (resp. west to east). Each box has a *hook length* which counts the number of boxes below it in its column and weakly to its right in its row. A $k + 1$ -*core* is a partition with no box of hook length $k + 1$. There is a bijection \mathfrak{p} [27] from the set of $k + 1$ -cores to Par^k mapping a $k + 1$ -core κ to the partition λ whose r -th row λ_r is the number of boxes in the r -th row of κ having hook length $\leq k$.

A *strong cover* $\tau \Rightarrow \kappa$ is a pair of $k + 1$ -cores such that $\tau \subset \kappa$ and $|\mathfrak{p}(\tau)| + 1 = |\mathfrak{p}(\kappa)|$. A *strong marked cover* $\tau \xrightarrow{r} \kappa$ is a strong cover $\tau \Rightarrow \kappa$ together with a positive integer r

which is allowed to be the smallest row index of any connected component of the skew shape κ/τ . Let $w = w_1 \cdots w_m$ be a word in the alphabet of positive integers. A *strong tableau T marked by w* is a sequence of strong marked covers of the form

$$\kappa^{(0)} \xrightarrow{w_m} \kappa^{(1)} \xrightarrow{w_{m-1}} \cdots \xrightarrow{w_1} \kappa^{(m)}.$$

We write $\text{inside}(T) = \mathbf{p}(\kappa^{(0)})$ and $\text{outside}(T) = \mathbf{p}(\kappa^{(m)})$. The set of strong tableaux marked by w with $\text{outside}(T) = \mu$ is denoted $\text{SMT}^k(w; \mu)$.

The *spin* of a strong marked cover $\tau \xrightarrow{r} \kappa$ is defined to be $c \cdot (h - 1) + N$, where c is the number of connected components of the skew shape κ/τ , h is the height (number of rows) of each component, and N is the number of components entirely contained in rows $> r$. For a strong tableau T marked by a word, $\text{spin}(T)$ is defined to be the sum of the spins of the strong marked covers comprising T .

2.2. Strong Pieri operators. Fix a positive integer k . The *strong Pieri operators* $u_1, u_2, \dots \in \text{End}_{\mathbb{Z}[t]}(\Lambda^k)$ are defined by their action on the basis $\{\mathfrak{s}_\mu^{(k)}\}_{\mu \in \text{Par}^k}$ as follows:

$$\mathfrak{s}_\mu^{(k)} \cdot u_p = \sum_{T \in \text{SMT}^k(p; \mu)} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(k)}. \quad (2.1)$$

We have found it more natural to define these as right operators for compatibility with conventions for tableau reading words (see Theorem 5.1). The set $\text{SMT}^k(p; \mu)$ in the sum is just another notation for the set of strong marked covers $\tau \xrightarrow{p} \kappa$ with $\mathbf{p}(\kappa) = \mu$.

By [5, Theorem 9.2], these operators are simply described in terms of Catalan functions: for any $\mu \in \text{Par}_\ell^k$ and $p \in [\ell]$,

$$\mathfrak{s}_\mu^{(k)} \cdot u_p = H(\Delta^k(\mu); \mu - \epsilon_p). \quad (2.2)$$

Let e_d^\perp be the linear operator on Λ that is adjoint to multiplication by e_d with respect to the Hall inner product. A main result of [5] (see Equation 2.6 and §9.2 therein) expresses e_d^\perp in terms of the strong Pieri operators. We need only the following special case:

Theorem 2.1. *For any $f \in \text{span}_{\mathbb{Z}[t]}\{\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t) \mid \mu \in \text{Par}_\ell^k\}$, $f \cdot e_\ell^\perp = f \cdot u_\ell u_{\ell-1} \cdots u_1$.*

Remark 2.2. In [5], we worked with symmetric functions over the coefficient ring $\mathbb{Q}(t)$ rather than $\mathbb{Z}[t]$, but it is easily checked that Theorem 2.1 and all other results of [5] cited here hold over $\mathbb{Z}[t]$.

3. CATALAN OPERATORS

Our results on modified Hall-Littlewood polynomials and k -split polynomials make use of symmetric function operators of Jing and Garsia [16, 13] and Shimozono-Zabrocki [38]. It is natural from our perspective to frame these in the context of more general Catalan operators, which recover Catalan functions upon action on 1.

Garsia's version [13] of Jing's Hall-Littlewood vertex operators [16] are the symmetric function operators defined for any $m \in \mathbb{Z}$ by

$$\mathbf{B}_m = \sum_{i, j \geq 0} (-1)^i t^j h_{m+i+j}(\mathbf{x}) e_i^\perp h_j^\perp \in \text{End}_{\mathbb{Z}[t]}(\Lambda). \quad (3.1)$$

These are creation operators for the modified Hall-Littlewood polynomials:

$$\mathbf{B}_m H_\mu(\mathbf{x}; t) = H_{(m, \mu)}(\mathbf{x}; t) \quad \text{for } m \geq \mu_1. \quad (3.2)$$

For $\alpha \in \mathbb{Z}^\ell$, set $\tilde{\mathbf{B}}_\alpha := \mathbf{B}_{\alpha_1} \mathbf{B}_{\alpha_2} \cdots \mathbf{B}_{\alpha_\ell}$.

Definition 3.1. The *Catalan operator* associated to an indexed root ideal (Ψ, γ) is the symmetric function operator given by

$$\mathbf{B}_\gamma^\Psi = \prod_{(i,j) \in \Delta^+ \setminus \Psi} (1 - t\mathbf{R}_{ij}) \tilde{\mathbf{B}}_\gamma \in \text{End}_{\mathbb{Z}[t]}(\Lambda), \quad (3.3)$$

where the raising operator \mathbf{R}_{ij} acts on the subscripts of the $\tilde{\mathbf{B}}_\alpha$ by $\mathbf{R}_{ij} \tilde{\mathbf{B}}_\alpha = \tilde{\mathbf{B}}_{\alpha + \epsilon_i - \epsilon_j}$.

Letting Catalan operators act on 1, we recover Catalan functions:

$$\mathbf{B}_\gamma^\Psi \cdot 1 = H(\Psi, \gamma) \quad \text{for any indexed root ideal } (\Psi, \gamma). \quad (3.4)$$

This holds by [5, Proposition 4.7]. The Catalan operators simultaneously generalize the iterated Garsia-Jing operators $\tilde{\mathbf{B}}_\alpha$ and the generalized Hall-Littlewood vertex operators \mathbf{B}_α of Shimozono-Zabrocki [38]: for any $\alpha \in \mathbb{Z}^\ell$,

$$\tilde{\mathbf{B}}_\alpha = \mathbf{B}_\alpha^{\Delta^+} \quad \text{and} \quad \mathbf{B}_\alpha = \mathbf{B}_\alpha^\emptyset. \quad (3.5)$$

The latter follows from, e.g., the description [25, Equation 6.7] of the \mathbf{B}_α operators.

Proposition 3.2. *Catalan operators obey the following composition law: for any indexed root ideals (Ψ, μ) and (Φ, ν) of lengths r and $\ell - r$, respectively, there holds*

$$\mathbf{B}_\mu^\Psi \mathbf{B}_\nu^\Phi = \mathbf{B}_{\mu\nu}^{\Psi \uplus \Phi}, \quad (3.6)$$

where \uplus is defined in (1.9).

Proof. We compute using Definition 3.1:

$$\begin{aligned} \mathbf{B}_{\mu\nu}^{\Psi \uplus \Phi} &= \prod_{(i,j) \in \Delta_\ell^+ \setminus (\Psi \uplus \Phi)} (1 - t\mathbf{R}_{ij}) \tilde{\mathbf{B}}_{\mu\nu} \\ &= \left(\prod_{(i,j) \in \Delta_r^+ \setminus \Psi} (1 - t\mathbf{R}_{ij}) \tilde{\mathbf{B}}_\mu \right) \left(\prod_{(i,j) \in \Delta_{\ell-r}^+ \setminus \Phi} (1 - t\mathbf{R}_{ij}) \tilde{\mathbf{B}}_\nu \right) = \mathbf{B}_\mu^\Psi \mathbf{B}_\nu^\Phi. \quad \square \end{aligned}$$

The $t = 1$ specializations of Catalan operators and functions can be made precise as follows: the ring homomorphism $\mathbb{Z}[t] \rightarrow \mathbb{Z}$, $t \mapsto 1$ makes \mathbb{Z} a $\mathbb{Z}[t]$ -algebra and $\mathbb{Z} \otimes_{\mathbb{Z}[t]} -$ is a functor from $\mathbb{Z}[t]\text{-Mod}$ to $\mathbb{Z}\text{-Mod}$ which we denote by $|_{t=1}$. Let $\Lambda_{\mathbb{Z}} = \Lambda|_{t=1} = \Lambda/(t-1) = \mathbb{Z}[h_1, h_2, \dots]$. Specializing $t = 1$ at the level of elements of Λ is then defined via the canonical ring homomorphism $\pi : \Lambda \rightarrow \Lambda/(t-1) = \Lambda_{\mathbb{Z}}$:

$$H(\Psi; \gamma)(\mathbf{x}; 1) := \pi(H(\Psi; \gamma)(\mathbf{x}; t)) \quad \text{and} \quad \mathfrak{s}_\mu^{(k)}(\mathbf{x}; 1) := \pi(\mathfrak{s}_\mu^{(k)}(\mathbf{x}; t)).$$

Proposition 3.3. *At $t = 1$, Catalan operators reduce to multiplication by Catalan functions: for any $g \in \Lambda_{\mathbb{Z}}$, $\mathbf{B}_\mu^\Psi |_{t=1}(g) = (\mathbf{B}_\mu^\Psi |_{t=1} \cdot 1) g = H(\Psi; \mu)(\mathbf{x}; 1) g$.*

Proof. The first equality follows from the fact that $\mathbf{B}_m|_{t=1} \in \text{End}_{\mathbb{Z}}(\Lambda_{\mathbb{Z}})$ is equal to multiplication by $h_m(\mathbf{x})$ (recall $h_m(\mathbf{x}) := 0$ for $m < 0$). The second holds by (3.4) and the general fact that for any $\mathbf{B} \in \text{End}_{\mathbb{Z}[t]}(\Lambda)$ and $f \in \Lambda$, $\pi(\mathbf{B}(f)) = \mathbf{B}|_{t=1}(\pi(f))$. \square

Corollary 3.4. *For any indexed root ideals (Ψ, μ) and (Φ, ν) ,*

$$H(\Psi; \mu)(\mathbf{x}; 1)H(\Phi; \nu)(\mathbf{x}; 1) = H(\Psi \uplus \Phi; \mu\nu)(\mathbf{x}; 1). \quad (3.7)$$

Proof. Apply the functor $|_{t=1}$ to $\mathbf{B}_\mu^\Psi \mathbf{B}_\nu^\Phi = \mathbf{B}_{\mu\nu}^{\Psi \uplus \Phi}$, use Proposition 3.3, and act on 1. \square

4. MODIFIED HALL-LITTLEWOOD POLYNOMIALS

We give a positive combinatorial formula for the k -Schur expansion of the modified Hall-Littlewood polynomials $H(\Delta^+; \mu) = H_\mu(\mathbf{x}; t)$, which is succinctly expressed in terms of the strong Pieri operators. This resolves the $q = 0$ specialization of the strengthened Macdonald positivity conjecture [24].

For a skew shape θ , the *superstandard tableau* Z_θ is the unique filling of θ whose i -th row consists entirely of i 's. The *column reading word* of a tableau T , denoted $\text{colword}(T)$, is the word obtained by concatenating the columns of T (reading each column bottom to top), starting with the leftmost column. For example, with $\theta = (4444)/(43220)$,

$$Z_\theta = \begin{array}{cccc} & & & 2 \\ & & 3 & 3 \\ & & 4 & 4 \\ 5 & 5 & 5 & 5 \end{array} \quad \text{and} \quad \text{colword}(Z_\theta) = 555435432.$$

Given a word $w = w_1 \cdots w_d$ in the positive integers, we write $u_w = u_{w_1} \cdots u_{w_d}$ for the corresponding monomial in the strong Pieri operators.

Theorem 4.1. *For any $\mu \in \text{Par}_\ell^k$, the k -Schur expansion of the modified Hall-Littlewood polynomial $H_\mu(\mathbf{x}; t)$ is given by*

$$H_\mu = \mathfrak{s}_{k^\ell}^{(k)} \cdot u_{\text{colword}(Z_{k^\ell/\mu})} = \sum_{T \in \text{SMT}^k(\text{colword}(Z_{k^\ell/\mu}); k^\ell)} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(k)}.$$

We give the proof now though it requires a result proved later, Lemma 10.2, which describes the interaction of the strong Pieri operators with the Garsia-Jing operators.

Proof. Set $\theta = k^\ell/\mu$. The proof is by induction on $\ell + |\theta|$. The base case $\ell = 0$ holds by $H_\emptyset = 1 = \mathfrak{s}_\emptyset^{(k)} \cdot u_{\text{colword}(Z_\emptyset)}$ ($\text{colword}(Z_\emptyset)$ is the empty word). Now suppose $\ell > 0$. If $\mu_1 < k$, then the rightmost column of Z_θ is a full column of length ℓ , and so $u_{\text{colword}(Z_\theta)} = u_{\text{colword}(Z_{\tilde{\theta}})} u_\ell u_{\ell-1} \cdots u_1$, where $\tilde{\theta} := k^\ell/(\mu + 1^\ell)$. By the inductive hypothesis,

$$H_{\mu+1^\ell} = \mathfrak{s}_{k^\ell}^{(k)} \cdot u_{\text{colword}(Z_{\tilde{\theta}})}. \quad (4.1)$$

Applying e_ℓ^\perp to both sides and using Theorem 2.1 gives

$$H_{\mu+1^\ell} \cdot e_\ell^\perp = \mathfrak{s}_{k^\ell}^{(k)} \cdot u_{\text{colword}(Z_{\tilde{\theta}})} u_\ell u_{\ell-1} \cdots u_1 = \mathfrak{s}_{k^\ell}^{(k)} \cdot u_{\text{colword}(Z_\theta)}.$$

Now $H(\Psi; \gamma) \cdot e_\ell^\perp = H(\Psi; \gamma - 1^\ell)$ for any indexed root ideal (Ψ, γ) of length ℓ , which follows directly from Definition 1.1 and the fact that $s_\gamma \cdot e_\ell^\perp = s_{\gamma-1^\ell}$ for any $\gamma \in \mathbb{Z}^\ell$. Hence $H_{\mu+1^\ell} \cdot e_\ell^\perp = H_\mu$, completing the proof in the $\mu_1 < k$ case.

Now suppose $\mu_1 = k$ and set $\hat{\theta} = k^{\ell-1}/(\mu_2, \dots, \mu_\ell)$. Then $\text{colword}(Z_\theta)$ does not contain any 1's and is obtained from $\text{colword}(Z_{\hat{\theta}})$ by adding 1 to each letter. Thus Lemma 10.2 with $r = 1$ yields

$$\mathfrak{s}_{k^\ell}^{(k)} \cdot u_{\text{colword}(Z_\theta)} = \mathbf{B}_k(\mathfrak{s}_{k^{\ell-1}}^{(k)} \cdot u_{\text{colword}(Z_{\hat{\theta}})}) = \mathbf{B}_k H_{(\mu_2, \dots, \mu_\ell)} = H_\mu,$$

where the second equality is by the inductive hypothesis and the third is by (3.2). \square

Example 4.2. According to Theorem 4.1, the 3-Schur expansion of H_{2211} is given by $\mathfrak{s}_{3333}^{(3)} \cdot u_4 u_3 u_4 u_3 u_2 u_1 = \sum_{\text{SMT}^3(434321; 3333)} t^{\text{spin}(T)} \mathfrak{s}_{\text{inside}(T)}^{(3)}$; this sum is over sequences of the form

$$\kappa^{(0)} \xrightarrow{1} \kappa^{(1)} \xrightarrow{2} \kappa^{(2)} \xrightarrow{3} \kappa^{(3)} \xrightarrow{4} \kappa^{(4)} \xrightarrow{3} \kappa^{(5)} \xrightarrow{4} \kappa^{(6)} = \mathbf{p}^{-1}(3333),$$

which are illustrated below by placing letter i in $\kappa^{(i)}/\kappa^{(i-1)}$.

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$$H_{2211} = t^4 \mathfrak{s}_{333}^{(3)} + t^3 \mathfrak{s}_{321}^{(3)} + t^2 \mathfrak{s}_{321}^{(3)} + t \mathfrak{s}_{3111}^{(3)} + t \mathfrak{s}_{222}^{(3)} + \mathfrak{s}_{2211}^{(3)}.$$

See also [5, Figure 1] for the 4-Schur expansion of H_{11111} .

5. SCHUR TIMES k -SCHUR

We give a positive combinatorial formula for the k -Schur expansion of a t -analog of a Schur function times a k -Schur function when the indexing partitions concatenate to a partition. The involved combinatorics has the same spirit as the k -Schur expansion of modified Hall-Littlewood polynomials, but rather than using the action of strong Pieri operators derived from one particular tableau word, it requires a set of tableaux words.

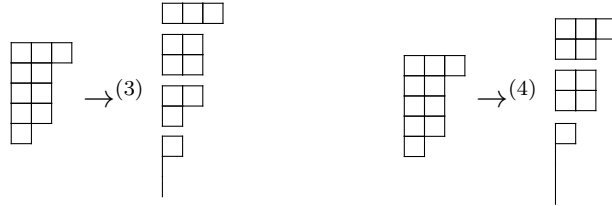
Even this special (Pieri) case of Theorem 5.1 is new and significant. Its specialization at $t = 1$ was previously known, but even here the story is interesting: the k -Pieri rule from [28] expresses $h_d \bar{s}_\nu^{(k)}$ in the k -Schur basis $\{\bar{s}_\mu^{(k)}\}$ using weak tableaux. On the other hand, (5.3) at $t = 1$ becomes $h_d \bar{s}_\nu^{(k)} = \bar{s}_{(k, \nu)}^{(k)} \cdot u_1^{k-d}|_{t=1}$, which (when $d \geq \nu_1$) agrees with a formulation of the k -Pieri rule from [11, Corollary 16]; only after some work (done in [11, Section 4]) can it be shown that these versions of the k -Pieri rule indeed compute the same thing.

6. k -SPLIT POLYNOMIALS

As mentioned in the introduction, Conjecture 1.5 predicts that for any tuple $(\lambda^1, \dots, \lambda^d)$ of partitions which concatenate to a partition, $\mathbf{B}_{\lambda^1} \mathbf{B}_{\lambda^2} \cdots \mathbf{B}_{\lambda^{d-1}} s_{\lambda^d}$ is k -Schur positive for $k = \max\{(\lambda^i)_1 + \ell(\lambda^i) - 1 \mid i \in [d]\}$, strengthening the corresponding Schur positivity conjecture of Broer and Shimozono-Weyman. We resolve this (stronger) conjecture for the k -split polynomials, a basis for Λ^k introduced in [25] to define the k -Schur candidate $\{\tilde{A}_\mu^{(k)}\}$. As a corollary, we deduce that the $\tilde{A}_\mu^{(k)}$ agree with the k -Schur Catalan functions.

6.1. k -split polynomials are k -Schur positive. For $\lambda \in \text{Par}^k$, the k -split of λ is the sequence $\lambda^{\rightarrow k} := (\lambda^1, \lambda^2, \dots, \lambda^d)$ obtained by decomposing λ (without rearranging entries) into partitions $\lambda^i \in \text{Par}_{r_i}^k$ so that $r_i = k - (\lambda^i)_1 + 1$ and $(\lambda^d)_1 > 0$. Thus the maximum hook length of λ^i is k for all $i < d$ and λ^d is padded with zeros so that it has a total of $k - (\lambda^d)_1 + 1$ entries. We adopt the convention that the k -split of the empty partition is the empty list of partitions ($d = 0$).

For example, $(322211)^{\rightarrow 3} = (3, 22, 21, 100)$ and $(322211)^{\rightarrow 4} = (32, 22, 1000)$:



Definition 6.1. For $\lambda \in \text{Par}^k$ with k -split $\lambda^{\rightarrow k} = (\lambda^1, \lambda^2, \dots, \lambda^d)$, the k -split polynomial indexed by λ is

$$G_\lambda^{(k)} := \mathbf{B}_{\lambda^1} \mathbf{B}_{\lambda^2} \cdots \mathbf{B}_{\lambda^{d-1}} s_{\lambda^d} \in \Lambda.$$

Theorem 6.2. Let $\lambda \in \text{Par}^k$ and $\lambda^{\rightarrow k} = (\lambda^1, \dots, \lambda^d)$ be the k -split of λ . For each $i \in [d]$, let $r_i = k - (\lambda^i)_1 + 1$, $U^i = (k - r_i + 1)^{r_i}$, $\theta^i = U^i / \lambda^i$, and $N_i \subset \mathbb{Z}_{\geq 1}$ be the interval of length r_i such that the restriction of λ to positions N_i is λ^i . Set $U = U^1 \cdots U^d$, the concatenation of the U^i . The k -Schur expansion of the k -split polynomial $G_\lambda^{(k)}$ is given by

$$G_\lambda^{(k)} = \mathfrak{s}_U^{(k)} \cdot \left(\sum_{T \in \text{SSYT}_{\theta^d}(N_d)} u_{\text{colword}(T)} \right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^2}(N_2)} u_{\text{colword}(T)} \right) \left(\sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)} \right). \tag{6.1}$$

The proof, given in Section 10, is by induction on d using Theorem 5.1.

6.2. k -Schur functions via k -split polynomials. By [25, Property 29], the k -split polynomials $\{G_\lambda^{(k)} \mid \lambda \in \text{Par}^k\}$ form a $\mathbb{Z}[t]$ -basis of Λ^k . We can thus write Λ^k as the direct sum of its free $\mathbb{Z}[t]$ -submodules

$$\tilde{\Omega}^{k,a} := \text{span}_{\mathbb{Z}[t]}\{G_\lambda^{(k)} \mid \lambda \in \text{Par}^k, \lambda_1 = a\} \quad (6.2)$$

over $0 \leq a \leq k$; note that $\tilde{\Omega}^{k,0} := \text{span}_{\mathbb{Z}[t]}\{G_\emptyset^{(k)}\}$ and $G_\emptyset^{(k)} = 1$. Let $\pi^{k,d} : \Lambda^k \rightarrow \Lambda^k$ denote the projection with kernel $\bigoplus_{a>d} \tilde{\Omega}^{k,a}$ and which is the identity on $\bigoplus_{a \leq d} \tilde{\Omega}^{k,a}$. A family of symmetric functions $\{\tilde{A}_\mu^{(k)} \mid \mu \in \text{Par}^k\} \subset \Lambda$ were introduced and conjectured to be k -Schur functions in [25]. They are defined inductively by

$$\tilde{A}_\mu^{(k)} = \begin{cases} \pi^{k,\mu_1}(\mathbf{B}_{\mu_1} \tilde{A}_{(\mu_2, \dots, \mu_\ell)}^{(k)}) & \text{if } \ell = \ell(\mu) > 0, \\ 1 & \text{if } \mu = \emptyset. \end{cases} \quad (6.3)$$

This k -Schur candidate agrees with the k -Schur Catalan functions (proof in Section 10).

Theorem 6.3. *For all $\mu \in \text{Par}^k$, $\tilde{A}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)}$.*

Hence, by combining this with [5, Theorem 2.4], we have identified three of the k -Schur candidates (Theorem 1.8). Properties of each are thus acquired by the others. As one application, the following *k -rectangle property* was proven for the $\tilde{A}_\mu^{(k)}$ [26, Theorem 3], so it is now established for the strong tableau k -Schur functions and k -Schur Catalan functions. A *k -rectangle* is a partition of the form $(k - r + 1)^r$ for $r \in [k]$.

Corollary 6.4. *Let $U = (k - r + 1)^r$ be a k -rectangle and $\mu \in \text{Par}^k$. Then*

$$\mathbf{B}_U s_\mu^{(k)} = \mathbf{B}_U \mathfrak{s}_\mu^{(k)} = \mathbf{B}_U \tilde{A}_\mu^{(k)} = t^d \tilde{A}_{U \sqcup \mu}^{(k)} = t^d \mathfrak{s}_{U \sqcup \mu}^{(k)} = t^d s_{U \sqcup \mu}^{(k)},$$

where d is the number of boxes in the diagram of μ in columns $> r$, and $U \sqcup \mu$ denotes the partition rearrangement of the parts of U and μ .

7. GROMOV-WITTEN INVARIANTS

Using a result of Lam-Shimozono [23], we identify Gromov-Witten invariants with certain k -Catalan-Kostka coefficients at $t = 1$ and apply Theorem 5.1 to give a positive combinatorial formula for a class of these invariants (Theorem 7.6).

We briefly introduce the quantum cohomology ring; further details can be found in [12, 17, 36]. Let Fl_{k+1} be the variety of complete flags in \mathbb{C}^{k+1} . Its cohomology ring $H^*(\text{Fl}_{k+1}) = H^*(\text{Fl}_{k+1}, \mathbb{C})$ has a basis of Schubert classes σ_w indexed by permutations $w \in \mathcal{S}_{k+1}$. The (small) *quantum cohomology ring* $QH^*(\text{Fl}_{k+1})$ is a commutative and associative algebra over $\mathbb{C}[\mathbf{q}] := \mathbb{C}[q_1, \dots, q_k]$. As a $\mathbb{C}[\mathbf{q}]$ -module, $QH^*(\text{Fl}_{k+1}) = \mathbb{C}[\mathbf{q}] \otimes H^*(\text{Fl}_{k+1})$ and thus has a $\mathbb{C}[\mathbf{q}]$ -basis of Schubert classes $\{\sigma_w\}_{w \in \mathcal{S}_{k+1}}$. The multiplication $*$ in $QH^*(\text{Fl}_{k+1})$ is determined by

$$\sigma_u * \sigma_v = \sum_{w \in \mathcal{S}_{k+1}} \sum_{\mathbf{d} \in \mathbb{Z}_{\geq 0}^k} c_{uv}^{w, \mathbf{d}} \mathbf{q}^{\mathbf{d}} \sigma_w, \quad (7.1)$$

where the coefficients $c_{uv}^{w, \mathbf{d}} = \langle \sigma_u, \sigma_v, \sigma_{w_\circ} \rangle_{\mathbf{d}}$ are the 3-point *Gromov-Witten invariants* of genus 0 and w_\circ is the longest element of \mathcal{S}_{k+1} . The $\mathbf{d} = \mathbf{0}$ Gromov-Witten invariants $c_{uv}^{w, \mathbf{0}}$ are the Schubert structure constants.

Let $\bar{\Lambda}^k = \mathbb{C}[h_1, \dots, h_k] = \mathbb{C} \otimes_{\mathbb{Z}[t]} \Lambda^k$, the map $\mathbb{Z}[t] \rightarrow \mathbb{C}$ given by $t \mapsto 1$. The ring $\bar{\Lambda}^k$ has \mathbb{C} -basis $\{\bar{s}_\mu^{(k)} \mid \mu \in \text{Par}^k\}$ by [28, Property 27], where the $\bar{s}_\mu^{(k)}$ are the weak tableau k -Schur functions—see §1.2. Note that $\bar{s}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)}(\mathbf{x}; 1)$ for all $\mu \in \text{Par}^k$ by Theorem 1.8.

Recall (§6.2) that a k -rectangle is a partition of the form $R_i := (i^{k+1-i})$ for $i \in [k]$; set $R_0 = R_{k+1} := \emptyset$. A k -bounded partition is *irreducible* if it has at most $k - i$ parts of size i (equivalently, it contains no k -rectangle as a subsequence).

The next result features the localization $\bar{\Lambda}^k[s_{R_1}^{-1}, \dots, s_{R_k}^{-1}]$ of $\bar{\Lambda}^k$ and it will be useful to have notation for a basis. Accordingly, let fPar^k denote the set of pairs (ν, \mathbf{a}) consisting of an irreducible k -bounded partition ν and a vector $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$. Also set $\mathbf{R}^{\mathbf{a}} := (\emptyset, \mathbf{a}) \in \text{fPar}^k$. For each $(\nu, \mathbf{a}) \in \text{fPar}^k$, define

$$\bar{s}_{(\nu, \mathbf{a})}^{(k)} := \bar{s}_\nu^{(k)} s_{R_1}^{a_1} \cdots s_{R_k}^{a_k} \in \bar{\Lambda}^k[s_{R_1}^{-1}, \dots, s_{R_k}^{-1}]. \quad (7.2)$$

For any $\mu \in \text{Par}^k$, there is a unique irreducible partition, denoted μ_\downarrow , obtained from μ by removing as many k -rectangles as possible; we identify μ with the element $(\mu_\downarrow, \mathbf{a}) \in \text{fPar}^k$, where a_i is the number of rectangles R_i removed. This identification makes the notation (7.2) consistent with earlier usage since the k -rectangle property (Corollary 6.4) at $t = 1$ yields $\bar{s}_\mu^{(k)} = \bar{s}_{\mu_\downarrow}^{(k)} s_{R_1}^{a_1} \cdots s_{R_k}^{a_k} = \bar{s}_{(\mu_\downarrow, \mathbf{a})}^{(k)}$. This computation also shows that $\bar{\Lambda}^k[s_{R_1}^{-1}, \dots, s_{R_k}^{-1}]$ has basis $\{\bar{s}_{(\nu, \mathbf{a})}^{(k)} \mid (\nu, \mathbf{a}) \in \text{fPar}^k\}$. The notation above conveniently handles some products: $\bar{s}_{(\nu, \mathbf{a})}^{(k)} \bar{s}_{\mathbf{R}^{\mathbf{b}}}^{(k)} = \bar{s}_{(\nu, \mathbf{a})}^{(k)} \bar{s}_{(\emptyset, \mathbf{b})}^{(k)} = \bar{s}_{(\nu, \mathbf{a} + \mathbf{b})}^{(k)}$.

For $w = w_1 \cdots w_{k+1} \in \mathcal{S}_{k+1}$ in one-line notation, the *inversion sequence* $\text{Inv}(w) \in \mathbb{Z}_{\geq 0}^{k+1}$ is given by $\text{Inv}_i(w) = |\{j > i : w_i > w_j\}|$. Define an injection $\zeta : \mathcal{S}_{k+1} \rightarrow \text{Par}^k$ as follows: the i -th column of $\zeta(w)$ is

$$\binom{k+1-i}{2} + \text{Inv}_i(w_\circ w) \quad (7.3)$$

for all $i \in [k]$. Set $\theta(w) = \zeta(w)_\downarrow$.

Example 7.1. For $k = 6$ and $w = 1246357$ in one-line notation, the conjugate of $\zeta(w)$ is $(21, 15, 9, 4, 3, 1)$ and $\theta(w) = (4, 3, 2)$.

For $w \in \mathcal{S}_{k+1}$, the *descent set* of w is $D(w) = \{i : w_i > w_{i+1}\}$, and the *descent vector* of w is $\mathbf{D}(w) := \sum_{i \in D(w)} \epsilon_i \in \mathbb{Z}_{\geq 0}^k$. Lam and Shimozono [23] combine powerful results of Givental, Kim, Ginzburg, Kostant, and Peterson [15, 17, 14, 18, 34] to obtain the following:

Theorem 7.2. *There is a ring isomorphism $\Phi : QH^*(\text{Fl}_{k+1})[q_1^{-1}, \dots, q_k^{-1}] \rightarrow \bar{\Lambda}^k[s_{R_1}^{-1}, \dots, s_{R_k}^{-1}]$ which maps the Schubert classes σ_w , $w \in \mathcal{S}_{k+1}$, and the q_i as follows:*

$$\Phi(\sigma_w) = \frac{\bar{s}_{\theta(w)}^{(k)}}{\prod_{i \in D(w)} s_{R_i}} = \frac{\bar{s}_{\theta(w)}^{(k)}}{\bar{s}_{\mathbf{R}^{\mathbf{D}(w)}}^{(k)}}, \quad \Phi(q_i) = \frac{s_{R_{i-1}} s_{R_{i+1}}}{s_{R_i}^2}. \quad (7.4)$$

Proof. This is Theorem 1.1 and Proposition 5.1 of [23] except that we have used an alternative description of $\theta(w)$ (denoted $\lambda(w)$ in [23]). The two descriptions are reconciled in Lemma 7.3 below. \square

Let $c_i = s_{k+1-i} \cdots s_k \in \mathcal{S}_{k+1}$. Any $w \in \mathcal{S}_{k+1}$ has a unique factorization $w = c_k^{m_0} c_{k-1}^{m_1} \cdots c_1^{m_{k-1}}$ with $0 \leq m_i \leq k-i$ for all $i = 0, \dots, k-1$. A word in the positive integers is *cyclically increasing* if some rotation of it is weakly increasing.

Lemma 7.3. *Let $w \in \mathcal{S}_{k+1}$ and set $(I_1, \dots, I_{k+1}) = \text{Inv}(w \circ w)$. For $i \in [k-1]$, the number of parts n_i of size i in $\theta(w)$ has the following descriptions:*

$$n_i = \begin{cases} I_i - I_{i+1} - 1 & \text{if } i \notin D(w), \\ k - i + I_i - I_{i+1} & \text{otherwise,} \end{cases} \quad (7.5)$$

and $n_i = m_i$ for m_i defined by the factorization $w = c_k^{m_0} c_{k-1}^{m_1} \cdots c_1^{m_{k-1}}$ above. This describes $\theta(w)$ completely since it is irreducible and thus only has parts of size $\leq k-1$.

Proof. Since a partition λ has $\lambda'_i - \lambda'_{i+1}$ parts of size i , there are $\tilde{n}_i := k - i + I_i - I_{i+1}$ parts of size i in $\zeta(w)$. If $i \notin D(w)$ then $k - i + 1 \geq I_i > I_{i+1}$, implying that $2(k - i + 1) > \tilde{n}_i > k - i$. On the other hand, if $i \in D(w)$ then $I_i \leq I_{i+1}$, implying $k - i \geq \tilde{n}_i$. Hence $\theta(w) = \zeta(w)_\downarrow$ is obtained from $\zeta(w)$ by removing one copy of the rectangle R_i for $i \notin D(w)$. The formula (7.5) follows.

Now consider the factorization $w = c_k^{m_0} c_{k-1}^{m_1} \cdots c_1^{m_{k-1}}$ as above. Let $i \in [k-1]$. The partial product $c_k^{m_0} c_{k-1}^{m_1} \cdots c_{k-i+1}^{m_{i-1}}$ in one-line notation has the form $\hat{w}v$ where $\hat{w} = w_1 \cdots w_i$, $v = v_1 \cdots v_{k+1-i}$, and $w_i v$ is cyclically increasing with $v_a > v_{a+1}$ for $a \in \{0, 1, \dots, k+1-i\}$ (for corner cases $a = 0$ and $a = k+1-i$, set $v_0 := w_1$ and $v_{k+2-i} := w_i$). It follows that $I_i = a$. Since right multiplication by $c_{k-i}^{m_i}$ cyclically shifts v to the left m_i positions,

$$c_k^{m_0} c_{k-1}^{m_1} \cdots c_{k-i}^{m_i} = \begin{cases} \hat{w} v_{m_i+1} < \cdots < v_a > v_{a+1} < \cdots < v_{k+1-i} < v_1 < \cdots < v_{m_i} & \text{if } i \notin D(w), \\ \hat{w} v_{m_i+1} < \cdots < v_{k+1-i} < v_1 < \cdots < v_a > v_{a+1} < \cdots < v_{m_i} & \text{otherwise,} \end{cases}$$

and $v_{m_i} < v_{m_i+1}$. Therefore, if $i \notin D(w)$, $I_{i+1} = a - m_i - 1$ implies $m_i = I_i - I_{i+1} - 1 = n_i$ and otherwise, $I_{i+1} = k - i - m_i + a$ implies $m_i = I_i - I_{i+1} + k - i = n_i$. \square

For a vector $\mathbf{d} \in \mathbb{Z}^k$, define $\tilde{\mathbf{d}} = \sum_{i \in [k]} d_i (\epsilon_{i-1} - 2\epsilon_i + \epsilon_{i+1}) \in \mathbb{Z}^k$ where $\epsilon_0 = \epsilon_{k+1} := 0$.

Theorem 7.4. *Gromov-Witten invariants are k -Catalan-Kostka coefficients at $t = 1$. Precisely, for $u, v, w \in \mathcal{S}_{k+1}$ and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^k$, let $\mu = \theta(u)$ and $\nu = \theta(v)$, $\Psi = \Delta^k(\mu) \uplus \Delta^k(\nu)$, and $\lambda = (\theta(w), \tilde{\mathbf{d}} + \mathbf{D}(u) + \mathbf{D}(v) - \mathbf{D}(w)) \in \text{fPar}^k$. Then*

$$c_{uv}^{w, \mathbf{d}} = K_{\lambda, \mu\nu}^{\Psi, k}(1)$$

provided λ is identified with an element of Par^k and $c_{uv}^{w, \mathbf{d}} = 0$ otherwise.

Note that $\lambda \in \text{Par}^k$ is equivalent to $\tilde{\mathbf{d}} + \mathbf{D}(u) + \mathbf{D}(v) - \mathbf{D}(w) \in \mathbb{Z}_{\geq 0}^k$.

Proof. By Theorem 7.2, applying Φ to (7.1) gives

$$\frac{\bar{s}_\mu^{(k)}}{\bar{s}_{\mathbf{R}^{\mathbf{D}(u)}}^{(k)}} \frac{\bar{s}_\nu^{(k)}}{\bar{s}_{\mathbf{R}^{\mathbf{D}(v)}}^{(k)}} = \sum_{w, \mathbf{d}} c_{uv}^{w, \mathbf{d}} \Phi(\mathbf{q}^{\mathbf{d}}) \frac{\bar{s}_{\theta(w)}^{(k)}}{\bar{s}_{\mathbf{R}^{\mathbf{D}(w)}}^{(k)}} = \sum_{w, \mathbf{d}} c_{uv}^{w, \mathbf{d}} \frac{\bar{s}_{\theta(w)}^{(k)}}{\bar{s}_{\mathbf{R}^{\mathbf{D}(w)}}^{(k)}} \prod_{i \in [k]} \left(\frac{s_{R_{i-1}} s_{R_{i+1}}}{s_{R_i}^2} \right)^{d_i} = \sum_{w, \mathbf{d}} c_{uv}^{w, \mathbf{d}} \bar{s}_{(\theta(w), \tilde{\mathbf{d}} - \mathbf{D}(w))}^{(k)},$$

where we have used the notation from (7.2). Clearing denominators we obtain

$$\sum_{w, \mathbf{d}} c_{uv}^{w, \mathbf{d}} \bar{s}_{(\theta(w), \tilde{\mathbf{d}} + \mathbf{D}(u) + \mathbf{D}(v) - \mathbf{D}(w))}^{(k)} = \bar{s}_\mu^{(k)} \bar{s}_\nu^{(k)} = H(\Psi; \mu\nu)(\mathbf{x}; 1) = \sum_{\lambda \in \text{Par}^k} K_{\lambda, \mu\nu}^{\Psi, k}(1) \bar{s}_\lambda^{(k)},$$

where the second equality is by Proposition 3.3 together with the fact $\bar{s}_\eta^{(k)}(\mathbf{x}) = \mathfrak{s}_\eta^{(k)}(\mathbf{x}; 1)$ for all $\eta \in \text{Par}^k$. \square

Lemma 7.5. *For $u \in \mathcal{S}_{k+1}$ with only one descent at position j ,*

$$\theta(u)' = (k + 1 - j - \text{Inv}_1(u), \dots, k + 1 - j - \text{Inv}_j(u)).$$

Proof. The partition $\mu = \theta(u)$ has $n_x := \mu'_x - \mu'_{x+1}$ parts of size x and $\ell(\mu) \leq k - 1$ since μ is irreducible. Let $(I_1, \dots, I_{k+1}) = \text{Inv}(w_\circ u)$. Since $D(u) = \{j\}$, (7.5) gives

$$n_i = \begin{cases} I_i - I_{i+1} - 1 & = \text{Inv}_{i+1}(u) - \text{Inv}_i(u) & \text{if } i < j, \\ k - i + I_i - I_{i+1} & = k + 1 - j - \text{Inv}_j(u) & \text{if } i = j, \\ I_i - I_{i+1} - 1 & = 0 & \text{if } i > j, \end{cases}$$

where we have used $I_x = k + 1 - x - \text{Inv}_x(u)$ for all x and $\text{Inv}_x(u) = 0$ for all $x > j$. Therefore $\mu'_i = 0$ for all $i > j$ and $\mu'_j = n_j = k + 1 - j - \text{Inv}_j(u)$. From there, $\mu'_{j-1} - \mu'_j = \text{Inv}_j(u) - \text{Inv}_{j-1}(u)$ implies $\mu'_{j-1} = k + 1 - j - \text{Inv}_{j-1}(u)$, and iterating gives the result. \square

Our main result on Gromov-Witten invariants is obtained by transferring our knowledge of the k -Catalan-Kostka coefficients obtained in Theorem 5.1.

Theorem 7.6. *Let $u, v, w \in \mathcal{S}_{k+1}$ and $\mathbf{d} \in \mathbb{Z}_{\geq 0}^k$. Suppose u has only one descent at position j and $v_{m+1} \cdots v_{k+1}$ is cyclically increasing, where m is the maximum index such that $\text{Inv}_1(u) = \cdots = \text{Inv}_m(u)$. Then*

$$c_{uv}^{w, \mathbf{d}} = \sum_{T \in \text{SSYT}_{U/\theta(u)}([r])} \sum_{\substack{S \in \text{SMT}^k(\text{colword}(T); U\theta(v)) \\ \text{inside}(S) = \lambda}} 1, \quad (7.6)$$

where $r = k + 1 - j - \text{Inv}_1(u)$, $U = R_{k+1-r}$, and $\lambda = (\theta(w), \tilde{\mathbf{d}} + \epsilon_j + \mathbf{D}(v) - \mathbf{D}(w)) \in \text{fPar}^k$.

Note that since the sum requires $\text{inside}(S) = \lambda$, $c_{uv}^{w, \mathbf{d}} = 0$ if $\lambda \notin \text{Par}^k$.

Proof. Let $\mu = \theta(u)$ and $\nu = \theta(v)$. Theorem 7.4 in the case $\mathbf{D}(u) = \epsilon_j$ gives $c_{uv}^{w, \mathbf{d}} = K_{\lambda, \mu\nu}^{\Psi, k}(1)$ with $\Psi = \Delta^k(\mu) \uplus \Delta^k(\nu)$ and λ as in the statement above.

The result then follows by computing $K_{\lambda, \mu\nu}^{\Psi, k}(1)$ using Theorem 5.1; to apply this theorem we need to check (1) $r = \ell(\mu)$ and $\mu_1 \leq k - r + 1$, and (2) $\mu\nu$ is a partition. Lemma 7.5 implies $r = \ell(\mu)$ and $\mu_1 \leq j = k - r + 1 - \text{Inv}_1(u)$, giving (1). For (2), since u has only one descent and $\text{Inv}_1(u) = \cdots = \text{Inv}_m(u)$, Lemma 7.5 implies that the parts of μ are $\geq m$. It remains to show that ν has no part $> m$. Letting $(I_1, \dots, I_{k+1}) = \text{Inv}(w_\circ v)$, we see from (7.5) that this is equivalent to (I) for $x > m$ with $x \in D(v)$, $I_x = I_{x+1} - (k - x)$, and (II) for $x > m$ with $x \notin D(v)$, $I_x = I_{x+1} + 1$. Condition (I) holds only if $I_x = 0$ and $I_{x+1} = k - x$ since $k - x \geq I_{x+1}$. One then checks that these conditions are equivalent to $v_{m+1} \cdots v_{k+1}$ being cyclically increasing. \square

Example 7.7. For $k = 6$, $u = 1246357$, and $v = 1734562$, the expansion of $\sigma_u * \sigma_v$ can be computed using Theorem 7.6 since the only descent of u is at position $j = 4$, $\text{Inv}_1(u) = \text{Inv}_2(u) = 0$, and $v_3 \cdots v_7$ is cyclically increasing. We have $\theta(u) = 432$ and $\theta(v) = 211111$, so we can make use of Example 5.3. In particular, for fixed λ (which depends on w and \mathbf{d} to be determined), we have already computed the right side of (7.6).

For example, for $\lambda = 442111111$, we deduce from (5.2) that the right side of (7.6) is 1. To see which w and \mathbf{d} produce this λ , consider

$$442111111 = \lambda = (\theta(w), \tilde{\mathbf{d}} + \epsilon_j + \mathbf{D}(v) - \mathbf{D}(w)) = (\theta(w), \tilde{\mathbf{d}} + \epsilon_4 + \epsilon_2 + \epsilon_6 - \mathbf{D}(w)).$$

Thus $\theta(w) = \lambda_{\downarrow} = 442$, which implies by Lemma 7.3 that $w = 1245367$ up to left multiplication by some power of $c_k = s_1 \cdots s_k$. For each possibility for w , we can solve for $\tilde{\mathbf{d}}$ and then for \mathbf{d} ; if the answer lies in $\mathbb{Z}_{\geq 0}^k$, then this yields a term of $\sigma_u * \sigma_v$. For example, with $w = 1245367$, this yields $\tilde{\mathbf{d}} = \epsilon_1 - \epsilon_2 - \epsilon_6$ and $\mathbf{d} = \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6$. This accounts for the fourth term below. The other terms are computed similarly.

$$\sigma_u * \sigma_v = \sigma_{1746352} + \sigma_{2745361} + \sigma_{2736451} + q_2 q_3 q_4 q_5 q_6 (\sigma_{1245367} + \sigma_{1236457} + \sigma_{2135467}).$$

8. THE COMBINATORICS OF CATALAN FUNCTIONS

We establish several important lemmas which express a Catalan function as the sum of other Catalan functions with similar indexed root ideals. We begin by introducing bounce graphs, a natural combinatorial object arising in these computations.

8.1. Bounce graphs. These definitions are a review from [5, §5.1]. We say that $\delta \in \Psi$ is a *removable root of Ψ* if $\Psi \setminus \delta$ is a root ideal.

Definition 8.1. Fix a root ideal $\Psi \in \Delta_{\ell}^+$ and $x \in [\ell]$. If there is a removable root (x, j) of Ψ , then define $\text{down}_{\Psi}(x) = j$; otherwise, $\text{down}_{\Psi}(x)$ is undefined. Similarly, if there is a removable root (i, x) of Ψ , then define $\text{up}_{\Psi}(x) = i$; otherwise, $\text{up}_{\Psi}(x)$ is undefined.

Definition 8.2. The *bounce graph* of a root ideal $\Psi \subset \Delta_{\ell}^+$ is the graph on the vertex set $[\ell]$ with edges $(r, \text{down}_{\Psi}(r))$ for each $r \in [\ell]$ such that $\text{down}_{\Psi}(r)$ is defined. The bounce graph of Ψ is a disjoint union of paths called *bounce paths of Ψ* .

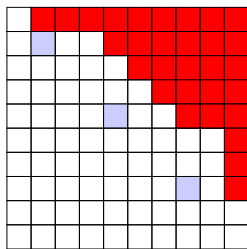
For each vertex $r \in [\ell]$, distinguish $\text{bot}_{\Psi}(r)$ (resp. $\text{top}_{\Psi}(r)$) to be the maximum (resp. minimum) element of the bounce path of Ψ containing r . For $a, b \in [\ell]$ in the same bounce path of Ψ with $a \leq b$, we define

$$\text{path}_{\Psi}(a, b) = \{a, \text{down}_{\Psi}(a), \text{down}_{\Psi}^2(a), \dots, b\},$$

i.e., the set of indices in this path lying between a and b . We also set $\text{downpath}_{\Psi}(r) = \text{path}(r, \text{bot}_{\Psi}(r))$ for any $r \in [\ell]$. For $b = \text{down}_{\Psi}^m(a)$, the *bounce* from a to b is

$$B_{\Psi}(a, b) := |\text{path}_{\Psi}(a, b)| - 1 = m.$$

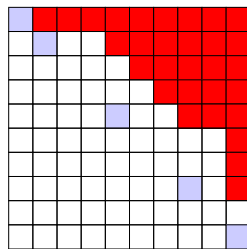
Example 8.3. Examples of path, downpath, and bounce for the root ideal Ψ below:



$$\text{path}_{\Psi}(2, 8) = \{2, 5, 8\}$$

$$\text{downpath}_{\Psi}(1) = \{1, 2, 5, 8, 10\}$$

$$B_{\Psi}(2, 8) = 2, B_{\Psi}(1, 10) = 4, B_{\Psi}(3, 6) = 1, \text{ and } B_{\Psi}(3, 3) = 0.$$



Definition 8.4. A root ideal Ψ is said to have

- a wall in rows $r, r + 1, \dots, r + d$ if the rows $r, \dots, r + d$ of Ψ have the same length,*
- a ceiling in columns $c, c + 1$ if columns c and $c + 1$ of Ψ have the same length, and*
- a mirror in rows $r, r + 1$ if Ψ has removable roots $(r, c), (r + 1, c + 1)$ for some $c > r + 1$.*

Example 8.5. The root ideal Ψ in the previous example has a ceiling in columns 2, 3, in columns 3, 4, and in columns 8, 9, a wall in rows 6, 7, 8 and in rows 9, 10, and a mirror in rows 2, 3, in rows 3, 4, and in rows 4, 5.

8.2. Mirror lemmas. The following lemmas give sufficient conditions for a Catalan function to be zero or for two Catalan functions to be equal.

Lemma 8.6 ([5, Lemma 6.2]). *Let (Ψ, η) be an indexed root ideal of length ℓ and $z \in [\ell - 1]$, and suppose*

$$\Psi \text{ has a ceiling in columns } z, z + 1; \tag{8.1}$$

$$\Psi \text{ has a wall in rows } z, z + 1; \tag{8.2}$$

$$\eta_z = \eta_{z+1} - 1. \tag{8.3}$$

Then $H(\Psi; \eta) = 0$.

Example 8.7. By Lemma 8.6 with $z = 2$, the following Catalan function is zero:

$$H(\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (3, 5)\}; 31211) = \begin{array}{|c|c|c|c|c|} \hline 3 & & & & \\ \hline & 1 & & & \\ \hline & & 2 & & \\ \hline & & & 1 & \\ \hline & & & & 1 \\ \hline \end{array} = 0.$$

Here and throughout the paper, we depict the Catalan function $H(\Psi; \gamma)$ of an indexed root ideal (Ψ, γ) of length ℓ by the $\ell \times \ell$ grid of boxes, labeled by matrix-style coordinates, with the boxes of Ψ shaded and the entries of γ written along the diagonal.

Lemma 8.8 ([5, Lemma 6.4]). *Let (Ψ, η) be an indexed root ideal of length ℓ , and let y, z, w be indices in the same bounce path of Ψ with $1 \leq y \leq z \leq w < \ell$, satisfying*

$$\Psi \text{ has a ceiling in columns } y, y + 1; \tag{8.4}$$

$$\Psi \text{ has a mirror in rows } x, x + 1 \text{ for all } x \in \text{path}_\Psi(y, w) \setminus \{w\}; \tag{8.5}$$

$$\Psi \text{ has a wall in rows } w, w + 1; \tag{8.6}$$

$$\eta_x = \eta_{x+1} \text{ for all } x \in \text{path}_\Psi(y, w) \setminus \{z\}; \tag{8.7}$$

$$\eta_z = \eta_{z+1} - 1. \tag{8.8}$$

Then $H(\Psi; \eta) = 0$.

Here is another useful variant, which is a simplified version of [5, Lemma 6.5].

Lemma 8.9. *Let (Ψ, η) be an indexed root ideal of length ℓ and $z \in [\ell - 1]$, and suppose*

$$\Psi \text{ has a ceiling in columns } z, z + 1; \tag{8.9}$$

$$\Psi \text{ has a wall in rows } z, z + 1; \tag{8.10}$$

$$\eta_z = \eta_{z+1}. \tag{8.11}$$

If Ψ has a removable root δ in row $z + 1$, then $H(\Psi; \eta) = H(\Psi \setminus \delta; \eta)$.

8.3. Downpath lemmas. We prove two lemmas which express a Catalan function as a sum over a downpath of similar Catalan functions. We first recall [5, Corollary 5.7]:

Lemma 8.10. *Let (Ψ, η) be an indexed root ideal of length ℓ and $p \in [\ell]$. Then*

$$H(\Psi; \eta) = \sum_{z \in \text{downpath}_{\Psi}(p)} t^{B_{\Psi}(p,z)} H(\Psi^z; \eta + \epsilon_p - \epsilon_z), \quad (8.12)$$

where $\Psi^z := \Psi \setminus \{(z, \text{down}_{\Psi}(z))\}$ for $z \neq \text{bot}_{\Psi}(p)$ and $\Psi^{\text{bot}_{\Psi}(p)} := \Psi$.

Lemma 8.11. *Let (Ψ, η) be an indexed root ideal of length ℓ , and $r \in [\ell]$, $p \in [r]$. Suppose*

$$\eta_r \geq \eta_{r+1} \geq \cdots \geq \eta_{\ell}; \quad (8.13)$$

$$\bar{\mathbf{r}}(\Psi)_i \geq r - i \text{ for } i \in [r]; \quad (8.14)$$

$$\text{band}(\Psi, \eta)_i \leq k \text{ for } i \leq r \text{ and } \text{band}(\Psi, \eta)_i = \min(k, \ell - i + \eta_i) \text{ for } i > r; \quad (8.15)$$

$$\text{band}(\Psi, \eta)_p < k; \quad (8.16)$$

$$(p < r \text{ and } \bar{\mathbf{r}}(\Psi)_p = \bar{\mathbf{r}}(\Psi)_{p+1}) \implies \bar{\mathbf{r}}(\Psi)_{p+1} \leq \bar{\mathbf{r}}(\Psi)_{p+2} \text{ and } \eta_p = \eta_{p+1} - 1. \quad (8.17)$$

Then

$$H(\Psi; \eta) = \sum_{\{z \in \text{downpath}_{\Psi}(p) \mid z=p \text{ or } \eta_z > \eta_{z+1}\}} t^{B_{\Psi}(p,z)} H(\Psi^z; \eta + \epsilon_p - \epsilon_z), \quad (8.18)$$

where $\Psi^z := \Psi \setminus \{(z, \text{down}_{\Psi}(z))\}$ for $z \neq \text{bot}_{\Psi}(p)$ and $\Psi^{\text{bot}_{\Psi}(p)} := \Psi$.

In the corner case $z = \ell$, we interpret the condition $\eta_z > \eta_{z+1}$ by setting $\eta_{\ell+1} = 0$ (though we still regard (Ψ, η) as an indexed root ideal of length ℓ , not $\ell + 1$).

Proof. Lemma 8.10 gives

$$H(\Psi; \eta) = \sum_{z \in \text{downpath}_{\Psi}(p)} t^{B_{\Psi}(p,z)} H(\Psi^z; \eta + \epsilon_p - \epsilon_z).$$

If the sum contains a term with $z = \ell > p$ and $\eta_{\ell} = 0$, then $H(\Psi^z; \eta + \epsilon_p - \epsilon_z) = 0$; this follows directly from Definition 1.1 since for $\gamma \in \mathbb{Z}^{\ell}$, $s_{\gamma} = 0$ whenever $\gamma_{\ell} < 0$ (see (1.1)).

This given, the result is now obtained by applying Lemma 8.8 (with Ψ^z in place of Ψ , $\tilde{\eta} := \eta + \epsilon_p - \epsilon_z$ in place of η , $w = z$, and y defined below) to show that $H(\Psi^z; \tilde{\eta}) = 0$ whenever $p < z < \ell$ and $\eta_z = \eta_{z+1}$. We verify the hypotheses (8.4)–(8.8): since $z \neq p$, (8.14) implies $z \geq \text{down}_{\Psi}(p) > r$. Hence $\eta_z = \eta_{z+1}$ and (8.15) imply Ψ^z has a wall in rows $z, z + 1$. Let $y + 1 = \text{top}_{\Psi^z}(z + 1)$. Since $\text{up}_{\Psi^z}(y + 1)$ is undefined, Ψ^z has a ceiling in columns $y, y + 1$.

Since $\text{down}_{\Psi}(p)$ is defined, $\bar{\mathbf{r}}(\Psi)_p \leq \bar{\mathbf{r}}(\Psi)_{p+1}$. If $p = r$, then $\text{band}(\Psi, \eta)_r < k$, $\text{band}(\Psi, \eta)_{r+1} = \min(k, \ell - i + \eta_{r+1})$, $\eta_r \geq \eta_{r+1}$, and $z < \ell$ imply $\bar{\mathbf{r}}(\Psi)_p < \bar{\mathbf{r}}(\Psi)_{p+1}$. The two cases below therefore exhaust all possibilities and complete the proof.

Case 1: $\bar{\mathbf{r}}(\Psi)_p < \bar{\mathbf{r}}(\Psi)_{p+1}$. In this case, $\text{up}_{\Psi}(q + 1)$ is undefined, where $q := \text{down}_{\Psi}(p)$. It then follows from $\text{down}_{\Psi}(p) > r$, (8.13), and (8.15) that $y + 1 \geq q + 1$ and

$$\{x + 1 \mid x \in \text{path}_{\Psi^z}(y, z)\} = \text{path}_{\Psi^z}(y + 1, z + 1). \quad (8.19)$$

Then (8.15) implies $\tilde{\eta}_x = \tilde{\eta}_{x+1}$ for all $x \in \text{path}_{\Psi^z}(y, z) \setminus \{z\}$. This verifies (8.5) and (8.7).

Case 2: $p < r$ and $\bar{\mathbf{r}}(\Psi)_p = \bar{\mathbf{r}}(\Psi)_{p+1}$. The same argument from case 1, using in addition that $\text{down}_{\Psi}(p) + 1 = \text{down}_{\Psi}(p + 1)$ and $\tilde{\eta}_p = \tilde{\eta}_{p+1}$ by (8.17), verifies (8.5) and (8.7). \square

The next result identifies a case where acting by a strong Pieri operator on a k -Schur function $\mathfrak{s}_\eta^{(k)}$ involves only strong marked covers with height 1 ribbon components; this is equivalent to the resulting sum of k -Schur functions being over only partitions contained in η (see Definition 7.9, the proof of Proposition 8.13, and Equation 9.6 of [5]).

Corollary 8.12. *Let $\eta \in \text{Par}_\ell^k$ and $\Phi = \Delta^k(\eta)$. Let $r \in [\ell]$ and $p \in [r]$. Assume that $\bar{\mathbf{r}}(\Phi)_i \geq r - i$ for $i \in [r]$ and that $p = r < \ell$ implies $\eta_r > \eta_{r+1}$. Then*

$$\mathfrak{s}_\eta^{(k)} \cdot u_p = H(\Phi; \eta - \epsilon_p) = \sum_{\{z \in \text{downpath}_\Phi(p) \mid \eta - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_\Phi(p,z)} \mathfrak{s}_{\eta - \epsilon_z}^{(k)}. \quad (8.20)$$

Proof. Lemma 8.11 yields

$$H(\Phi; \eta - \epsilon_p) = \sum_{\{z \in \text{downpath}_\Phi(p) \mid z=p \text{ or } \eta - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_\Phi(p,z)} H(\Phi^z; \eta - \epsilon_z), \quad (8.21)$$

where Φ^z is as in (8.18). For the terms with $\eta - \epsilon_z \in \text{Par}_\ell^k$, we have $\Phi^z = \Delta^k(\eta - \epsilon_z)$ and hence $H(\Phi^z; \eta - \epsilon_z) = \mathfrak{s}_{\eta - \epsilon_z}^{(k)}$. It remains to show the $z = p$ term is 0 when $\eta - \epsilon_p \notin \text{Par}_\ell^k$. If $p = \ell$, then $\eta_p = 0$ and $H(\Phi^p; \eta - \epsilon_p) = 0$ follows directly from Definition 1.1 since for $\gamma \in \mathbb{Z}^\ell$, $s_\gamma = 0$ whenever $\gamma_\ell < 0$ (see (1.1)). If $p < \ell$, then $\eta_p = \eta_{p+1}$. So $p < r$ by the second assumption and then Φ has a ceiling in columns $p, p + 1$ by the first. Thus $H(\Phi^p; \eta - \epsilon_p) = 0$ by Lemma 8.6. \square

9. SCHUR TIMES k -SCHUR: PROOF OF THEOREM 5.1

In Theorem 9.4 below, we give a positive combinatorial formula for the k -Schur expansion of a large class of Catalan functions which extrapolate between the Catalan functions corresponding to Schur times k -Schur of Theorem 5.1 and single k -Schur functions. This result strengthens Theorem 5.1 in a way that allows for a proof by induction on the size of the root ideal. We begin by introducing a generalization of the tableaux $\text{SSYT}_{R/\mu}([r])$ from Theorem 5.1 needed for the more general Theorem 9.4.

9.1. Flagged tableaux. A *pseudopartition* of length r is a sequence

$$\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}_{\geq 0}^r \quad \text{satisfying} \quad \alpha_1 + r - 1 \geq \alpha_2 + r - 2 \geq \dots \geq \alpha_r.$$

For a pseudopartition α of length r with $m = \max(\alpha)$, define

$$\text{diagram}(\alpha) := \{(i, j) \in [r] \times [m] \mid m - \alpha_i < j \leq m\}.$$

Definition 9.1. For a pseudopartition α of length r and flagging $\mathbf{n} = (n_1, \dots, n_r) \in [r]^r$, define $\text{FTAB}_\alpha(\mathbf{n})$ to be the set of fillings of $\text{diagram}(\alpha)$ with integers which are (I) weakly increasing left to right along rows and (II) strictly increasing down columns, and satisfy the following flag conditions: (III) entries in row i lie in $\{n_i, n_i + 1, \dots, r\}$, and (IV) if $i \in \{2, 3, \dots, r\}$, $(i - 1, j) \in \text{diagram}(\alpha)$, and $(i, j) \notin \text{diagram}(\alpha)$, then the entry in box $(i - 1, j)$ is less than n_i . In the case $\alpha = 0^r$ (and \mathbf{n} any element of $[r]^r$), $\text{FTAB}_\alpha(\mathbf{n})$ consists of a single tableau with no boxes.

For $T \in \text{FTAB}_\alpha(\mathbf{n})$, the *column reading word* of T , denoted $\text{colword}(T)$, is the word obtained by concatenating the columns of T (reading each column bottom to top), starting with the leftmost column.

If $T \in \text{FTAB}_\alpha(\mathbf{n})$, $x \in \mathbb{Z}$, and $i \in [\ell(\alpha)]$, then $\boxed{x}_i \sqcup T$ denotes the filling of $\text{diagram}(\alpha + \epsilon_i)$ obtained from T by adding a box with entry x to row i on the left.

Example 9.2. With $r = 4$, $\alpha = 2121$, and $\mathbf{n} = 1224$, we have

$$\text{FTAB}_\alpha(\mathbf{n}) = \left\{ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 2 \\ \hline 2 & 3 \\ \hline & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 2 \\ \hline 3 & 3 \\ \hline & 4 \\ \hline \end{array} \right\}.$$

See also Example 9.7. If T is the tableau on the left, then $\boxed{2}_3 \sqcup T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline & 2 \\ \hline 2 & 2 & 3 \\ \hline & & 4 \\ \hline \end{array}$.

Proposition 9.3. *Let $i \in [r]$. Let α and $\alpha^- = \alpha - \epsilon_i$ be pseudopartitions of length r with $\alpha_i \geq \alpha_{i-1}$. Let $\mathbf{n} = (n_1, n_2, \dots, n_r) \in [r]^r$ and $\mathbf{n}^+ = \mathbf{n} + \epsilon_i$ be weakly increasing. Then*

$$\text{FTAB}_\alpha(\mathbf{n}) = \text{FTAB}_\alpha(\mathbf{n}^+) \sqcup \left\{ \boxed{n_i}_i \sqcup U : U \in \text{FTAB}_{\alpha^-}(\mathbf{n}) \right\}.$$

Proof. Consider $T \in \text{FTAB}_\alpha(\mathbf{n})$. It either contains n_i in the leftmost box in the i -th row or not. If it contains n_i , then removing this box yields an element T^- of $\text{FTAB}_{\alpha^-}(\mathbf{n})$ (note T^- satisfies (IV) by column strictness of T), and if not, then $T \in \text{FTAB}_\alpha(\mathbf{n}^+)$. This establishes the inclusion \subset . To show the inclusion \supset , we show that $\text{FTAB}_\alpha(\mathbf{n}^+)$ and $\left\{ \boxed{n_i}_i \sqcup U : U \in \text{FTAB}_{\alpha^-}(\mathbf{n}) \right\}$ are each contained in $\text{FTAB}_\alpha(\mathbf{n})$. The former comes down to the following: $\alpha_i \geq \alpha_{i-1}$ implies that condition (IV) for $T \in \text{FTAB}_\alpha(\mathbf{n}^+)$ yields condition (IV) for T considered as an element of $\text{FTAB}_\alpha(\mathbf{n})$. For the latter, let $T^- \in \text{FTAB}_{\alpha^-}(\mathbf{n})$; we must verify conditions (I)–(IV) for $\boxed{n_i}_i \sqcup T^-$: (I) and (III) are straightforward and (IV) follows from $n_i < n_{i+1}$; (II) follows from $n_i < n_{i+1}$ and condition (IV) for T^- . \square

9.2. A generalization of Schur times k -Schur. For a sequence $\beta \in \mathbb{Z}^r$, define $\text{crop}(\beta) \in \mathbb{Z}^r$ to be the partition given by

$$\text{crop}(\beta)_i := \min\{\beta_j \mid j \in [i]\}.$$

For example, $\text{crop}(45653213) = 44443211$.

Theorem 9.4. *Fix positive integers k, ℓ and $r \in [\ell]$. Let $(\Psi, \mu\nu)$ be an indexed root ideal of length ℓ satisfying*

$$\mu \text{ is a pseudopartition of length } r; \tag{9.1}$$

$$\nu \in \text{Par}_{\ell-r}^k \text{ and } \mu_r \geq \nu_1; \tag{9.2}$$

$$\bar{\mathbf{r}}(\Psi)_i \geq r - i \text{ for } i \in [r]; \tag{9.3}$$

$$\text{band}(\Psi, \mu\nu)_i \leq k \text{ for } i \leq r \text{ and } \text{band}(\Psi, \mu\nu)_i = \min(k, \ell - i + (\mu\nu)_i) \text{ for } i > r. \tag{9.4}$$

Define $\zeta = (k - \bar{\mathbf{r}}(\Psi)_1, k - \bar{\mathbf{r}}(\Psi)_2, \dots, k - \bar{\mathbf{r}}(\Psi)_r)$, $\lambda = \text{crop}(\zeta)$, and $\alpha = \lambda - \mu$. Let $\mathbf{n} = (n_1, \dots, n_r) \in [r]^r$ be the sequence given by $n_i = i - (\zeta_i - \lambda_i)$. Also assume

$$\alpha_i \geq 0 \text{ for all } i \in [r]. \tag{9.5}$$

Then

$$H(\Psi; \mu\nu) = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} \mathfrak{s}_{\lambda\nu}^{(k)} \cdot u_{\text{colword}(T)}. \tag{9.6}$$

Note that $\zeta \in \mathbb{Z}_{\geq 0}^r$ by (9.4) and then $\lambda\nu \in \text{Par}_\ell^k$ since $\lambda_1 = \zeta_1 \leq k$ and $\lambda_r \geq \mu_r \geq \nu_1$. Also $\mathbf{n} \in [r]^r$ as required by Definition 9.1 since $n_1 = 1 - (\zeta_1 - \lambda_1) = 1$ and \mathbf{n} is weakly increasing by Proposition 9.5 (e) below.

Let us now see how Theorem 5.1 is obtained as a special case of Theorem 9.4: let $\mu \in \text{Par}_r^{k-r+1}$ and $\nu \in \text{Par}^k$ be as in Theorem 5.1; set $\ell = \max(\ell(\mu\nu), r)$ so that $\nu \in \text{Par}_{\ell-r}^k$ and $\mu\nu \in \text{Par}_\ell^{k-r+1}$ (note that we allow $\nu = \emptyset$). Set $\Psi = \emptyset_r \uplus \Delta^k(\nu)$. With this input to Theorem 9.4, we have $\lambda = (k-r+1)^r$ and $\mathbf{n} = (1, \dots, 1)$. The latter implies $\text{FTAB}_\alpha(\mathbf{n}) = \text{SSYT}_{\lambda/\mu}([r])$, which matches the right side of (5.1) with the right side of (9.6); the left sides match by Proposition 3.2 and (3.4). The hypotheses of Theorem 9.4 are satisfied: (9.1)–(9.3) are clear, (9.4) holds by definition of Ψ and $\mu_1 \leq k-r+1$, and (9.5) holds by $\alpha_i = \lambda_i - \mu_i \geq k-r+1 - \mu_1 \geq 0$.

In the next proposition, we handle several technical preliminaries for the proof of Theorem 9.4. For statement (a), we define the root ideal $\Delta^k(\gamma)$ for pseudopartitions $\gamma \in \mathbb{Z}_{\leq k}^\ell$ just as in Definition 1.2.

Proposition 9.5. *Maintain the notation of Theorem 9.4. Set $\Phi = \Delta^k(\lambda\nu)$.*

- (a) $\Delta^k(\lambda\nu) = \Phi \subset \Psi = \Delta^k(\zeta\nu)$.
- (b) For $j \in [r-1]$, $\zeta_j < \zeta_{j+1} \iff \Psi$ has a wall in rows $j, j+1$.
- (c) For $j \in [r-1]$, $\alpha_j > \alpha_{j+1}$ and $\zeta_j \leq \zeta_{j+1}$ imply $\mu_j = \mu_{j+1} - 1$ and $\alpha_j = \alpha_{j+1} + 1$.
- (d) $(r, \text{down}_\Psi(r))$ is a removable root of Ψ provided $\ell > r+k$.
- (e) The sequence \mathbf{n} is weakly increasing.
- (f) $\lambda_{n_p} = \lambda_{n_p+1} = \dots = \lambda_p$.
- (g) $\text{downpath}_\Phi(n_p) \setminus \{n_p\} = \text{downpath}_\Psi(p) \setminus \{p\}$ provided $\ell > r+k$.

Proof. We first prove (a). We have $\Delta^k(\zeta\nu) = \Psi$ by (9.4) and the definitions of ζ and Δ^k . By definition of crop, $\zeta_i \geq \lambda_i$ for all $i \in [r]$. Then by definition of Δ^k , $\bar{\mathbf{r}}(\Phi)_i = k - \lambda_i \geq k - \zeta_i = \bar{\mathbf{r}}(\Psi)_i$ for $i \in [r]$. Hence $\Phi \subset \Psi$. Statement (b) is clear from the fact that Ψ is a root ideal. For (c), $\zeta_j \leq \zeta_{j+1}$ and the definition of crop imply $\lambda_j = \lambda_{j+1}$. Then $\alpha_j > \alpha_{j+1}$ and $\mu = \lambda - \alpha$ yield $\mu_j < \mu_{j+1}$. Since μ is a pseudopartition, we obtain $\mu_j = \mu_{j+1} - 1$ and $\alpha_j = \alpha_{j+1} + 1$. Statement (d) follows from $\mu_r \geq \nu_1$ and $\text{band}(\Psi, \mu\nu)_r \leq k = \text{band}(\Psi, \mu\nu)_{r+1}$, where the latter holds by (9.4) and $\ell > r+k$.

Let $i \in [r-1]$. Since $\bar{\mathbf{r}}(\Psi)_{i+1} \geq \bar{\mathbf{r}}(\Psi)_i - 1$, we have $\zeta_{i+1} - \zeta_i \leq 1$ and hence

$$\zeta_{i+1} - \lambda_{i+1} - (\zeta_i - \lambda_i) = \begin{cases} -(\zeta_i - \lambda_i) & \text{if } \zeta_{i+1} = \min\{\zeta_j \mid j \in [i+1]\} = \lambda_{i+1} \\ \zeta_{i+1} - \zeta_i & \text{if } \zeta_{i+1} > \min\{\zeta_j \mid j \in [i]\} = \lambda_i \end{cases}$$

is also ≤ 1 . Statement (e) follows.

Let $a \in [p]$ be any index such that $\zeta_a = \lambda_p := \min_{j \in [p]} \{\zeta_j\}$. Then λ is constant on the interval $\{a, a+1, \dots, p\}$. Since $\zeta_{p+1} - \zeta_p \leq 1$ and $\zeta_p - \zeta_a \leq p-a$, there holds

$$n_p = p - (\zeta_p - \lambda_p) = p - \zeta_p + \zeta_a \geq a.$$

Statement (f) follows.

Since Φ and Ψ agree in rows $> r$, statement (g) follows from the computation

$$\text{down}_\Phi(n_p) - 1 = n_p + \bar{\mathbf{r}}(\Phi)_{n_p} = p - (\zeta_p - \lambda_p) + k - \lambda_{n_p} = p + k - \zeta_p = p + \bar{\mathbf{r}}(\Psi)_p = \text{down}_\Psi(p) - 1,$$

where the third equality is by (f). \square

We also need the following result to adjust trailing zeros of ν in the proof below.

Proposition 9.6 ([5, Proposition 4.12]). *If (Ψ, γ) is an indexed root ideal of length ℓ with $\gamma_\ell = 0$, then $H(\Psi; \gamma) = H(\Psi \cap \Delta_{\ell-1}^+; (\gamma_1, \dots, \gamma_{\ell-1}))$.*

Proof of Theorem 9.4. The proof is by induction, beginning with several reductions, and then addressing the left side of (9.6) using Lemma 8.11 (Step 1). The terms in the sum are evaluated in Steps 2 and 3. Then in Step 4, the right side of (9.6) is evaluated using Corollary 8.12 and matched with the result from Step 3. Steps 2–4 split into two cases.

Step 0: Base case and reductions.

First, we can reduce to the case $\ell > r + k$ since adding extra zeros to ν does not change (9.6): precisely, if $\ell \leq r + k$ and we let $\tilde{\nu} = (\nu, 0^{r+k+1-\ell})$ and $\tilde{\Psi} \subset \Delta_{r+k+1}^+$ be the root ideal with $\bar{\mathbf{r}}(\tilde{\Psi})_i = \bar{\mathbf{r}}(\Psi)_i$ for $i \leq r$ and $\bar{\mathbf{r}}(\tilde{\Psi})_i = \min(k - (\mu\tilde{\nu})_i, r + k + 1 - i)$ for $i > r$, then

$$H(\Psi; \mu\nu) = H(\tilde{\Psi}; \mu\tilde{\nu}) = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} \mathfrak{s}_{\lambda\tilde{\nu}}^{(k)} \cdot u_{\text{colword}(T)} = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} \mathfrak{s}_{\lambda\nu}^{(k)} \cdot u_{\text{colword}(T)}.$$

The first equality is by Proposition 9.6, noting that $\tilde{\Psi} \cap \Delta_\ell^+ = \Psi$ follows from (9.4); the second is by the result in the $\ell > r + k$ case; the third holds by $\mathfrak{s}_{\lambda\nu}^{(k)} = \mathfrak{s}_{(\lambda\nu, 0)}^{(k)} = \mathfrak{s}_{(\lambda\nu, 0, 0)}^{(k)} = \dots$ (by Proposition 9.6 again) and the definition (2.1) of the strong Pieri operators.

From now on we assume $\ell > r + k$. We proceed by induction on $|\Psi| + |\alpha|$. By Proposition 9.5 (a), the base case is $\alpha = 0^r$ and $\Psi = \Delta^k(\lambda\nu)$. The desired (9.6) holds in this case since $H(\Psi; \mu\nu) = H(\Delta^k(\lambda\nu); \lambda\nu) = \mathfrak{s}_{\lambda\nu}^{(k)}$ and $\text{FTAB}_\alpha(\mathbf{n})$ consists of a single tableau with no boxes.

We now establish that for any $i \in [r - 1]$,

$$\alpha_i > \alpha_{i+1} \text{ and } \Psi \text{ has a wall in rows } i, i + 1 \implies \text{both sides of (9.6) are 0.} \quad (9.7)$$

Accordingly, suppose the left side of (9.7) is true. We have

$$\Psi \text{ has a wall in rows } i, i + 1 \implies \zeta_i = \zeta_{i+1} - 1 \implies \lambda_i = \lambda_{i+1}. \quad (9.8)$$

This gives $\mu_i = \mu_{i+1} - 1$ by Proposition 9.5 (c) as well as

$$n_i = i - (\zeta_i - \lambda_i) = i + 1 - (\zeta_{i+1} - \lambda_{i+1}) = n_{i+1}.$$

Therefore the right side of (9.6) is 0 since $\text{FTAB}_\alpha(\mathbf{n}) = \emptyset$ whenever $\alpha_i > \alpha_{i+1}$ and $n_i = n_{i+1}$ (by condition (IV) in Definition 9.1) and the left side is 0 by Lemma 8.6 (with $z = i$). Hence (9.7) is established.

We next establish that for any $i \in [r - 1]$,

$$\Psi \text{ has a wall in rows } i, i + 1 \text{ and } \alpha_i = \max\{\alpha_i, \dots, \alpha_r\} \implies (9.6) \text{ holds.} \quad (9.9)$$

Let $j \in [i + 1, r]$ be the largest number such that Ψ has a wall in rows $i, i + 1, \dots, j$. By (9.7) and $\alpha_i = \max\{\alpha_i, \dots, \alpha_r\}$, we can assume $\alpha_i = \dots = \alpha_j$. By Proposition 9.5 (b), and the definition of crop, $\lambda_i = \dots = \lambda_j$. Then $\mu_{j-1} = \mu_j$ (using $\mu = \lambda - \alpha$). By definition

of j (and by Proposition 9.5 (d) if $j = r$), $\delta := (j, \text{down}_\Psi(j))$ is a removable root of Ψ . The last two sentences show that we can apply Lemma 8.9 (with $z = j - 1$) to obtain

$$H(\Psi; \mu\nu) = H(\Psi \setminus \delta; \mu\nu) = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n}^+)} \mathfrak{s}_{\lambda\nu}^{(k)} \cdot u_{\text{colword}(T)} = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} \mathfrak{s}_{\lambda\nu}^{(k)} \cdot u_{\text{colword}(T)}.$$

The second equality is by the inductive hypothesis with data $\zeta - \epsilon_j$ in place of ζ , $\mathbf{n}^+ = \mathbf{n} + \epsilon_j$ in place of \mathbf{n} , and μ, λ, α unchanged (the hypotheses (9.1)–(9.3) and (9.5) are clear while (9.4) follows from $\text{band}(\Psi, \mu\nu)_j = \text{band}(\Psi, \mu\nu)_{j-1} - 1 < k$). The third equality is by $\text{FTAB}_\alpha(\mathbf{n}^+) = \text{FTAB}_\alpha(\mathbf{n})$, which follows from $\alpha_{j-1} = \alpha_j$ and

$$n_{j-1} = j - 1 - (\zeta_{j-1} - \lambda_{j-1}) = j - 1 - (\zeta_j - 1 - \lambda_j) = n_j.$$

Next suppose $\alpha = 0^r$. Since we have already handled the base case, we may assume $\Delta^k(\zeta\nu) = \Psi \supseteq \Delta^k(\lambda\nu) = \Delta^k(\mu\nu)$ (see Proposition 9.5 (a)). Then $\lambda \neq \zeta$, so there exists an $i \in [r - 1]$ such that $\zeta_i < \zeta_{i+1}$. Hence Ψ has a wall in rows $i, i + 1$, so we are done by (9.9).

Step 1: expand the left side of (9.6).

We can now assume $|\alpha| > 0$. Set

$$p = \max\{j \in [r] \mid \alpha_j = \max(\alpha)\}.$$

If $p = r$, then $\delta := (p, \text{down}_\Psi(p))$ is a removable root of Ψ by Proposition 9.5 (d). If $p < r$ and Ψ has a wall in rows $p, p + 1$, then we are done by (9.7). So from now on we may assume δ is a removable root of Ψ . In addition, by (9.9), we may assume that if $p + 1 < r$ and $\alpha_{p+1} = \max\{\alpha_{p+1}, \dots, \alpha_r\}$, then $\bar{\mathbf{r}}(\Psi)_{p+1} \leq \bar{\mathbf{r}}(\Psi)_{p+2}$.

Lemma 8.11 yields

$$H(\Psi; \mu\nu) = \sum_{\{z \in \text{downpath}_\Psi(p) \mid z=p \text{ or } (\mu\nu)_z > (\mu\nu)_{z+1}\}} t^{B_\Psi(p,z)} H(\Psi^z; \mu\nu + \epsilon_p - \epsilon_z), \quad (9.10)$$

where $\Psi^z := \Psi \setminus \{(z, \text{down}_\Psi(z))\}$ for $z \neq \text{bot}_\Psi(p)$ and $\Psi^{\text{bot}_\Psi(p)} := \Psi$. We must verify the hypotheses of the lemma: (8.13) is clear from (9.1) and (9.2), (8.14) is immediate from (9.3), (8.15) is immediate from (9.4), and (8.16) follows from

$$\text{band}(\Psi, \mu\nu)_p = \bar{\mathbf{r}}(\Psi)_p + \mu_p \leq k - \lambda_p + \mu_p < k, \quad (9.11)$$

where the first inequality is by $\lambda_p \leq \zeta_p$ and the second is by $\lambda_p - \mu_p = \alpha_p > 0$. Finally, we verify (8.17): suppose $p < r$ and $\bar{\mathbf{r}}(\Psi)_p = \bar{\mathbf{r}}(\Psi)_{p+1}$. By Proposition 9.5 (c), $\mu_p = \mu_{p+1} - 1$ and $\alpha_p = \alpha_{p+1} + 1$. The latter implies $\alpha_{p+1} = \max\{\alpha_{p+1}, \dots, \alpha_r\}$. The conclusion $\bar{\mathbf{r}}(\Psi)_{p+1} \leq \bar{\mathbf{r}}(\Psi)_{p+2}$ thus follows from the previous paragraph if $p + 1 < r$ and from Proposition 9.5 (d) if $p + 1 = r$.

Step 2: apply the inductive hypothesis to the terms arising in Step 1.

We proceed by applying the inductive hypothesis to any term from the sum in (9.10) with $z > p$ (with $\mu^+ = \mu + \epsilon_p$ in place of μ , $\alpha^- = \alpha - \epsilon_p$ in place of α , Ψ^z in place of Ψ , and $\lambda, \zeta, \mathbf{n}$ unchanged); the hypotheses are readily verified: (9.2) and (9.3) are clear, (9.5) holds since $\alpha_p > 0$, and (9.4) holds by (9.11) and the fact that Ψ^z and $\Delta^k(\mu\nu + \epsilon_p - \epsilon_z)$ agree in rows $> r$. For (9.1), we have

$$\lambda_{p-1} \geq \lambda_p \text{ and } \alpha_{p-1} \leq \alpha_p \text{ and } \mu = \lambda - \alpha \implies \mu_{p-1} \geq \mu_p \implies \mu^+ \text{ is a pseudopartition.}$$

Hence by the inductive hypothesis,

$$H(\Psi^z; \mu\nu + \epsilon_p - \epsilon_z) = \mathfrak{s}_{\lambda\nu - \epsilon_z}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}. \quad (9.12)$$

We next consider the $z = p$ term $H(\Psi \setminus \delta; \mu\nu)$ of (9.10). Set

$$\zeta^- = \zeta - \epsilon_p \quad \text{and} \quad \lambda^- = \text{crop}(\zeta^-),$$

which is part of the new data for this term (the new α and \mathbf{n} will be addressed later). The remainder of the proof separates into the cases $\lambda \neq \lambda^-$ (case 1) and $\lambda = \lambda^-$ (case 2); case 1 breaks into the two subcases below.

Case 1A: λ^- and λ differ in more than one position.

It follows from the definition of crop that $\lambda_p = \lambda_{p+1}$, $\lambda_p^- = \lambda_{p+1}^- = \lambda_p - 1$, and $\zeta_p^- = \lambda_p^-$. This gives $\zeta_{p+1}^- = \zeta_{p+1} \geq \lambda_{p+1} > \lambda_p^- = \zeta_p^-$, and thus $\Psi \setminus \delta$ has a wall in rows $p, p+1$ by Proposition 9.5 (b). Moreover, $\lambda_p = \lambda_{p+1}$, $\alpha_p > \alpha_{p+1}$, and μ a pseudopartition give $\mu_p = \mu_{p+1} - 1$. Hence by Lemma 8.6 (with $z = p$),

$$H(\Psi \setminus \delta; \mu\nu) = 0. \quad (9.13)$$

Case 1B: $\lambda^- = \lambda - \epsilon_p$.

We apply the inductive hypothesis to the term $H(\Psi \setminus \delta; \mu\nu)$ with data ζ^- , λ^- as mentioned above, $\alpha^- = \lambda^- - \mu = \alpha - \epsilon_p$, and \mathbf{n} unchanged since

$$\zeta^- - \text{crop}(\zeta^-) = \zeta - \epsilon_p - \lambda^- = \zeta - \epsilon_p - (\lambda - \epsilon_p) = \zeta - \lambda.$$

The hypotheses are satisfied: (9.1), (9.2), and (9.3) are clear, (9.4) follows from (9.11), and (9.5) holds since $\alpha_p > 0$. The inductive hypothesis thus gives

$$H(\Psi \setminus \delta; \mu\nu) = \mathfrak{s}_{\lambda^- \nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}. \quad (9.14)$$

Case 2: $\lambda^- = \lambda$.

We apply the inductive hypothesis to the term $H(\Psi \setminus \delta; \mu\nu)$ with data ζ^- , λ^- as mentioned above, α unchanged, and $\mathbf{n}^+ := \mathbf{n} + \epsilon_p$, which follows from

$$\zeta^- - \lambda^- = \zeta - \epsilon_p - \lambda^- = (\zeta - \lambda) - \epsilon_p.$$

The hypotheses are satisfied: (9.1), (9.2), (9.3), and (9.5) are clear, and (9.4) follows from (9.11). Hence by the inductive hypothesis,

$$H(\Psi \setminus \delta; \mu\nu) = \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n}^+)} u_{\text{colword}(T)}. \quad (9.15)$$

Step 3 (case 1): assemble the terms from step 2.

In this case ($\lambda \neq \lambda^-$), combining (9.10), (9.12), (9.13), and (9.14) yields

$$H(\Psi; \mu\nu) = \sum_{\{z \in \text{downpath}_{\Psi}(p) \mid \lambda\nu - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_{\Psi}(p,z)} \left(\mathfrak{s}_{\lambda\nu - \epsilon_z}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} \right). \quad (9.16)$$

Note that this holds in both subcases of case 1: in case 1A, $\lambda_p = \lambda_{p+1}$ and so there is no $z = p$ term in this sum as required by (9.13); in case 1B, $\lambda^- \nu = \lambda \nu - \epsilon_p$ is a partition, and therefore this sum has a term for $z = p$ and it agrees with the right side of (9.14).

Step 4 (case 1): evaluate the right side of (9.6) to match it with (9.16).

It follows from the definition of crop and $\lambda^- \neq \lambda$ that $\lambda_p := \min_{j \in [p]}(\zeta_j) = \zeta_p$; in addition, if $p < r$, then $\zeta_p \geq \zeta_{p+1}$ (since δ is removable) and therefore $\lambda_{p+1} := \min_{j \in [p+1]}(\zeta_j) = \zeta_{p+1}$. Hence $n_p = p$ and, if $p < r$, $n_{p+1} = p + 1$. Then by Proposition 9.3, $\text{FTAB}_\alpha(\mathbf{n}) = \{\overline{[n_p]}_p \sqcup T : T \in \text{FTAB}_{\alpha^-}(\mathbf{n})\}$ (condition (IV) of Definition 9.1 implies $\text{FTAB}_\alpha(\mathbf{n} + \epsilon_p) = \emptyset$). The consequence for column reading words is thus

$$\sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} u_{\text{colword}(T)} = u_{n_p} \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}. \quad (9.17)$$

We need to compute the action of the operator in (9.17) on $\mathfrak{s}_{\lambda \nu}^{(k)}$. Set $\Phi = \Delta^k(\lambda \nu)$. By Corollary 8.12 and the description (2.2) of the strong Pieri operators,

$$\mathfrak{s}_{\lambda \nu}^{(k)} \cdot u_{n_p} = H(\Phi; \lambda \nu - \epsilon_{n_p}) = \sum_{\{z \in \text{downpath}_\Phi(n_p) \mid \lambda \nu - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_\Phi(n_p, z)} \mathfrak{s}_{\lambda \nu - \epsilon_z}^{(k)}. \quad (9.18)$$

The hypotheses of the corollary hold by (9.3) and the fact $n_p = p = r$ implies $\lambda_r > \mu_r \geq \nu_1$.

Finally, applying the operator $\sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}$ to (9.18) yields

$$\mathfrak{s}_{\lambda \nu}^{(k)} \cdot u_{n_p} \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} = \sum_{\{z \in \text{downpath}_\Phi(n_p) \mid \lambda \nu - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_\Phi(n_p, z)} \mathfrak{s}_{\lambda \nu - \epsilon_z}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}.$$

The left side agrees with the right side of (9.6) by (9.17), and the right side agrees with the right side of (9.16) since $n_p = p$ and $\zeta_p = \lambda_p$ implies $\text{downpath}_\Phi(p) = \text{downpath}_\Psi(p)$. This completes the proof in case 1.

Step 3 (case 2): assemble terms from step 2.

In this case ($\lambda^- = \lambda$), combining (9.10), (9.12), and (9.15) yields

$$\begin{aligned} H(\Psi; \mu \nu) &= \mathfrak{s}_{\lambda \nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_\alpha(\mathbf{n}^+)} u_{\text{colword}(T)} \\ &+ \sum_{\{z \in \text{downpath}_\Psi(p) \setminus \{p\} \mid (\mu \nu)_z > (\mu \nu)_{z+1}\}} t^{B_\Psi(p, z)} \left(\mathfrak{s}_{\lambda \nu - \epsilon_z}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} \right). \end{aligned} \quad (9.19)$$

Step 4 (case 2): evaluate the right side of (9.6) to match it with (9.19).

Since \mathbf{n}^+ is weakly increasing (Proposition 9.5 (e)), we can use Proposition 9.3 to obtain

$$\sum_{T \in \text{FTAB}_\alpha(\mathbf{n})} u_{\text{colword}(T)} = \sum_{T \in \text{FTAB}_\alpha(\mathbf{n}^+)} u_{\text{colword}(T)} + u_{n_p} \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}. \quad (9.20)$$

We now compute the action of the operator in (9.20) on $\mathfrak{s}_{\lambda \nu}^{(k)}$. Set $\Phi = \Delta^k(\lambda \nu)$. By Corollary 8.12 (the hypotheses hold by (9.3) and $n_p < n_p^+ \leq r$),

$$\mathfrak{s}_{\lambda \nu}^{(k)} \cdot u_{n_p} = H(\Phi; \lambda \nu - \epsilon_{n_p}) = \sum_{\{z \in \text{downpath}_\Phi(n_p) \mid \lambda \nu - \epsilon_z \in \text{Par}_\ell^k\}} t^{B_\Phi(n_p, z)} \mathfrak{s}_{\lambda \nu - \epsilon_z}^{(k)}. \quad (9.21)$$

Using (9.20) followed by (9.21), we obtain

$$\begin{aligned} & \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n})} u_{\text{colword}(T)} = \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \left(\sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n}^+)} u_{\text{colword}(T)} + u_{n_p} \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} \right) \\ &= \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n}^+)} u_{\text{colword}(T)} + \sum_{\{z \in \text{downpath}_{\Phi}(n_p) \mid \lambda\nu - \epsilon_z \in \text{Par}_{\ell}^k\}} t^{B_{\Phi}(n_p, z)} \mathfrak{s}_{\lambda\nu - \epsilon_z}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)}. \end{aligned}$$

This agrees with the right side of (9.19) by Proposition 9.5 (g) and because there is no $z = n_p$ term in the second sum; the latter follows from $\lambda_{n_p} = \lambda_{n_p+1}$, which comes from Proposition 9.5 (f) and $n_p < n_p^+ \leq p$. This completes the proof in case 2. \square

Example 9.7. We illustrate case 1B and case 2 from the proof of Theorem 9.4. To keep the examples from getting too large, we have not appended zeros to ν to force $\ell > k + r$ as is done in Step 0. However, we can match the examples below exactly with the proof by appending $k + r + 1 - \ell$ zeros to the weights of all Catalan functions and modifying all root ideals as in the first paragraph of Step 0.

Example for case 1B. Let $k = 8$, $r = 4$, $\mu = 3212$, $\nu = 22221$, and Ψ be the root ideal on the left in (9.23). Hence $\zeta = 4543$, $\lambda = 4443$, $\alpha = 1231$, $\mathbf{n} = 1134$, and

$$\text{FTAB}_{\alpha}(\mathbf{n}) = \left\{ \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 2 & 2 & \\ \hline 3 & 3 & 3 \\ \hline & & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 1 & 2 & \\ \hline 3 & 3 & 3 \\ \hline & & 4 \\ \hline \end{array} \right\}.$$

According to Theorem 9.4,

$$H(\Psi; \mu\nu) = \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n})} u_{\text{colword}(T)} = \mathfrak{s}_{444322221}^{(8)} \cdot (u_{3324321} + u_{3314321}). \quad (9.22)$$

Tracing through the proof of Theorem 9.4 for this example, we have $p = 3$, $\lambda^- = 4433$, and so case 1B applies. In the present example, (9.10) becomes

$$H(\Psi; \mu\nu) = \begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 2 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 2 \\ \hline & & & & & & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 2 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 2 \\ \hline & & & & & & 1 \\ \hline \end{array} + t \begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline 2 & & & & & \\ \hline & 2 & & & & \\ \hline & & 2 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 2 \\ \hline & & & & & & 1 \\ \hline \end{array}. \quad (9.23)$$

For the first term on the right, the inductive hypothesis gives (as in (9.14))

$$\begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline 2 & & & & & \\ \hline & 1 & & & & \\ \hline & & 2 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 2 \\ \hline & & & & & & 1 \\ \hline \end{array} = \mathfrak{s}_{\lambda^- \nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} = \mathfrak{s}_{443322221}^{(8)} \cdot (u_{324321} + u_{314321}), \quad (9.24)$$

where $\alpha^- = 1221$ and $\text{FTAB}_{\alpha^-}(\mathbf{n}) = \text{FTAB}_{1221}(1134) = \left\{ \begin{array}{|c|c|} \hline 1 & \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & \\ \hline 1 & 2 \\ \hline 3 & 3 \\ \hline & 4 \\ \hline \end{array} \right\}.$

For the second term on the right of (9.23), the inductive hypothesis yields (as in (9.12))

$$\begin{array}{|c|c|c|c|c|c|} \hline 3 & & & & & \\ \hline & 2 & & & & \\ \hline & & 2 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 2 \\ \hline & & & & & & 1 \\ \hline & & & & & & & 1 \\ \hline \end{array} = \mathfrak{s}_{\lambda\nu-\epsilon_8}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} = \mathfrak{s}_{444322211}^{(8)} \cdot (u_{324321} + u_{314321}). \quad (9.25)$$

The right side of (9.24) plus t times the right side of (9.25) agrees with the right side of (9.22) as is shown in Step 4 (case 1) of the proof of Theorem 9.4.

Example for case 2. Let $k = 8$, $r = 4$, $\mu = 5442$, $\nu = 211$, and Ψ be the root ideal on the left in (9.27). Hence $\zeta = 5666$, $\lambda = 5555$, $\alpha = 0113$, $\mathbf{n} = 1123$, and

$$\text{FTAB}_{\alpha}(\mathbf{n}) = \left\{ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \ 3 \ 3 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \ 3 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 3 \ 3 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 3 \ 3 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 3 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 3 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \right\}.$$

According to Theorem 9.4,

$$H(\Psi; \mu\nu) = \mathfrak{s}_{5555211}^{(8)} \cdot \left(u_{33321} + u_{33421} + u_{33431} + u_{33432} + u_{34421} + u_{34431} + u_{34432} + u_{44421} + u_{44431} + u_{44432} \right). \quad (9.26)$$

Tracing through the proof of Theorem 9.4 for this example, we have $p = 4$, $\lambda^- = \lambda = 5555$ and so case 2 applies. In the present example, (9.10) becomes

$$H(\Psi; \mu\nu) = \begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline & 4 & & & & \\ \hline & & 4 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline & 4 & & & & \\ \hline & & 4 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 1 \\ \hline \end{array} + t \begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline & 4 & & & & \\ \hline & & 4 & & & \\ \hline & & & 3 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 0 \\ \hline \end{array}. \quad (9.27)$$

For the first term on the right, the inductive hypothesis gives (as in (9.15))

$$\begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline & 4 & & & & \\ \hline & & 4 & & & \\ \hline & & & 2 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 1 \\ \hline \end{array} = \mathfrak{s}_{\lambda\nu}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha}(\mathbf{n}^+)} u_{\text{colword}(T)} = \mathfrak{s}_{5555211}^{(8)} \cdot (u_{44421} + u_{44431} + u_{44432}), \quad (9.28)$$

where $\mathbf{n}^+ = \mathbf{n} + \epsilon_p = 1124$ and $\text{FTAB}_{\alpha}(\mathbf{n}^+) = \text{FTAB}_{0113}(1124) = \left\{ \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \ 4 \ 4 \\ \hline \end{array} \right\}.$

For the second term on the right of (9.27), the inductive hypothesis yields (as in (9.12))

$$\begin{array}{|c|c|c|c|c|c|} \hline 5 & & & & & \\ \hline & 4 & & & & \\ \hline & & 4 & & & \\ \hline & & & 3 & & \\ \hline & & & & 2 & \\ \hline & & & & & 1 \\ \hline & & & & & & 0 \\ \hline \end{array} = \mathfrak{s}_{\lambda\nu-\epsilon_7}^{(k)} \cdot \sum_{T \in \text{FTAB}_{\alpha^-}(\mathbf{n})} u_{\text{colword}(T)} = \mathfrak{s}_{5555210}^{(8)} \cdot (u_{3321} + u_{3421} + u_{3431} + u_{3432} + u_{4421} + u_{4431} + u_{4432}), \quad (9.29)$$

where $\alpha^- = \alpha - \epsilon_p = 0112$ and

$$\text{FTAB}_{\alpha^-}(\mathbf{n}) = \text{FTAB}_{0112}(1123) = \left\{ \begin{array}{c} | \\ \hline 1 \\ | \\ 2 \\ | \\ 3 \end{array} \begin{array}{c} | \\ \hline 1 \\ | \\ 2 \\ | \\ 3 \end{array} \begin{array}{c} | \\ \hline 1 \\ | \\ 3 \\ | \\ 4 \end{array} \begin{array}{c} | \\ \hline 2 \\ | \\ 3 \\ | \\ 4 \end{array} \begin{array}{c} | \\ \hline 1 \\ | \\ 2 \\ | \\ 4 \end{array} \begin{array}{c} | \\ \hline 1 \\ | \\ 3 \\ | \\ 4 \end{array} \begin{array}{c} | \\ \hline 2 \\ | \\ 3 \\ | \\ 4 \end{array} \right\}.$$

The right side of (9.28) plus t times the right side of (9.29) agrees with the right side of (9.26) as is shown in Step 4 (case 2) of the proof of Theorem 9.4.

10. PROOF OF THEOREMS 6.2 AND 6.3

We give the proofs of Theorems 6.2 and 6.3 after three lemmas. The proof of Theorem 6.2 is by an induction on the number of partitions in the k -split using Theorem 5.1, and Theorem 6.3 follows easily from Theorem 6.2 and its proof.

Our first lemma is related to the k -rectangle property (Corollary 6.4). Note that we do not yet know that the k -Schur Catalan functions have the k -rectangle property since Corollary 6.4 relies on what we are now proving.

Lemma 10.1. *Let $U = (k - r + 1)^r$ be a k -rectangle and (Ψ, γ) an indexed root ideal of length $\ell - r$ such that $\bar{\mathbf{r}}(\Psi)_1 \geq \min(r - 1, \ell - r - 1)$. Let $\tilde{\Psi} \subset \Delta_\ell^+$ be the root ideal defined by $\bar{\mathbf{r}}(\tilde{\Psi})_i = \min(r - 1, \ell - i)$ for $i \leq r$ and $\bar{\mathbf{r}}(\tilde{\Psi})_i = \bar{\mathbf{r}}(\Psi)_{i-r}$ for $i > r$. Then*

$$\mathbf{B}_U H(\Psi; \gamma) = H(\tilde{\Psi}; U\gamma). \quad (10.1)$$

In particular, if $\nu \in \text{Par}_{\ell-r}^{k-r+1}$, then $\mathbf{B}_U \mathfrak{s}_\nu^{(k)} = \mathfrak{s}_{U\nu}^{(k)}$.

Proof. By Proposition 3.2 and (3.4), $\mathbf{B}_U H(\Psi; \gamma) = H(\emptyset_r \uplus \Psi; U\gamma)$. Then applying Lemma 8.9 with $z = r - 1, r - 2, \dots, 1$ shows that $H(\emptyset_r \uplus \Psi; U\gamma) = H(\Phi; U\gamma)$, where the root ideal $\Phi \subset \Delta_\ell^+$ is obtained from $\emptyset_r \uplus \Psi$ by removing one root in rows $2, \dots, r$. We can again modify Φ using Lemma 8.9 with $z = r - 1, r - 2, \dots, 2$ to remove one root in rows $3, \dots, r$. Continuing in this way, we obtain the root ideal $\tilde{\Psi}$. \square

Throughout this section we will work with the following free $\mathbb{Z}[t]$ -submodules of Λ^k :

$$\begin{aligned} \Omega^{k,a} &:= \text{span}_{\mathbb{Z}[t]} \{ \mathfrak{s}_\mu^{(k)} \mid \mu \in \text{Par}^k, \mu_1 = a \}, \\ \Lambda^{k,d} &:= \bigoplus_{a \leq d} \Omega^{k,a} = \text{span}_{\mathbb{Z}[t]} \{ \mathfrak{s}_\mu^{(k)} \mid \mu \in \text{Par}^d \}. \end{aligned}$$

The next result shows that the strong Pieri operators commute with certain generalized Hall-Littlewood vertex operators \mathbf{B}_U , up to a shift in indices. Be aware that although the strong Pieri operators act on the right and the generalized Hall-Littlewood vertex operators act on the left, they do not commute, so we must take care with parentheses.

Lemma 10.2. *Let $U = (k - r + 1)^r$ be a k -rectangle and $\nu \in \text{Par}_{\ell-r}^{k-r+1}$ so that $U\nu \in \text{Par}_\ell^k$. Let $w = w_1 \cdots w_d$ be a word in letters $[\ell - r]$ and let $w^+ = (w_1 + r) \cdots (w_d + r)$. Suppose*

$$\mathfrak{s}_\nu^{(k)} \cdot u_{w_1} \cdots u_{w_j} \in \Lambda^{k, k-r+1} \quad \text{for all } j \in [d]. \quad (10.2)$$

Then

$$\mathbf{B}_U (\mathfrak{s}_\nu^{(k)} \cdot u_w) = (\mathbf{B}_U \mathfrak{s}_\nu^{(k)}) \cdot u_{w^+} = \mathfrak{s}_{U\nu}^{(k)} \cdot u_{w^+}. \quad (10.3)$$

Additionally, $\mathfrak{s}_{U\nu}^{(k)} \cdot u_{w^+} \in \text{span}_{\mathbb{Z}[t]} \{ \mathfrak{s}_{U\beta}^{(k)} \mid \beta \in \text{Par}^{k-r+1} \}$.

Proof. The proof of (10.3) is by induction on d . The second equality holds by Lemma 10.1; this given, the base case $d = 0$ is trivial and it remains to prove that for $d > 0$ the left and right sides of (10.3) agree. Set $v = w_1 \cdots w_{d-1}$, $v^+ = (w_1 + r) \cdots (w_{d-1} + r)$, and $p = w_d$. By (10.2), we can write $\mathfrak{s}_\nu^{(k)} \cdot u_v = \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathfrak{s}_\eta^{(k)}$ for $c_\eta \in \mathbb{N}[t]$, where we have used $\mathfrak{s}_\mu^{(k)} = \mathfrak{s}_{(\mu,0)}^{(k)} = \mathfrak{s}_{(\mu,0,0)}^{(k)} = \cdots$ to write the sum over $\text{Par}_{\ell-r}^{k-r+1}$ rather than Par^{k-r+1} . Then by the inductive hypothesis and Lemma 10.1,

$$\mathfrak{s}_{U\nu}^{(k)} \cdot u_{v^+} = \mathbf{B}_U(\mathfrak{s}_\nu^{(k)} \cdot u_v) = \mathbf{B}_U\left(\sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathfrak{s}_\eta^{(k)}\right) = \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathfrak{s}_{U\eta}^{(k)}. \quad (10.4)$$

We compute

$$\begin{aligned} \mathbf{B}_U(\mathfrak{s}_\nu^{(k)} \cdot u_v u_p) &= \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathbf{B}_U(\mathfrak{s}_\eta^{(k)} \cdot u_p) = \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathbf{B}_U H(\Delta^k(\eta); \eta - \epsilon_p) \\ &= \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta H(\Delta^k(U\eta); U\eta - \epsilon_{p+r}) = \sum_{\eta \in \text{Par}_{\ell-r}^{k-r+1}} c_\eta \mathfrak{s}_{U\eta}^{(k)} \cdot u_{p+r} = \mathfrak{s}_{U\nu}^{(k)} \cdot u_{w^+}. \end{aligned}$$

The second and fourth equalities are by the description (2.2) of the strong Pieri operators, the third equality is by Lemma 10.1, and the fifth is by (10.4).

For the second statement, (10.3), (10.2), and Lemma 10.1 yield

$$\mathfrak{s}_{U\nu}^{(k)} \cdot u_{w^+} = \mathbf{B}_U(\mathfrak{s}_\nu^{(k)} \cdot u_w) \in \mathbf{B}_U \Lambda^{k,k-r+1} \subset \text{span}_{\mathbb{Z}[t]} \{ \mathfrak{s}_{U\beta}^{(k)} \mid \beta \in \text{Par}^{k-r+1} \}. \quad \square$$

It turns out that under a mild assumption, the strong Pieri operators in Theorem 5.1 do not change the first part of the partition:

Lemma 10.3. *As in Theorem 5.1, let $\mu \in \text{Par}_r^{k-r+1}$ and $\nu \in \text{Par}^k$ be such that $\mu\nu$ is a partition; set $U = (k - r + 1)^r$. Suppose the first row of U/μ is empty, i.e., $\mu_1 = U_1$. Let $T \in \text{SSYT}_{U/\mu}([r])$ and $w = w_1 \cdots w_{|U/\mu|} = \text{colword}(T)$. Then for any $j \leq |U/\mu|$,*

$$\mathfrak{s}_{U\nu}^{(k)} \cdot u_{w_1} \cdots u_{w_j} \in \Omega^{k,k-r+1}. \quad (10.5)$$

Proof. Write $\mathfrak{s}_{U\nu}^{(k)} \cdot u_{w_1} \cdots u_{w_j} = \sum_{\tau \in \text{Par}^k} c_\tau \mathfrak{s}_\tau^{(k)}$ with coefficients $c_\tau \in \mathbb{N}[t]$. We apply Corollary 8.12 for each strong Pieri operator u_{w_1}, \dots, u_{w_j} . The assumptions of the corollary are satisfied: any term in the k -Schur function expansion of $\mathfrak{s}_{U\nu}^{(k)} \cdot u_{w_1} \cdots u_{w_i}$ for $i \in [j]$ is of the form $d_\eta \mathfrak{s}_\eta^{(k)}$ for $\eta \subset U\nu$ and thus $\bar{\mathbf{r}}(\Delta^k(\eta))_i \geq \bar{\mathbf{r}}(\Delta^k(U\nu))_i \geq r - i$ for $i \in [r]$; the second assumption ($p = r < \ell \implies \eta_r > \eta_{r+1}$) follows from $\mu_r \geq \nu_1$ and the fact that the number of r 's in the word w is at most $U_r - \mu_r$. The corollary then tells us that $c_\tau = 0$ unless there exists a saturated chain $U\nu = \tau^0 \supset \tau^1 \supset \cdots \supset \tau^j = \tau$ in Young's lattice with $\tau^a = \tau^{a-1} - \epsilon_{b_a}$ for all $a \in [j]$, where the integers b_1, \dots, b_j satisfy $b_1 \geq w_1, \dots, b_j \geq w_j$. It follows that if some $c_\tau \neq 0$ with $\tau_1 \neq U_1 = k - r + 1$, then $w_1 \cdots w_j$ contains $r - 1 \cdots 2 - 1$ as a subsequence. But this is impossible since the first row of U/μ is empty and any strictly decreasing subsequence of $w = \text{colword}(T)$ uses at most one entry from each row of T . This proves that $\tau_1 = U_1 = k - r + 1$ whenever $c_\tau \neq 0$, as desired. \square

Proof of Theorem 6.2. We prove by induction on d the following stronger claim: in addition to (6.1), for a u -monomial $u_w = u_{w_1} \cdots u_{w_m}$ arising in the expansion of

$\left(\sum_{T \in \text{SSYT}_{\theta^d}(N_d)} u_{\text{colword}(T)}\right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)}\right)$ as a sum of u -monomials, we have $\mathfrak{s}_U^{(k)} \cdot u_{w_1} \cdots u_{w_j} \in \Omega^{k, k-r_1+1} = \Omega^{k, \lambda_1}$ for all $j \in [m]$.

It makes sense to take the base case to be $d = 0$ which occurs exactly when λ is the empty partition, and then $G_\emptyset^{(k)} = 1 = \mathfrak{s}_\emptyset^{(k)}$ is the desired result. Now assume $d > 0$.

Since $(\lambda^2, \dots, \lambda^d)$ is the k -split of $(\lambda_{r_1+1}, \lambda_{r_1+2}, \dots)$, the inductive hypothesis yields

$$\mathbf{B}_{\lambda^2} \cdots \mathbf{B}_{\lambda^{d-1}} s_{\lambda^d} = \mathfrak{s}_{\hat{U}}^{(k)} \cdot \left(\sum_{T \in \text{SSYT}_{\theta^d}(\hat{N}_d)} u_{\text{colword}(T)} \right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^2}(\hat{N}_2)} u_{\text{colword}(T)} \right) \quad (10.6)$$

where \hat{U} is the partition U restricted to rows $> r_1$ and $\hat{N}_j := \{i - r_1 \mid i \in N_j\}$ for $j = 2, \dots, d$. Moreover, by the inductive hypothesis for the strengthened claim, the right side of (10.6) is a sum of k -Schur functions $\sum_{\nu} c_{\nu} \mathfrak{s}_{\nu}^{(k)}$ over $\{\nu \in \text{Par}^k \mid \nu_1 = (\lambda^2)_1\}$ provided $d > 1$ (if $d = 1$, (10.6) simply reads $G_\emptyset^{(k)} = \mathfrak{s}_\emptyset^{(k)}$). Hence for each ν appearing in this sum, Theorem 5.1 applies and yields

$$\mathbf{B}_{\lambda^1} \mathfrak{s}_{\nu}^{(k)} = \mathfrak{s}_{U^1 \nu}^{(k)} \cdot \sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)} = (\mathbf{B}_{U^1} \mathfrak{s}_{\nu}^{(k)}) \cdot \sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)}, \quad (10.7)$$

where the second equality is by Lemma 10.1.

Applying \mathbf{B}_{λ^1} to both sides of (10.6) yields

$$\begin{aligned} G_{\lambda}^{(k)} &= \mathbf{B}_{\lambda^1} \left(\mathfrak{s}_{\hat{U}}^{(k)} \cdot \left(\sum_{T \in \text{SSYT}_{\theta^d}(\hat{N}_d)} u_{\text{colword}(T)} \right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^2}(\hat{N}_2)} u_{\text{colword}(T)} \right) \right) \\ &= \left(\mathbf{B}_{U^1} \left(\mathfrak{s}_{\hat{U}}^{(k)} \cdot \left(\sum_{T \in \text{SSYT}_{\theta^d}(\hat{N}_d)} u_{\text{colword}(T)} \right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^2}(\hat{N}_2)} u_{\text{colword}(T)} \right) \right) \right) \cdot \sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)} \\ &= \mathfrak{s}_U^{(k)} \cdot \left(\sum_{T \in \text{SSYT}_{\theta^d}(N_d)} u_{\text{colword}(T)} \right) \cdots \left(\sum_{T \in \text{SSYT}_{\theta^2}(N_2)} u_{\text{colword}(T)} \right) \left(\sum_{T \in \text{SSYT}_{\theta^1}(N_1)} u_{\text{colword}(T)} \right), \end{aligned}$$

where the second equality is by (10.7) and the third is by Lemma 10.2 (the assumption (10.2) holds by the inductive hypothesis for the strengthened claim). This proves (6.1).

To complete the proof of the strengthened claim, let $u_w = u_{w_1} \cdots u_{w_j}$, $j \in [m]$, be as in the first paragraph of the proof. Set $m' = \min(|\theta^d| + \cdots + |\theta^2|, j)$. Write $w_1 \cdots w_j = v v'$, where $v = w_1 \cdots w_{m'}$ and $v' = w_{m'+1} \cdots w_j$ (v or v' may be empty). By the second statement in Lemma 10.2 applied with $r = r_1$, $U = U^1$, $\nu = \hat{U}$, and $w^+ = v$ (again, the assumption (10.2) holds by the inductive hypothesis for the strengthened claim),

$$\mathfrak{s}_U^{(k)} \cdot u_v = \sum_{\eta \in \text{Par}^{k-r_1+1}} d_{\eta} \mathfrak{s}_{U^1 \eta}^{(k)} \quad \text{for } d_{\eta} \in \mathbb{N}[t].$$

Lemma 10.3 then yields (the first row of $\theta^1 = U^1/\lambda^1$ is empty since $k - r_1 + 1 = \lambda_1$)

$$\mathfrak{s}_U^{(k)} \cdot u_{w_1} \cdots u_{w_j} = \mathfrak{s}_U^{(k)} \cdot u_v u_{v'} = \left(\sum_{\eta \in \text{Par}^{k-r_1+1}} d_{\eta} \mathfrak{s}_{U^1 \eta}^{(k)} \right) \cdot u_{v'} \in \Omega^{k, k-r_1+1}. \quad \square$$

Proof of Theorem 6.3. We prove $\tilde{A}_\mu^{(k)} = \mathfrak{s}_\mu^{(k)}$ by induction on $\ell(\mu)$. The base case $\ell(\mu) = 0$ holds since $\tilde{A}_\emptyset^{(k)} = 1 = \mathfrak{s}_\emptyset^{(k)}$. Now assume $\ell(\mu) > 0$ and set $\hat{\mu} := (\mu_2, \mu_3, \dots, \mu_\ell)$. By the inductive hypothesis, $\tilde{A}_{\hat{\mu}}^{(k)} = \mathfrak{s}_{\hat{\mu}}^{(k)}$. We evaluate definition (6.3) by first computing $\mathbf{B}_{\mu_1} \tilde{A}_{\hat{\mu}}^{(k)} = \mathbf{B}_{\mu_1} \mathfrak{s}_{\hat{\mu}}^{(k)}$. Corollary 5.4 yields the first equality below:

$$\mathbf{B}_{\mu_1} \mathfrak{s}_{\hat{\mu}}^{(k)} = \mathfrak{s}_{(k, \hat{\mu})}^{(k)} \cdot u_1^{k-\mu_1} = \mathfrak{s}_\mu^{(k)} + \sum_{\{\nu \in \text{Par}^k \mid \nu_1 > \mu_1\}} c_\nu \mathfrak{s}_\nu^{(k)}, \quad \text{for coefficients } c_\nu \in \mathbb{N}[t]. \quad (10.8)$$

The second equality follows from applying Corollary 8.12 $k - \mu_1$ times with $r = 1$, noting that the assumption $\eta_r > \eta_{r+1}$ holds by $\mu_1 \geq \mu_2$.

By the strengthened claim in the proof of Theorem 6.2, $G_\lambda^{(k)} \in \Omega^{k, \lambda_1}$ for all $\lambda \in \text{Par}^k$. It follows that

$$\tilde{\Omega}^{k, a} = \text{span}_{\mathbb{Z}[t]} \{G_\lambda^{(k)} \mid \lambda \in \text{Par}^k, \lambda_1 = a\} = \text{span}_{\mathbb{Z}[t]} \{\mathfrak{s}_\lambda^{(k)} \mid \lambda \in \text{Par}^k, \lambda_1 = a\} = \Omega^{k, a}$$

for all $a \in [0, k]$. Hence applying the projection $\pi^{k, d}$ from definition (6.3) to the right side of (10.8) sends the term $\mathfrak{s}_\mu^{(k)}$ to itself and sends each term $c_\nu \mathfrak{s}_\nu^{(k)}$ in the sum to 0, giving

$$\tilde{A}_\mu^{(k)} = \pi^{k, \mu_1}(\mathbf{B}_{\mu_1} \tilde{A}_{\hat{\mu}}^{(k)}) = \pi^{k, \mu_1}(\mathbf{B}_{\mu_1} \mathfrak{s}_{\hat{\mu}}^{(k)}) = \mathfrak{s}_\mu^{(k)}. \quad \square$$

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