KRONECKER COEFFICIENTS AND NONCOMMUTATIVE SUPER SCHUR FUNCTIONS

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Abstract. The theory of noncommutative Schur functions can be used to obtain positive combinatorial formulae for the Schur expansion of various classes of symmetric functions, as shown by Fomin and Greene [11]. We develop a theory of noncommutative super Schur functions and use it to prove a positive combinatorial rule for the Kronecker coefficients $g_{\lambda\mu\nu}$ where one of the partitions is a hook, recovering previous results of the two authors [7, 22]. This method also gives a precise connection between this rule and a heuristic for Kronecker coefficients first investigated by Lascoux [19].

1. Introduction

Let M_λ be the irreducible representation of the symmetric group $\mathcal{S}_n$ corresponding to the partition λ. Given three partitions λ, μ, and ν of n, the *Kronecker coefficient* $g_{\lambda\mu\nu}$ is the multiplicity of M_ν in the tensor product $M_\lambda \otimes M_\mu$. A longstanding open problem in algebraic combinatorics, called the *Kronecker problem*, is to find a positive combinatorial formula for these coefficients. See [2, 7, 9, 10, 16, 19, 22, 23, 24, 25, 26] for some known special cases.

Our story begins with the work of Lascoux [19], wherein he gave a formula for the Kronecker coefficients $g_{\lambda\mu\nu}$ when two of the partitions are hooks by considering products of permutations in certain Knuth equivalence classes. Though this rule no longer holds outside the hook-hook case, it seems to approximate Kronecker coefficients amazingly well for *any three partitions* and therefore gives a useful heuristic.

Several years ago, the first author [7] gave a rule for Kronecker coefficients when one of the partitions is a hook. This rule was discovered using Lascoux's heuristic, but it was left as an open problem to give a precise statement relating it to the heuristic. Recently, the second author [22] gave a simplified description and proof of this rule.

We develop a theory of noncommutative super Schur functions based on work of Fomin-Greene [11] and the first author [5]. Using this we

- reprove and strengthen the rule from [22],
- establish a precise connection between this rule and the Lascoux heuristic, and
- uncover a surprising parallel between this rule and combinatorics underlying transformed Macdonald polynomials indexed by a 3-column shape, as described in [14, 5].

Key words and phrases. Kronecker coefficients, noncommutative Schur functions, super Schur functions, colored tableaux.

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2. Main results

In this section, we state our main theorem on noncommutative super Schur functions (Theorem 2.3), and show how it can be used to recover the rule from [22] for Kronecker coefficients where one of the partitions is a hook. Proofs will be deferred to later sections.

2.1. Colored words and the algebra $\mathcal{U}$.

Let $\mathcal{A}_{\varnothing} = \{1, 2, \ldots, N\}$ denote the alphabet of unbarred letters and $\mathcal{A}_{\text{-}} = \{\bar{1}, \bar{2}, \ldots, \overline{N}\}$ the alphabet of barred letters. A *colored word* is a word in the total alphabet $\mathcal{A} = \mathcal{A}_{\varnothing} \sqcup \mathcal{A}_{\text{-}}$.

We will consider total orders $\lessdot$ on $\mathcal{A}$ such that $1 \lessdot 2 \lessdot \cdots \lessdot N$ and $\bar{1} \lessdot \bar{2} \lessdot \cdots \lessdot \overline{N}$; we call such orders *shuffle orders*. Two shuffle orders we will work with frequently are

the *natural order* $<$ given by $1 < \bar{1} < 2 < \bar{2} < \cdots < N < \overline{N}$, and

the *big bar order* $\prec$ given by $1 \prec 2 \prec \cdots \prec N \prec \bar{1} \prec \bar{2} \prec \cdots \prec \overline{N}$.

Let $\mathcal{U}$ be the free associative $\mathbb{Z}$-algebra in the noncommuting variables u_x, $x \in \mathcal{A}$. Equivalently, $\mathcal{U}$ is the tensor algebra of $\mathbb{Z}\mathcal{A}$, the $\mathbb{Z}$-module freely spanned by the elements of $\mathcal{A}$. We identify the monomials of $\mathcal{U}$ with colored words and frequently write x for the variable u_x and $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_t = u_{w_1} \cdots u_{w_t}$ for a colored word/monomial.

For the natural order $<$, it is useful to have a notation for "going down by one." Accordingly, for any $x \in \mathcal{A}$, define

$$x{\downarrow} = \begin{cases} \overline{a-1} & \text{if } x = a,\ a \in \{2, \ldots, N\}, \\ a & \text{if } x = \bar{a},\ a \in \{1, \ldots, N\}. \end{cases} \tag{1}$$

2.2. Quotients of $\mathcal{U}$.

Often we will be interested in performing computations in a quotient of $\mathcal{U}$. One important such quotient is the *$<$-colored plactic algebra*, denoted $\mathcal{U}/I^{<}_{\text{plac}}$, which is the quotient of $\mathcal{U}$ by the relations

$$\mathsf{xzy} = \mathsf{zxy} \quad \text{for } x, y, z \in \mathcal{A},\ x < y < z, \tag{2}$$

$$\mathsf{yxz} = \mathsf{yzx} \quad \text{for } x, y, z \in \mathcal{A},\ x < y < z, \tag{3}$$

$$\mathsf{yyx} = \mathsf{yxy} \quad \text{for } x \in \mathcal{A},\ y \in \mathcal{A}_{\varnothing},\ x < y, \tag{4}$$

$$\mathsf{zyy} = \mathsf{yzy} \quad \text{for } z \in \mathcal{A},\ y \in \mathcal{A}_{\varnothing},\ y < z, \tag{5}$$

$$\mathsf{yyz} = \mathsf{yzy} \quad \text{for } z \in \mathcal{A},\ y \in \mathcal{A}_{\text{-}},\ y < z, \tag{6}$$

$$\mathsf{xyy} = \mathsf{yxy} \quad \text{for } x \in \mathcal{A},\ y \in \mathcal{A}_{\text{-}},\ x < y; \tag{7}$$

let $I^{<}_{\text{plac}}$ denote the corresponding two-sided ideal of $\mathcal{U}$. The $<$-colored plactic algebra behaves much like the ordinary plactic algebra.

For applications to Kronecker coefficients, we need the following variant of the $<$-colored plactic algebra. Let $\mathcal{U}/I_{\text{Kron}}$ denote the quotient of $\mathcal{U}$ by the relations (4), (5), (6), (7), and

$$(\mathsf{xz} - \mathsf{zx})\mathsf{y} = \mathsf{y}(\mathsf{xz} - \mathsf{zx}) \quad \text{for } x, y, z \in \mathcal{A},\ x = y{\downarrow} = z{\downarrow\downarrow}, \tag{8}$$

$$\mathsf{xz} = \mathsf{zx} \quad \text{for } x, z \in \mathcal{A},\ x < z{\downarrow\downarrow}; \tag{9}$$

let I_{Kron} denote the corresponding two-sided ideal of $\mathcal{U}$. We refer to (9) as the *far commutation relations*.

2.3. **Main theorem.** For any two letters $x, y \in \mathcal{A}$, write $y \geq_{\text{col}} x$ to mean either $y > x$, or y and x are equal barred letters. The *noncommutative super elementary symmetric functions* are defined by

$$e_k(\mathbf{u}) = \sum_{\substack{z_1 \geq_{\text{col}} z_2 \geq_{\text{col}} \cdots \geq_{\text{col}} z_k \\ z_1, \ldots, z_k \in \mathcal{A}}} u_{z_1} u_{z_2} \cdots u_{z_k} \ \in \mathcal{U}$$

for any positive integer k; set $e_0(\mathbf{u}) = 1$ and $e_k(\mathbf{u}) = 0$ for $k < 0$.

We will prove that $e_k(\mathbf{u})$ and $e_l(\mathbf{u})$ commute for all k and l in both $\mathcal{U}/I_{\text{Kron}}$ and $\mathcal{U}/I_{\text{plac}}^{<}$ in Propositions 8.1 and 8.3. Similar results are proved in [20, 11, 18, 6, 17].

Using these elementary functions, we can define the noncommutative super Schur functions as follows.

Definition 2.1. Let $\nu = (\nu_1, \nu_2, \ldots)$ be a partition. Let ν' be the conjugate partition, which has $t = \nu_1$ parts. The *noncommutative super Schur function* $\mathfrak{J}_\nu(\mathbf{u})$ is given by the following noncommutative version of the Kostka-Naegelsbach/Jacobi-Trudi formula:

$$\mathfrak{J}_\nu(\mathbf{u}) = \sum_{\pi \in \mathcal{S}_t} \operatorname{sgn}(\pi)\, e_{\nu'_1+\pi(1)-1}(\mathbf{u}) e_{\nu'_2+\pi(2)-2}(\mathbf{u}) \cdots e_{\nu'_t+\pi(t)-t}(\mathbf{u}) \ \in \mathcal{U}.$$

As explained in [11] and as we will see below, establishing monomial positivity of $\mathfrak{J}_\nu(\mathbf{u})$ in various quotients of $\mathcal{U}$ has important consequences for proving manifestly positive combinatorial formulae for structure coefficients.

It follows easily from [11, Lemma 3.2] that $\mathfrak{J}_\nu(\mathbf{u})$ is equal to a positive sum of monomials in $\mathcal{U}/I_{\text{plac}}^{<}$ (Theorem 3.10). We conjecture that this can be strengthened to monomial positivity of $\mathfrak{J}_\nu(\mathbf{u})$ in an algebra $\mathcal{U}/I_{\text{Kron-K}}$ (see §3.1) which has both $\mathcal{U}/I_{\text{plac}}^{<}$ and $\mathcal{U}/I_{\text{Kron}}$ as quotients. Our main result is a weaker version of this, a proof that $\mathfrak{J}_\nu(\mathbf{u})$ is monomial positive in $\mathcal{U}/I_{\text{Kron}}$. (We believe that $\mathcal{U}/I_{\text{Kron-K}}$ is the natural algebra for the applications described below, but we work with $\mathcal{U}/I_{\text{Kron}}$ instead since this makes statements easier to prove.) We now give the precise statement of our main result.

A $<$-*colored tableau* is a tableau with entries in $\mathcal{A}$ such that each row and column is weakly increasing with respect to the natural order $<$, while the unbarred letters in each column and the barred letters in each row are strictly increasing. Let $\text{CT}_\nu^{<}$ denote the set of $<$-colored tableaux of shape ν.

Definition 2.2. For a $<$-colored tableau T, define the colored word $\text{arwread}(T)$ as follows: let $D^1, D^2, \ldots, D^k$ be the diagonals of T, starting from the southwest. Let w^{i} be the result of reading the unbarred entries of D^i in the direction $\nwarrow$, followed by the barred entries of D^i in the direction $\searrow$. Set $\text{arwread}(T) = \mathsf{w}^1 \mathsf{w}^2 \cdots \mathsf{w}^{\mathsf{k}}$.

The word $\text{arwread}(T)$ is a particular choice of an *arrow respecting reading word* of T, which will be defined in §5.3.

For example, for $\nu = (5,4,4)$,

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & \bar{3} & \bar{4} & 6 \\ \hline \bar{2} & 3 & 4 & \bar{4} & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & \bar{3} & \bar{4} & 5 & \multicolumn{1}{c}{} \\ \cline{1-4} \end{array} \in \mathrm{CT}_\nu^<,$$

$$\operatorname{arwread}(T) = \mathsf{3\,\bar{2}\,\bar{3}\,3\,1\,\bar{4}\,5\,4\,1\,\bar{3}\,\bar{4}\,\bar{4}\,6}.$$

We now state our main theorem. Section 6 is devoted to its proof.

Theorem 2.3. *In the algebra $\mathcal{U}/I_{\mathrm{Kron}}$, the noncommutative super Schur function $\mathfrak{J}_\nu(\mathbf{u})$ is equal to the following positive sum of monomials:*

$$\mathfrak{J}_\nu(\mathbf{u}) = \sum_{T \in \mathrm{CT}_\nu^<} \operatorname{arwread}(T) \qquad \text{in } \mathcal{U}/I_{\mathrm{Kron}}. \tag{10}$$

In [5], the first author used the theory of noncommutative Schur functions to prove a positive combinatorial formula for the Schur expansion of LLT polynomials indexed by 3-tuples of skew shapes and thereby settled a conjecture of Haglund [14] on transformed Macdonald polynomials indexed by 3-column shapes. The noncommutative Schur function computation required for this work ([5, Theorem 1.1]) is quite similar to Theorem 2.3 (though it seems neither result can be obtained from the other; see §7.2). It is quite surprising that LLT polynomials for 3-tuples of skew shapes and Kronecker coefficients for one hook shape have such similar underlying combinatorics.

2.4. **Applications.** The machinery of noncommutative Schur functions as developed in [11, 8] can be used to study symmetric functions in the usual sense, i.e., formal power series in infinitely many (commuting) variables $\mathbf{x} = (x_1, x_2, \dots)$ which are symmetric under permutations of these variables.

Let $\mathsf{w} = \mathsf{w_1} \cdots \mathsf{w_t}$ be a colored word. For any shuffle order $\lessdot$ on $\mathcal{A}$, define the *$\lessdot$-descent set of* w, denoted $\mathrm{Des}_\lessdot(\mathsf{w})$, by

$$\mathrm{Des}_\lessdot(\mathsf{w}) := \big\{ i \in [t-1] \mid \mathsf{w_i} \gtrdot \mathsf{w_{i+1}} \text{ or } (\mathsf{w_i} \text{ and } \mathsf{w_{i+1}} \text{ are equal barred letters}) \big\}.$$

The associated *fundamental quasisymmetric function* (Gessel [13]) is given by

$$Q_{\mathrm{Des}_\lessdot(\mathsf{w})}(\mathbf{x}) = \sum_{\substack{1 \le i_1 \le \cdots \le i_t \\ j \in \mathrm{Des}_\lessdot(\mathsf{w}) \implies i_j < i_{j+1}}} x_{i_1} \cdots x_{i_t}.$$

Definition 2.4. For any $\gamma = \sum_{\mathsf{w}} \gamma_{\mathsf{w}} \mathsf{w} \in \mathcal{U}$ (where the sum is over colored words w, and $\gamma_{\mathsf{w}} \in \mathbb{Z}$), define

$$F_\gamma^\lessdot(\mathbf{x}) = \sum_{\mathsf{w}} \gamma_{\mathsf{w}}\, Q_{\mathrm{Des}_\lessdot(\mathsf{w})}(\mathbf{x}) \in \mathbb{Z}[[x_1, x_2, \dots]],$$

and for a set of colored words W, define $F_W^\lessdot(\mathbf{x}) = F_\gamma^\lessdot(\mathbf{x})$ where $\gamma = \sum_{\mathsf{w} \in W} \mathsf{w}$.

Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on $\mathcal{U}$ for which the monomials (colored words) form an orthonormal basis. Note that any element of $\mathcal{U}/I$ has a well-defined pairing with any element of $I^\perp$ for any two-sided ideal I of $\mathcal{U}$.

The next result is a straightforward adaptation of the theory of noncommutative Schur functions [11, 8] to the super setting. It also requires the fact mentioned above that the noncommutative super elementary symmetric functions commute in $\mathcal{U}/I_{\text{Kron}}$. (A full proof is given by Theorem 8.4 and Proposition 8.1.)

Theorem 2.5. *For any $\gamma \in (I_{\text{Kron}})^{\perp}$, the function $F^{<}_{\gamma}(\mathbf{x})$ is symmetric and*

$$F^{<}_{\gamma}(\mathbf{x}) = \sum_{\nu} s_{\nu}(\mathbf{x})\langle \mathfrak{J}_{\nu}(\mathbf{u}), \gamma\rangle.$$

Theorems 2.5 and 2.3 then immediately yield the following.

Corollary 2.6. *For any set of colored words W such that $\sum_{\mathsf{w}\in W} \mathsf{w} \in (I_{\text{Kron}})^{\perp}$,*

$$\Big(\textit{the coefficient of } s_{\nu}(\mathbf{x}) \textit{ in } F^{<}_{W}(\mathbf{x})\Big) = \big|\big\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in W\big\}\big|.$$

We will soon see how Corollary 2.6 can be used to prove a positive combinatorial formula for the Kronecker coefficients indexed by one hook shape and two arbitrary shapes, recovering results of [22, 7]. It is more powerful, however, than this main application. In §3.1, we give a way of visualizing the implications of Corollary 2.6 and give examples to better indicate its full strength.

2.5. **Relation to Kronecker coefficients.** To apply Corollary 2.6 to the Kronecker problem, we define a set W of colored Yamanouchi words such that the coefficient of $s_{\nu}(\mathbf{x})$ in $F^{<}_{W}(\mathbf{x})$ is given in terms of Kronecker coefficients. The proofs of the results in this subsection are deferred to §4.

Definition 2.7 (Colored Yamanouchi words)**.** Let w be a colored word. Define w^{brgt} to be the ordinary word formed from w by shuffling the barred letters to the right, reversing this subword of barred letters and removing their bars. We say that w is *Yamanouchi* of content λ if w^{brgt} is Yamanouchi of content λ. Define $\mathrm{CYW}_{\lambda,d}$ to be the set of colored Yamanouchi words of content λ having exactly d barred letters.[1]

For example, if $\mathsf{w} = 2\bar{1}\bar{2}1\bar{3}\bar{1}21$, then $\mathsf{w}^{\text{brgt}} = 21211321$, and these are Yamanouchi of content $(4,3,1)$. See Figure 2.

For $0 \le d \le n-1$, let $\mu(d)$ denote the hook partition $(n-d, 1^d)$. The following is in some sense well known (see §4.1 for a proof).

Proposition 2.8. *For any partitions λ, ν of n and $d \le n$,*

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = \Big(\textit{the coefficient of } s_{\nu}(\mathbf{x}) \textit{ in } F^{\prec}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})\Big).$$

(By convention, set $g_{\lambda\,\mu(n)\,\nu} = g_{\lambda\,\mu(-1)\,\nu} = 0$.)

We use a trick called word conversion to convert from the big bar order $\prec$ to the natural order $<$ (see §4.2).

Proposition 2.9. *For any partition λ of n and $d \le n$, $F^{\prec}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x}) = F^{<}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})$.*

By showing that $\sum_{\mathsf{w}\in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I_{\text{Kron}})^{\perp}$ (see §4.3) and applying Propositions 2.8 and 2.9 and Corollary 2.6, we obtain the following result, first proved in [22] using the conversion operation on colored tableaux (see Lemma 3.1 and Remark 3.3 of [22]).

[1]Warning: the colored Yamanouchi words defined here are not the same as those defined in [7].

Corollary 2.10. *For any partitions λ, ν of n and $d \leq n$,*

$$g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu} = \Big(\textit{the coefficient of } s_\nu(\mathbf{x}) \textit{ in } F^{<}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})\Big)$$
$$= \Big|\big\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}\big\}\Big|.$$

This result demonstrates the importance of I_{Kron} since it is not always the case that $\sum_{\mathsf{w}\in\mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I^{<}_{\mathrm{plac}})^{\perp}$. For example, $\sum_{\mathsf{w}\in\mathrm{CYW}_{\lambda,d}} \mathsf{w} \notin (I^{<}_{\mathrm{plac}})^{\perp}$ for $\lambda = (2,2)$, $d = 2$ since $\bar{1}\bar{2}21 \in \mathrm{CYW}_{\lambda,d}$, but $\bar{2}\bar{1}21 \notin \mathrm{CYW}_{\lambda,d}$, whereas $\bar{1}\bar{2}21 \equiv \bar{2}\bar{1}21 \mod I^{<}_{\mathrm{plac}}$.

Corollary 2.10 easily implies an explicit combinatorial formula for $g_{\lambda\mu(d)\nu}$. Partition $\mathrm{CYW}_{\lambda,d}$ into sets $\mathrm{CYW}^{-}_{\lambda,d}$ and $\mathrm{CYW}^{+}_{\lambda,d}$ consisting of the words ending in an unbarred letter or barred letter, respectively. Since there is a bijection from $\mathrm{CYW}^{+}_{\lambda,d+1}$ to $\mathrm{CYW}^{-}_{\lambda,d}$ given by removing the bar from the last letter, and this bijection respects the set of words of the form $\mathrm{arwread}(T)$, this partition separates the two Kronecker coefficients appearing in Corollary 2.10, giving the following result (see §4.4 for a full proof):

Corollary 2.11 ([22])**.** *For any partitions λ, ν of n and $d \leq n-1$,*

$$g_{\lambda\mu(d)\nu} = \Big|\big\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in \mathrm{CYW}^{-}_{\lambda,d}\big\}\Big|.$$

2.6. **The Lascoux heuristic.** We now describe a heuristic for computing Kronecker coefficients first investigated by Lascoux in [19] and relate it to Corollary 2.11.

For any partition λ, let $Z^{\mathrm{st}}_{\lambda}$ be the *(standardized) superstandard tableau* of shape λ whose boxes are labeled in order across rows from top to bottom. Let Γ_λ be the set of permutations whose insertion tableau is $Z^{\mathrm{st}}_{\lambda}$. For any two partitions λ and μ, consider the multiset of permutations

$$\Gamma_\lambda \circ \Gamma_\mu = \big\{u \circ v \mid u \in \Gamma_\lambda, v \in \Gamma_\mu\big\}, \tag{11}$$

where $\circ$ denotes ordinary composition of permutations.

For example, take

$$\lambda = (3,1), \qquad Z^{\mathrm{st}}_{\lambda} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}, \quad \Gamma_\lambda = \{4123, 1423, 1243\};$$

$$\mu = (2,1,1), \quad Z^{\mathrm{st}}_{\mu} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}, \qquad \Gamma_\mu = \{4312, 4132, 1432\}.$$

Then $\Gamma_\lambda \circ \Gamma_\mu$ consists of the nine products in the multiplication table:

$\circ$	4312	4132	1432
4123	3241	3421	4321
1423	3214	3124	1324
1243	3412	3142	1342

In this case, $\Gamma_\lambda \circ \Gamma_\mu$ is a union of four Knuth equivalence classes with insertion tableaux

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline\end{array}.$$

Moreover, $g_{\lambda\mu\nu} = 1$ if $\nu = (2,1,1)$, $(2,2)$, $(3,1)$, or $(1,1,1,1)$, and $g_{\lambda\mu\nu} = 0$ otherwise, so the classes present in $\Gamma_\lambda \circ \Gamma_\mu$ precisely describe the Kronecker coefficients $g_{\lambda\mu\nu}$.

Lascoux [19] showed that if λ and μ are both hooks, then this phenomenon occurs in general.

Theorem 2.12 (Lascoux's Kronecker Rule [19])**.** *If λ and μ are hook shapes, then $\Gamma_\lambda \circ \Gamma_\mu$ is a union of Knuth equivalence classes, and $g_{\lambda\mu\nu}$ is the number of these classes with insertion tableau of shape ν.*

Lascoux [19] and Garsia-Remmel [12, §6–7] both investigate the extent to which this rule generalizes to other shapes. They give examples showing that it does not extend beyond the hook-hook case. Nevertheless, computations suggest that this rule is often close to holding even when it fails, and therefore it provides a valuable heuristic (see [7, §1]).

For example, when only μ is a hook, $\Gamma_\lambda \circ \Gamma_\mu$ may not be a union of Knuth equivalence classes, but the corresponding quasisymmetric function $F^{<}_{\Gamma_\lambda \circ \Gamma_\mu}(\mathbf{x})$ is still symmetric and equal to $\sum_\nu g_{\lambda\mu\nu} s_\nu(\mathbf{x})$. (This follows from Propositions 2.8 and 2.9 and (12) below.)

We can rephrase Lascoux's rule using colored words and standardization. For a colored word w, the *standardization of* w, denoted w^{st}, is the permutation obtained from w by first relabeling, from left to right (resp. right to left), the occurrences of the smallest letter (with respect to $<$) in w by $1, \dots, k$ if this letter is unbarred (resp. barred), then relabeling the occurrences of the next smallest letter of w by $k+1, \dots, k+k'$, and so on. For example, $(2\bar{1}\bar{2}1\bar{2}\bar{1}21)^{\mathrm{st}} = 54817362$. (For more about standardization, see §4.5.)

For any colored word $\mathsf{w} \in \mathrm{CYW}_{\lambda,d}$, $\mathsf{w}^{\mathrm{st}} = (\mathsf{w}^{\mathrm{brgt}})^{\mathrm{st}} \circ \mathsf{v}^{\mathrm{st}}$, where v is the colored word obtained from w by replacing all unbarred letters with 1 and all barred letters with $\bar{1}$. Since $\mathsf{w}^{\mathrm{brgt}}$ is Yamanouchi, $(\mathsf{w}^{\mathrm{brgt}})^{\mathrm{st}} \in \Gamma_\lambda$. Similarly, $\mathsf{v}^{\mathrm{st}} \in \Gamma_{\mu(d)}$ if w ends with an unbarred letter, and $\mathsf{v}^{\mathrm{st}} \in \Gamma_{\mu(d-1)}$ if w ends with a barred letter.

Hence the set of standardizations of words in $\mathrm{CYW}_{\lambda,d}$ is

$$(\mathrm{CYW}_{\lambda,d})^{\mathrm{st}} = (\mathrm{CYW}^-_{\lambda,d})^{\mathrm{st}} \sqcup (\mathrm{CYW}^+_{\lambda,d})^{\mathrm{st}} = (\Gamma_\lambda \circ \Gamma_{\mu(d)}) \sqcup (\Gamma_\lambda \circ \Gamma_{\mu(d-1)}). \tag{12}$$

Lascoux's rule then implies that when λ is a hook, $(\mathrm{CYW}_{\lambda,d})^{\mathrm{st}}$ is a union of Knuth equivalence classes and hence $\mathrm{CYW}_{\lambda,d}$ is a union of $<$-colored plactic equivalence classes; that is, $\sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I^{<}_{\mathrm{plac}})^{\perp}$. These equivalence classes give an explicit partition of $\mathrm{CYW}_{\lambda,d}$ from which one can directly obtain the coefficients $g_{\lambda\mu(d)\nu}$.

Unfortunately, this does not typically hold when λ is not a hook (for example, see the discussion after Corollary 2.10). However, in proving Corollary 2.10 we show that $\sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I_{\mathrm{Kron}})^{\perp}$ for all λ. Although we cannot explicitly partition $\mathrm{CYW}_{\lambda,d}$ into classes to find the Schur expansion of $F^{<}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})$, we still get an explicit description of its Schur expansion from Corollary 2.6 using the subset of words of the form $\mathrm{arwread}(T)$. It would be interesting to determine if there exists an explicit partition (say, by way of a modified insertion algorithm) that gives this Schur expansion as in the case λ is a hook.

2.7. **Organization.** The remainder of this paper is organized as follows. Section 3 supports the main results above by giving examples of Corollaries 2.6 and 2.10; we also prove two strengthenings of Corollary 2.10 and investigate a connection between noncommutative super Schur functions and conversion (§3.3). Section 4 fills in the missing proofs from §2.5 and shows how the colored plactic algebra and ordinary plactic algebra are related via standardization. Section 5 develops tableau combinatorics for the proof of the main theorem, and Section 6 is devoted to its proof. In Section 7, we conjecture a strengthening of the main theorem and compare it to results about LLT polynomials from [5]. Finally, Section 8 lays out the basic theory of noncommutative super Schur functions, including two results about when elementary functions commute.

3. Switchboards and conversion

This section supports the results of the previous section with examples, further context, and strengthenings. We introduce certain graphs called $I_{\text{Kron-K}}$-switchboards and use them to illustrate Corollaries 2.6 and 2.10 (§3.1), put the results of the previous section in the context of similar, easier results for colored plactic algebras (§3.2), and prove two strengthenings of Corollary 2.10, which hint at a mysterious connection between noncommutative super Schur functions and conversion (§3.3).

3.1. $I_{\text{Kron-K}}$-switchboards.

Here we introduce certain graphs to give an intuitive understanding of Corollaries 2.6 and 2.10 and to give examples of these corollaries.

Let $I_{\text{Kron-K}}$ be the two-sided ideal of $\mathcal{U}$ corresponding to the relations (4), (5), (6), (7), (8), and

$$\mathsf{xzy} = \mathsf{zxy} \qquad \text{for } x, y, z \in \mathcal{A},\ x < y < z,\ x < z{\downarrow\downarrow}, \tag{13}$$

$$\mathsf{yxz} = \mathsf{yzx} \qquad \text{for } x, y, z \in \mathcal{A},\ x < y < z,\ x < z{\downarrow\downarrow}. \tag{14}$$

We conjecture that Theorem 2.3 (and hence Corollary 2.6) holds with $I_{\text{Kron-K}}$ in place of I_{Kron} (see Conjecture 7.1). The graphs we introduce below are better suited to studying these strengthenings than the original statements.

Definition 3.1. Let $\mathsf{w} = \mathsf{w_1} \cdots \mathsf{w_n}$ and $\mathsf{w}' = \mathsf{w}'_1 \cdots \mathsf{w}'_n$ be two colored words of the same length n in the alphabet $\mathcal{A}$. We say that w and w' are related by a *switch* in position i if

- $\mathsf{w}_j = \mathsf{w}'_j$ for any $j \notin \{i-1, i, i+1\}$;
- the unordered pair $\{\mathsf{w_{i-1}w_iw_{i+1}}, \mathsf{w'_{i-1}w'_iw'_{i+1}}\}$ fits one of the following patterns:
  - (a) $\{\mathsf{xzy}, \mathsf{zxy}\}$ or $\{\mathsf{yxz}, \mathsf{yzx}\}$ for $x < y < z$, or
  - (b) $\{\mathsf{yxz}, \mathsf{xzy}\}$ or $\{\mathsf{yzx}, \mathsf{zxy}\}$ for $x = y{\downarrow} = z{\downarrow\downarrow}$, or
  - (c) $\{\mathsf{yyx}, \mathsf{yxy}\}$ for $y \in \mathcal{A}_\varnothing$, $x < y$, or
  - (d) $\{\mathsf{zyy}, \mathsf{yzy}\}$ for $y \in \mathcal{A}_\varnothing$, $y < z$, or
  - (e) $\{\mathsf{yyz}, \mathsf{yzy}\}$ for $y \in \mathcal{A}_{\text{-}}$, $y < z$, or
  - (f) $\{\mathsf{xyy}, \mathsf{yxy}\}$ for $y \in \mathcal{A}_{\text{-}}$, $x < y$.

We refer to the switches in (b) as *rotation switches* and the other switches as *Knuth switches.*

The next definition is an adaptation of the switchboards of [8] (which are based on the D graphs of Assaf [1]) to the super setting and the ideal $I_{\text{Kron-K}}$.

Figure 1: On the left, an $I_{\text{Kron-K}}$-switchboard whose edges are Knuth switches; on the right, an $I_{\text{Kron-K}}$-switchboard on the same vertex set whose edges are rotation switches. Knuth (resp. rotation) switches in position i are labeled i (resp. $\tilde{i}$).

Definition 3.2. An $I_{\text{Kron-K}}$-*switchboard* is an edge-labeled graph Γ on a vertex set of colored words of fixed length n in the alphabet $\mathcal{A}$ with edge labels from the set $\{2, 3, \dots, n-1\}$ such that each edge labeled i corresponds to a switch in position i, and each vertex in Γ which has exactly one $<$-descent in positions $i-1$ and i belongs to exactly one i-edge.

Note that there can be more than one $I_{\text{Kron-K}}$-switchboard on a given vertex set since the conditions in (a) and (b) of Definition 3.1 are not mutually exclusive—see Figure 1.

The $I_{\text{Kron-K}}$-switchboards give a convenient and intuitive understanding of the condition that a $(0,1)$-vector $\gamma \in \mathcal{U}$ lies in ${I_{\text{Kron-K}}}^{\perp}$.

Proposition 3.3. *For a set of colored words W of the same length, the following are equivalent:*

- $\sum_{\mathsf{w} \in W} \mathsf{w} \in {I_{\text{Kron-K}}}^{\perp}$*;*
- *W is the vertex set of an $I_{\text{Kron-K}}$-switchboard.*

We omit the proof of Proposition 3.3, which is not difficult; a similar result is proved in [6, Proposition-Definition 3.2].

Proposition 3.4. *There is an $I_{\text{Kron-K}}$-switchboard with vertex set $\text{CYW}_{\lambda,d}$ for any partition λ of n and $d \leq n$. Moreover, there is a unique $I_{\text{Kron-K}}$-switchboard with this vertex set in which every i-edge $\{\mathsf{w}, \mathsf{w}'\}$ such that $\{\mathsf{w_{i-1}}, \mathsf{w_i}, \mathsf{w_{i+1}}\} = \{\mathsf{x}, \mathsf{y}, \mathsf{z}\}$ with $x = y{\downarrow} = z{\downarrow\downarrow}$ is a rotation switch.*

Proof. By Proposition 4.6, $\sum_{\mathsf{w} \in \text{CYW}_{\lambda,d}} \mathsf{w} \in {I_{\text{Kron}}}^{\perp} \subset {I_{\text{Kron-K}}}^{\perp}$. Hence the first statement follows from Proposition 3.3.

Since the set $\text{CYW}_{\lambda,d}$ is closed under shuffling barred letters past unbarred letters, if $\mathsf{w} \in \text{CYW}_{\lambda,d}$ and $\{\mathsf{w_{i-1}w_iw_{i+1}}\}$ fits one of the patterns $\mathsf{yxz}, \mathsf{xzy}, \mathsf{yzx}, \mathsf{zxy}$ with $x = y{\downarrow} = z{\downarrow\downarrow}$, then the unique w' such that w and w' are related by a rotation switch in position i belongs to $\text{CYW}_{\lambda,d}$. The second statement follows. $\square$

Example 3.5. Let $\lambda = (3,2)$, $d = 2$. The set $\text{CYW}_{\lambda,d}$ is shown in Figure 2, along with the $I_{\text{Kron-K}}$-switchboard on this vertex set that is described in Proposition 3.4. This $I_{\text{Kron-K}}$-switchboard gives a way to "see" that $\sum_{\mathsf{w} \in \text{CYW}_{\lambda,d}} \mathsf{w} \in {I_{\text{Kron-K}}}^{\perp}$. Since only letters $\leq \bar{2}$ appear in $\text{CYW}_{\lambda,d}$, this also implies $\sum_{\mathsf{w} \in \text{CYW}_{\lambda,d}} \mathsf{w} \in I_{\text{Kron}}$.

To compute the symmetric function $F^{<}_{\text{CYW}_{\lambda,d}}(\mathbf{x})$, i.e., the sum of quasisymmetric functions associated to the $<$-descent sets of $\text{CYW}_{\lambda,d}$, we apply Corollary 2.6, which says that the coefficient of $s_\nu(\mathbf{x})$ in $F^{<}_{\text{CYW}_{\lambda,d}}(\mathbf{x})$ is equal to the number of $<$-colored tableaux T of shape

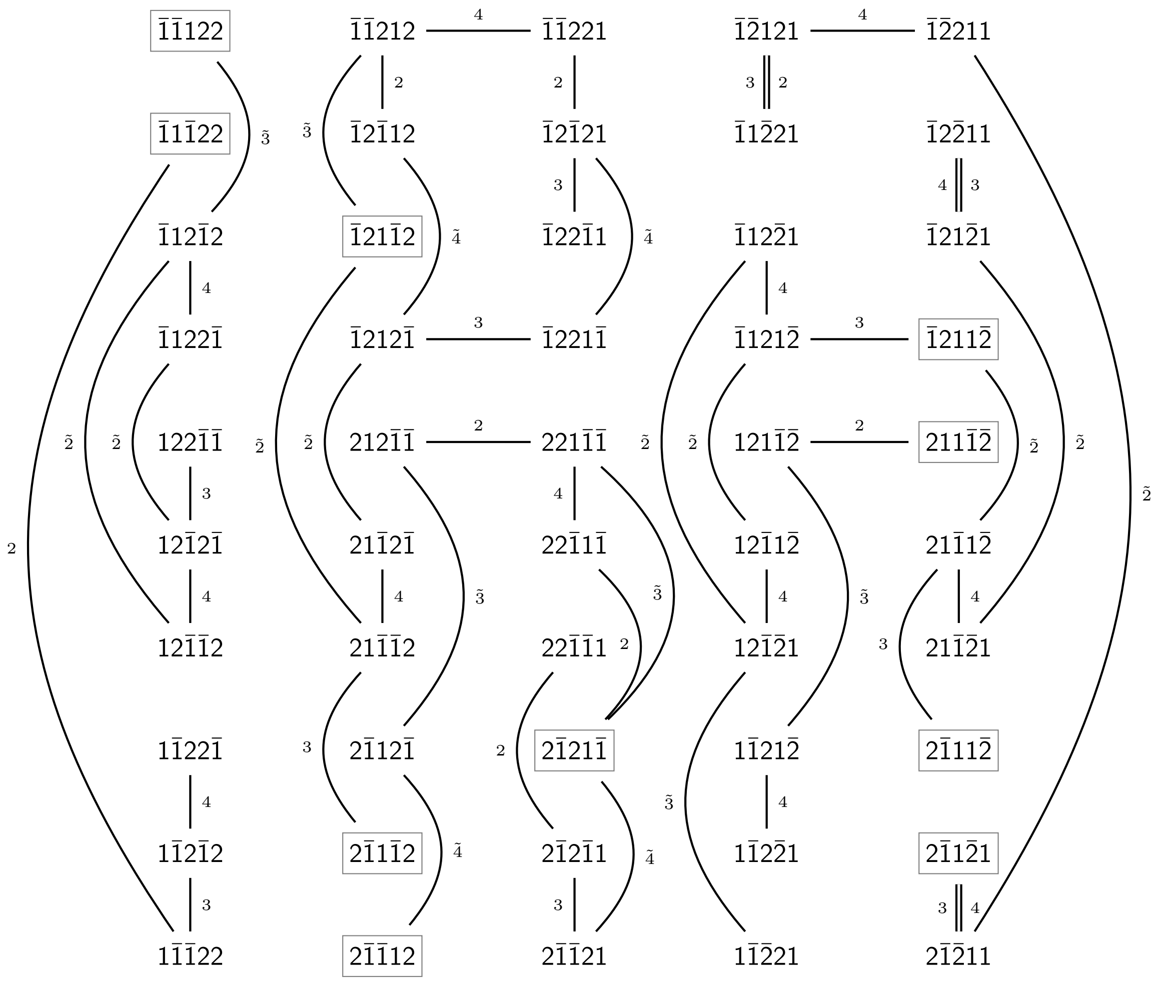
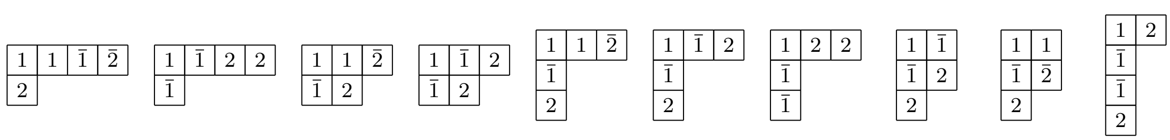


Figure 2: An $I_{\text{Kron-K}}$-switchboard on the vertex set $\mathrm{CYW}_{(3,2),2}$. The Schur expansion of the symmetric function $F^{<}_{\mathrm{CYW}_{(3,2),2}}(\mathbf{x})$ can be read off from the outlined words; see Example 3.5.

ν such that $\mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}$. The next line gives the $<$-colored tableau T such that $\mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}$, and below each tableau T is the colored word $\mathrm{arwread}(T)$ (these are the outlined words in Figure 2).



$211\bar1\bar2$ $\quad \bar11\bar122$ $\quad \bar1211\bar2$ $\quad \bar121\bar12$ $\quad 2\bar111\bar2$ $\quad 2\bar11\bar12$ $\quad \bar1\bar1122$ $\quad 2\bar121\bar1$ $\quad 2\bar11\bar21$ $\quad 2\bar1\bar112$

Hence $F^{<}_{\mathrm{CYW}_{\lambda,d}} = 2s_{41} + 2s_{32} + 3s_{311} + 2s_{221} + s_{2111}$.

Definition 3.6. For a colored word w and shuffle order $\lessdot$ on $\mathcal{A}$, *the $\lessdot$-insertion tableau of* w, denoted $P^{\lessdot}(\mathsf{w})$, is defined using the usual Schensted insertion algorithm using the order $\lessdot$ except that when a barred letter $\bar{a}$ is inserted into a row, it bumps the smallest letter $\gtrdot\!\!= \bar{a}$ (an unbarred letter a bumps the smallest letter $\gtrdot a$ as usual). The resulting tableau is indeed a $\lessdot$-colored tableau (see [4, §3] for details).

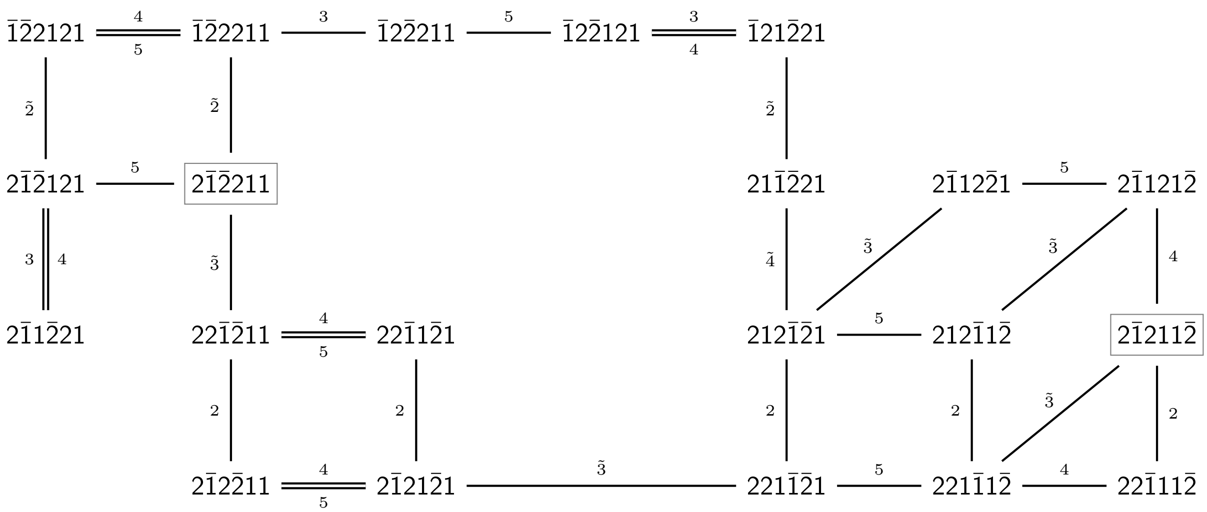


Figure 3: The $I_{\text{Kron-K}}$-switchboard Γ_1 from Example 3.8. As explained in the example, its symmetric function $F^{<}_{\text{Vert}(\Gamma_1)} = s_{321} + s_{222}$ can be computed from the outlined words.

Remark 3.7. It is easy to show that a colored word w is of the form $\text{arwread}(T)$ for $T \in \text{CT}^{<}_{\nu}$ if and only if $\mathsf{w} = \text{arwread}(P^{<}(\mathsf{w}))$. This gives a good way in practice to compute the subset of words of the form $\text{arwread}(T)$ of a given set of colored words.

Example 3.8. Let $\lambda = (3, 3)$, $d = 2$. The $I_{\text{Kron-K}}$-switchboard with vertex set $\text{CYW}_{\lambda,d}$ that is described in Proposition 3.4 has four components $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. The component Γ_1 is depicted in Figure 3. It follows from Proposition 3.3 and the fact that $\text{CYW}_{\lambda,d}$ only involves letters $\leq \bar{2}$, that $\sum_{\mathsf{w} \in \text{Vert}(\Gamma_i)} \mathsf{w} \in {I_{\text{Kron}}}^{\perp} \subseteq {I_{\text{Kron-K}}}^{\perp}$. Hence by Corollary 2.6, the symmetric function $F^{<}_{\text{Vert}(\Gamma_i)}(\mathbf{x})$ is computed by finding all $<$-colored tableaux T such that $\text{arwread}(T) \in \text{Vert}(\Gamma_i)$.

In the table below, we give the symmetric functions $F^{<}_{\text{Vert}(\Gamma_i)}(\mathbf{x})$, the sets

$$\{<\text{-colored tableau } T \mid \text{arwread}(T) \in \text{Vert}(\Gamma_i)\},$$

and the words $\text{arwread}(T)$ for each tableau T in these sets.

Γ_1	Γ_2	Γ_3	Γ_4
$s_{321} + s_{222}$	s_{42}	$s_{42} + s_{411}$	$s_{321} + s_{3111}$
$\begin{array}{lll} 1 & 1 & \bar2 \\ \bar1 & 2 & \\ 2 & & \end{array}$ $\begin{array}{ll} 1 & 1 \\ \bar1 & 2 \\ 2 & \bar2 \end{array}$	$\begin{array}{llll} 1 & 1 & \bar1 & \bar2 \\ 2 & 2 & & \end{array}$	$\begin{array}{llll} 1 & \bar1 & 2 & 2 \\ \bar1 & 2 & & \end{array}$ $\begin{array}{llll} 1 & \bar1 & 2 & 2 \\ \bar1 & & & \\ 2 & & & \end{array}$	$\begin{array}{lll} 1 & \bar1 & 2 \\ \bar1 & 2 & \\ 2 & & \end{array}$ $\begin{array}{lll} 1 & 2 & 2 \\ \bar1 & & \\ \bar1 & & \\ 2 & & \end{array}$
$2\bar1211\bar2$ $2\bar1\bar2211$	$2211\bar1\bar2$	$\bar121\bar122$ $2\bar11\bar122$	$2\bar121\bar12$ $2\bar1\bar1122$

This yields an expression for $F^{<}_{\text{CYW}_{\lambda,d}}$ as a sum of four Schur positive expressions: $F^{<}_{\text{CYW}_{\lambda,d}} = \sum_i F^{<}_{\text{Vert}(\Gamma_i)}$. This shows that Corollary 2.6 is stronger than Corollary 2.10.

3.2. **Noncommutative super Schur functions in colored plactic algebras.** Here we compare Theorem 2.3 and Corollaries 2.6 and 2.10 to analogous, but easier results for colored plactic algebras for any shuffle order (defined below).

We first extend several definitions involving the natural order (from §2.2–2.3) to an arbitrary shuffle order on $\mathcal{A}$. Throughout this subsection, fix a shuffle order $\lessdot$ on $\mathcal{A}$.

The $\lessdot$-*colored plactic algebra*, denoted $\mathcal{U}/I^{\lessdot}_{\text{plac}}$, is the quotient of $\mathcal{U}$ by the relations

$$\begin{aligned}
&\mathsf{xzy} = \mathsf{zxy} && \text{for } x,y,z \in \mathcal{A},\ x \lessdot y \lessdot z, && (15)\\
&\mathsf{yxz} = \mathsf{yzx} && \text{for } x,y,z \in \mathcal{A},\ x \lessdot y \lessdot z, && (16)\\
&\mathsf{yyx} = \mathsf{yxy} && \text{for } x \in \mathcal{A},\ y \in \mathcal{A}_{\varnothing},\ x \lessdot y, && (17)\\
&\mathsf{zyy} = \mathsf{yzy} && \text{for } z \in \mathcal{A},\ y \in \mathcal{A}_{\varnothing},\ y \lessdot z, && (18)\\
&\mathsf{yyz} = \mathsf{yzy} && \text{for } z \in \mathcal{A},\ y \in \mathcal{A}_{-},\ y \lessdot z, && (19)\\
&\mathsf{xyy} = \mathsf{yxy} && \text{for } x \in \mathcal{A},\ y \in \mathcal{A}_{-},\ x \lessdot y; && (20)
\end{aligned}$$

let $I^{\lessdot}_{\text{plac}}$ denote the corresponding two-sided ideal of $\mathcal{U}$. The relations (15)–(20) are the same as (2)–(7) with $\lessdot$ in place of the natural order $<$.

Definition 3.9 (*Noncommutative super Schur functions for shuffle orders*)**.** For any two letters $x, y \in \mathcal{A}$, write $y \geq_{\text{col}} x$ to mean either $y \gtrdot x$, or y and x are equal barred letters. Define the following generalization of the $e_k(\mathbf{u})$:

$$e^{\lessdot}_k(\mathbf{u}) = \sum_{\substack{z_1 \geq_{\text{col}} z_2 \geq_{\text{col}} \cdots \geq_{\text{col}} z_k \\ z_1,\dots,z_k \in \mathcal{A}}} u_{z_1} u_{z_2} \cdots u_{z_k} \ \in \mathcal{U}$$

for any positive integer k; set $e^{\lessdot}_0(\mathbf{u}) = 1$ and $e^{\lessdot}_k(\mathbf{u}) = 0$ for $k < 0$. Now define the noncommutative super Schur functions $\mathfrak{J}^{\lessdot}_{\nu}(\mathbf{u})$ exactly as in Definition 2.1, except with $e^{\lessdot}_k(\mathbf{u})$ in place of $e_k(\mathbf{u})$.

A $\lessdot$-*colored tableau* is a tableau with entries in $\mathcal{A}$ such that each row and column is weakly increasing with respect to the order $\lessdot$, while the unbarred letters in each column and the barred letters in each row are strictly increasing. Let $\text{CT}^{\lessdot}_{\nu}$ denote the set of $\lessdot$-colored tableaux of shape ν.

For a $\lessdot$-colored tableaux T, the *column reading word* of T, denoted $\text{colword}(T)$, is the colored word obtained by concatenating the columns of T (reading each column bottom to top), starting with the leftmost column. For example,

$$\text{colword}\left(\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \bar{3} & \bar{4} & 6 \\ \hline \bar{2} & 3 & 4 & \bar{4} \\ \cline{1-4} 3 & \bar{3} & \bar{4} & 5 \\ \cline{1-4}\end{array}\right) = \mathsf{3\,\bar{2}\,1\,\bar{3}\,3\,1\,\bar{4}\,4\,\bar{3}\,5\,\bar{4}\,\bar{4}\,6}.$$

Also, for any $\lessdot$-colored tableaux T, we define the colored word $\text{arwread}(T)$ exactly as in Definition 2.2.

The next theorem is an analog of Theorem 2.3 for the $\lessdot$-colored plactic algebra.

Theorem 3.10. *For any partition* ν*,*

$$\mathfrak{J}^{\lessdot}_{\nu}(\mathbf{u}) = \sum_{T \in \text{CT}^{\lessdot}_{\nu}} \text{colword}(T) \qquad \text{in} \ \ \mathcal{U}/I^{\lessdot}_{\text{plac}}.$$

This theorem follows from a similar result for the ordinary plactic algebra and a standardization argument. So as not to interrupt the discussion, we postpone the proof to §4.5.

Remark 3.11. The word colword(T) in Theorem 3.10 can be replaced by any colored word that has $\lessdot$-insertion tableau T (see Definition 3.6). In particular, it can be replaced by arwread(T). Here we are using that $\lessdot$-colored plactic equivalence classes are the same as sets of colored words with a fixed $\lessdot$-insertion tableau; this can be shown by a standardization argument similar to that in Proposition 4.8.

Similar to Theorem 2.5, the next result is by Proposition 8.3 and Theorem 8.4 (it is also a straightforward adaptation of the setup of [11]). Recall from Definition 2.4 the notation $F^{\lessdot}_{\gamma}(\mathbf{x}) = \sum_{\mathsf{w}} \gamma_{\mathsf{w}} Q_{\mathrm{Des}_{\lessdot}(\mathsf{w})}(\mathbf{x})$ for $\gamma = \sum_{\mathsf{w}} \gamma_{\mathsf{w}} \mathsf{w}$, and $F^{\lessdot}_{W}(\mathbf{x}) = \sum_{\mathsf{w} \in W} Q_{\mathrm{Des}_{\lessdot}(\mathsf{w})}(\mathbf{x})$ for a set of colored words W.

Theorem 3.12. *For any $\gamma \in (I^{\lessdot}_{\mathrm{plac}})^{\perp}$, the function $F^{\lessdot}_{\gamma}(\mathbf{x})$ is symmetric and*

$$F^{\lessdot}_{\gamma}(\mathbf{x}) = \sum_{\nu} s_{\nu}(\mathbf{x}) \langle \mathfrak{J}^{\lessdot}_{\nu}(\mathbf{u}), \gamma \rangle.$$

Theorems 3.12 and 3.10 and Remark 3.11 then have the following consequence, which is an analog of Corollary 2.6 for the ideal $I^{\lessdot}_{\mathrm{plac}}$.

Corollary 3.13. *For any set of colored words W such that $\sum_{\mathsf{w} \in W} \mathsf{w} \in (I^{\lessdot}_{\mathrm{plac}})^{\perp}$,*

$$\Big(\text{the coefficient of } s_{\nu}(\mathbf{x}) \text{ in } F^{\lessdot}_{W}(\mathbf{x})\Big) = \Big|\Big\{T \in \mathrm{CT}^{\lessdot}_{\nu} \mid \mathrm{arwread}(T) \in W\Big\}\Big|.$$

Corollary 3.13 can also be proven directly from the fact [13] that $s_{\nu}(\mathbf{x}) = F^{\lessdot}_{C}(\mathbf{x})$ for any ordinary Knuth equivalence class C with insertion tableau of shape ν, and a standardization argument; however, we have presented it this way to parallel Corollary 2.6 and to give a sense of the comparative strengths of Theorems 2.3 and 3.10.

We also have the (partial) analog of Corollary 2.10 for the $\prec$-colored plactic algebra.

Corollary 3.14. *For any partitions λ, ν of n and $d \leq n$,*

$$\Big(\text{the coefficient of } s_{\nu}(\mathbf{x}) \text{ in } F^{\prec}_{\mathrm{CYW}_{\lambda,d}}(\mathbf{x})\Big) = \Big|\Big\{T \in \mathrm{CT}^{\prec}_{\nu} \mid \mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}\Big\}\Big|.$$

Proof. This follows from Corollary 3.13 with $W = \mathrm{CYW}_{\lambda,d}$, provided we verify $\sum_{\mathsf{w} \in W} \mathsf{w} \in (I^{\prec}_{\mathrm{plac}})^{\perp}$. We must check that if colored words v and w differ by a single application of one of the relations (15)–(20), then neither or both belong to $\mathrm{CYW}_{\lambda,d}$. This holds in the case when v and w differ by a relation involving three barred letters or three unbarred letters because the set of Yamanouchi words of content λ is an ordinary Knuth equivalence class. On the other hand, if v and w differ by a relation involving three letters not all barred nor all unbarred, then the smallest letter (for $\prec$) must be unbarred and the largest must be barred, hence swapping these two letters preserves the property of being in $\mathrm{CYW}_{\lambda,d}$. □

Note that the order $\prec$ is important here since it is not always true that $\sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in (I^{\lessdot}_{\mathrm{plac}})^{\perp}$ for shuffle orders $\lessdot$ other than $\prec$ (see the discussion after Corollary 2.10).

3.3. **Conversion.** Conversion is an operation on colored tableaux that gives a bijection between the sets $CT^{<}_{\nu}$ and $CT^{\prec}_{\nu}$ (see Definition 3.17 below). We first state a strengthening of Corollary 2.10 and then relate this to conversion. We believe that the results here are the first glimpses of a deep connection between conversion and the $\mathfrak{J}^{\lessdot}_{\nu}(\mathbf{u})$ for different orders $\lessdot$.

A set of colored words W is *shuffle closed* if for every $\mathsf{w} \in W$, any word obtained from w by swapping a barred letter with an adjacent unbarred letter also lies in W.

Corollary 3.15. *For any shuffle closed set of colored words W such that $\sum_{\mathsf{w}\in W} \mathsf{w} \in (I_{\text{Kron}})^{\perp}$, there holds*

$$\left|\left\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in W\right\}\right| \tag{21}$$
$$= \textit{the coefficient of } s_\nu(\mathbf{x}) \textit{ in } F^{<}_{W}(\mathbf{x}) \tag{22}$$
$$= \textit{the coefficient of } s_\nu(\mathbf{x}) \textit{ in } F^{\prec}_{W}(\mathbf{x}) \tag{23}$$
$$= \left|\left\{T \in \mathrm{CT}^{\prec}_{\nu} \mid \mathrm{arwread}(T) \in W\right\}\right|. \tag{24}$$

Remark 3.16. Here is a useful way to rephrase the condition on W in Corollary 3.15.

Let $\mathcal{U}/J$ be the quotient of $\mathcal{U}$ by the relations (4)–(7), (9), and

$$\mathsf{xz} = \mathsf{zx}, \qquad x \in \mathcal{A}_{\varnothing}, z \in \mathcal{A}_{-}. \tag{25}$$

(We do not require $x < z$ in (25).) Note that $\mathcal{U}/J$ is a monoid algebra and J is homogeneous, hence the space $J^{\perp}$ has $\mathbb{Z}$-basis given by $\sum_{\mathsf{w}\in C} \mathsf{w}$, as C ranges over equivalence classes of colored words modulo J. It is easy to show that

$$\Big(W \text{ is shuffle closed and } \textstyle\sum_{\mathsf{w}\in W} \mathsf{w} \in (I_{\text{Kron}})^{\perp}\Big) \iff W \text{ is a union of equivalence classes mod } J.$$

Proof of Corollary 3.15. First note that by Remark 3.16 and the fact $I^{\prec}_{\text{plac}} \subseteq J$, it follows that $\sum_{\mathsf{w}\in W} \mathsf{w} \in (I^{\prec}_{\text{plac}})^{\perp}$ as well. Hence the third equality in the statement is by Corollary 3.13. The first equality is by Corollary 2.6 and the second is by Proposition 4.5. □

In light of Corollary 3.15, it is natural to conjecture that the sets of tableaux appearing in (21) and (24) can be obtained from each other by conversion. Theorem 3.18 shows that this is indeed the case; this is a generalization of [22, Lemma 3.1] (which is Corollary 2.10) and its proof uses conversion and is similar to that of [22, Lemma 3.1]. We do not know how to prove this using noncommutative super Schur function machinery since this machinery does not easily lend itself to bijective results. We suspect that there is something deeper lurking here that would connect the $\mathfrak{J}^{\lessdot}_{\nu}(\mathbf{u})$ for different orders $\lessdot$ (considered in different algebras depending on $\lessdot$) via something like conversion.

Definition 3.17 (Conversion)**.** Suppose $\lessdot$ and $\lessdot'$ are shuffle orders on $\mathcal{A}$ that are identical except for the order of b and $\bar{a}$, say $b \lessdot \bar{a}$ but $\bar{a} \lessdot' b$. Then there is a natural bijection between $\lessdot$-colored tableaux and $\lessdot'$-colored tableaux via a process called *conversion*, introduced by Haiman [15] (see also [3], [22, §2.3], [7, §2.6]).

Let $T_{\lessdot}$ be a $\lessdot$-colored tableau, so that the boxes containing b or $\bar{a}$ form a ribbon. For each component of the ribbon, there is a unique way to refill it with the same number of b's and $\bar{a}$'s in a way that is compatible with $\bar{a} \lessdot' b$. (If the northeast corner contains b, then shift each b to the bottom of its column; if the northeast corner contains $\bar{a}$, shift each b

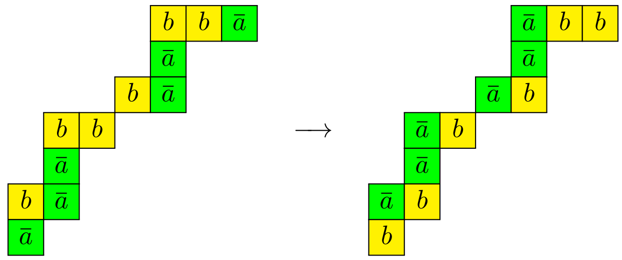


Figure 4: Conversion from $b \lessdot \bar{a}$ to $\bar{a} \lessdot' b$. The ribbon above has two connected components with six boxes each.

once to the right within its row. See Figure 4.) Replacing each component in this manner yields a $\lessdot'$-colored tableau denoted $T_{\lessdot}(\lessdot \to \lessdot')$.

Given any two shuffle orders $\lessdot$ and $\lessdot'$ on $\mathcal{A}$, one can be obtained from the other by repeatedly swapping the order of a consecutive unbarred letter and barred letter. Therefore, one can iterate the process above to convert any $\lessdot$-colored tableau $T_{\lessdot}$ to a $\lessdot'$-colored tableau denoted $T_{\lessdot}(\lessdot \to \lessdot')$. The resulting tableau is well defined, i.e., it does not depend on the sequence of swaps used to go from $\lessdot$ to $\lessdot'$ (see [3]).

Theorem 3.18. *Suppose that a set of colored words W is shuffle closed and satisfies $\sum_{\mathsf{w} \in W} \mathsf{w} \in (I_{\mathrm{Kron}})^{\perp}$. Let $\mathrm{CT}^{\prec}_{\nu}(W)$ and $\mathrm{CT}^{<}_{\nu}(W)$ denote the sets of colored tableaux $\{T \in \mathrm{CT}^{\prec}_{\nu} \mid \mathrm{arwread}(T) \in W\}$ and $\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in W\}$, respectively. Then*

$$\{T(\prec \to <) \mid T \in \mathrm{CT}^{\prec}_{\nu}(W)\} = \mathrm{CT}^{<}_{\nu}(W). \tag{26}$$

Proof. Throughout the proof, we write $f \equiv g$ to mean that f and g are congruent modulo I_{Kron} (for $f, g \in \mathcal{U}$).

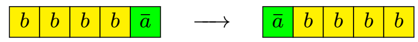
Fix some $T \in \mathrm{CT}^{\prec}_{\nu}(W)$, and for any shuffle order $\lessdot$ on $\mathcal{A}$, let $\mathsf{w}^{\lessdot} = \mathrm{arwread}(T(\prec \to \lessdot))$. Let $\mathsf{w}^{\lessdot}_{\varnothing}$ (resp. $\mathsf{w}^{\lessdot}_{-}$) be the subsequence of $\mathsf{w}^{\lessdot}$ consisting of the unbarred (resp. barred) letters of $\mathsf{w}^{\lessdot}$. We need to show that $\mathsf{w}^{<}$ lies in W. Since W is shuffle closed, we can instead show $\mathsf{w}^{<}_{\varnothing}\mathsf{w}^{<}_{-} \in W$, and we know that $\mathsf{w}^{\prec}_{\varnothing}\mathsf{w}^{\prec}_{-} \in W$. Hence the result follows if we show $\mathsf{w}^{\prec}_{\varnothing} \equiv \mathsf{w}^{<}_{\varnothing}$ and $\mathsf{w}^{\prec}_{-} \equiv \mathsf{w}^{<}_{-}$. We prove the first congruence below; the other is similar.

We can convert from $\prec$ to $<$ by swapping the order of $\bar{1}$ with $N, N-1, \ldots, 2$, then swapping $\bar{2}$ with $N, N-1, \ldots, 3$, and so forth. Each step performs a switch between consecutive letters of the form

$$\cdots \lessdot b-1 \lessdot b \lessdot \bar{a} \lessdot b+1 \lessdot \cdots \quad \to \quad \cdots \lessdot' b-1 \lessdot' \bar{a} \lessdot' b \lessdot' b+1 \lessdot' \cdots .$$

Denote by d^i the decreasing word consisting of the unbarred letters in the i-th diagonal of $T(\prec \to \lessdot)$, along with b (if it does not already appear); denote by $\hat{\mathsf{d}}^i$ the word obtained from d^i by removing b. Thus $\mathsf{w}^{\lessdot}_{\varnothing}$ and $\mathsf{w}^{\lessdot'}_{\varnothing}$ are products over all i of either d^i or $\hat{\mathsf{d}}^i$. An easy calculation shows that since d^i contains b and is decreasing, b and d^i commute in the (ordinary) plactic algebra (defined in §4.5); hence they commute in $\mathcal{U}/I_{\mathrm{Kron}}$ as well.

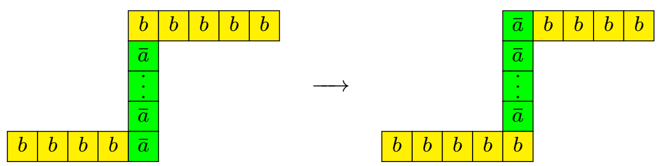
In the conversion of $T(\prec \to \lessdot)$ to $T(\prec \to \lessdot')$, first consider the occurrences of b that get shifted one box to the right. We can handle such b one row at a time from top to bottom, and each step looks as follows:



We need to show that $\mathsf{d}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j \equiv \hat{\mathsf{d}}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\mathsf{d}^j$, where the boxes shown above lie in diagonals i through j. Since the b appearing in d^i is the leftmost b in its row, d^i cannot contain $\mathsf{b}-1$, so the far commutation relations (9) imply $\mathsf{d}^i \equiv \hat{\mathsf{d}}^i\mathsf{b}$. Similarly, d^j cannot contain $\mathsf{b}+1$, so $\mathsf{d}^j \equiv \mathsf{b}\hat{\mathsf{d}}^j$. Thus

$$\mathsf{d}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j \equiv \hat{\mathsf{d}}^i\mathsf{b}\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j \equiv \hat{\mathsf{d}}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\mathsf{b}\hat{\mathsf{d}}^j \equiv \hat{\mathsf{d}}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\mathsf{d}^j.$$

Now consider, in the conversion of $T(\prec\to\lessdot)$ to $T(\prec\to\lessdot')$, the occurrences of b that get shifted down. If we handle such b one at a time from top to bottom, then each step looks as follows:



Let $i \le j < k \le l$ be the indices such that the boxes shown above lie in diagonals i through l, and j and k are the diagonals of the boxes whose entries change upon conversion. We need to show that

$$\mathsf{d}^i\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j\cdots\hat{\mathsf{d}}^{k-1}\mathsf{d}^k\cdots\mathsf{d}^\ell \equiv \mathsf{d}^i\cdots\mathsf{d}^j\hat{\mathsf{d}}^{j+1}\cdots\hat{\mathsf{d}}^k\mathsf{d}^{k+1}\cdots\mathsf{d}^\ell.$$

As in the previous case, $\mathsf{d}^i \equiv \hat{\mathsf{d}}^i\mathsf{b}$, $\mathsf{d}^j \equiv \mathsf{b}\hat{\mathsf{d}}^j$, $\mathsf{d}^k \equiv \hat{\mathsf{d}}^k\mathsf{b}$, and $\mathsf{d}^\ell \equiv \mathsf{b}\hat{\mathsf{d}}^\ell$. Moreover, the diagonals $j+1,\dots,k-1$ cannot contain $\mathsf{b}-1$ or $\mathsf{b}+1$, so b commutes with $\hat{\mathsf{d}}^{j+1},\dots,\hat{\mathsf{d}}^{k-1}$ in $\mathcal{U}/I_{\mathrm{Kron}}$. Thus, assuming $i<j<k<l$ (the other cases are similar and easier), we have

$$\begin{aligned}
\mathsf{d}^i\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j\cdots\hat{\mathsf{d}}^{k-1}\mathsf{d}^k\cdots\mathsf{d}^\ell &\equiv \hat{\mathsf{d}}^i\mathsf{b}\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\hat{\mathsf{d}}^j\cdots\hat{\mathsf{d}}^{k-1}\mathsf{d}^k\cdots\mathsf{d}^{\ell-1}\mathsf{b}\hat{\mathsf{d}}^\ell\\
&\equiv \hat{\mathsf{d}}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\mathsf{b}\hat{\mathsf{d}}^j\cdots\hat{\mathsf{d}}^{k-1}\mathsf{b}\mathsf{d}^k\cdots\mathsf{d}^{\ell-1}\hat{\mathsf{d}}^\ell\\
&\equiv \hat{\mathsf{d}}^i\mathsf{d}^{i+1}\cdots\mathsf{d}^{j-1}\mathsf{d}^j\hat{\mathsf{d}}^{\mathsf{j}+1}\cdots\hat{\mathsf{d}}^{k-1}\mathsf{b}\hat{\mathsf{d}}^k\mathsf{b}\mathsf{d}^{k+1}\cdots\mathsf{d}^{\ell-1}\hat{\mathsf{d}}^\ell\\
&\equiv \hat{\mathsf{d}}^i\mathsf{b}\mathsf{d}^{i+1}\cdots\mathsf{d}^j\hat{\mathsf{d}}^{\mathsf{j}+1}\cdots\hat{\mathsf{d}}^k\mathsf{d}^{k+1}\cdots\mathsf{d}^{\ell-1}\mathsf{b}\hat{\mathsf{d}}^\ell\\
&\equiv \mathsf{d}^i\cdots\mathsf{d}^j\hat{\mathsf{d}}^{\mathsf{j}+1}\cdots\hat{\mathsf{d}}^k\mathsf{d}^{k+1}\cdots\mathsf{d}^\ell,
\end{aligned}$$

as desired. □

Example 3.19. As mentioned above, Theorem 3.18 is a generalization of Corollary 2.10; this is clear since the sets $\mathrm{CYW}_{\lambda,d}$ are shuffle closed and satisfy $\sum_{\mathsf{w}\in\mathrm{CYW}_{\lambda,d}}\mathsf{w} \in {I_{\mathrm{Kron}}}^\perp$ (Proposition 4.6). To see that Theorem 3.18 is indeed an interesting generalization, consider the set $\mathrm{CYW}_{(3,2),2}$ depicted in Figure 2. The subset W of colored words in the first column is shuffle closed and satisfies $\sum_{\mathsf{w}\in W}\mathsf{w} \in {I_{\mathrm{Kron}}}^\perp$; the same goes for the colored words in the second and third columns, as well as the colored words in the fourth and fifth columns. None of these subsets is of the form $\mathrm{CYW}_{\lambda,d}$.

4. Proofs for Section 2 and Theorem 3.10

We now fill in the proofs of Proposition 2.8, Proposition 2.9, Corollary 2.10, Corollary 2.11, and Theorem 3.10 in §4.1, §4.2, §4.3, §4.4, and §4.5, respectively. The only genuinely new content here is the word conversion trick used to prove Proposition 2.9.

4.1. **Proof of Proposition 2.8.**

Proposition 2.8 is obtained by combining Corollary 3.14 and the following result. The proof below follows that of [22, Proposition 4.1] (which is in turn based on [7, Proposition 3.1]), modified to the conventions of this paper.

Proposition 4.1. *For any partitions λ, ν of n and $d \leq n$,*

$$g_{\lambda\, \mu(d)\, \nu} + g_{\lambda\, \mu(d-1)\, \nu} = \left|\left\{T \in \mathrm{CT}_\nu^{\prec} \mid \mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}\right\}\right|.$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the Hall inner product on the ring of symmetric functions, so that, for instance, $\langle s_\lambda, s_\mu s_\nu \rangle$ is the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\,\nu}$. Let $*$ denote the internal product on symmetric functions defined by $\langle s_\lambda, s_\mu * s_\nu \rangle = g_{\lambda\mu\nu}$. For a partition α, let α' denote the conjugate partition of α. We have

$$\begin{aligned} g_{\lambda\, \mu(d)\, \nu} + g_{\lambda\, \mu(d-1)\, \nu} &= \big\langle s_\lambda, s_{\mu(d)} * s_\nu \big\rangle + \big\langle s_\lambda, s_{\mu(d-1)} * s_\nu \big\rangle \\ &= \big\langle s_\lambda, \big(s_{\mu(d)} + s_{\mu(d-1)}\big) * s_\nu \big\rangle \\ &= \big\langle s_\lambda, \big(s_{(n-d)} \cdot s_{(1^d)}\big) * s_\nu \big\rangle \\ &= \sum_{\alpha \vdash n-d,\ \beta \vdash d} \big\langle s_\lambda, c^{\nu}_{\alpha\,\beta} \big(s_{(n-d)} * s_\alpha\big)\big(s_{(1^d)} * s_\beta\big) \big\rangle \\ &= \sum_{\alpha \vdash n-d,\ \beta \vdash d} \big\langle s_\lambda, c^{\nu'}_{\alpha'\,\beta'}\, s_\alpha s_{\beta'} \big\rangle \\ &= \sum_{\alpha \vdash n-d} \langle s_\lambda, s_\alpha s_{\nu'/\alpha'} \rangle. \end{aligned}$$

The fourth equality is by a general result of Littlewood [21] which relates the internal and ordinary products of the symmetric group; the fifth equality uses the symmetry $c^{\nu}_{\alpha\,\beta} = c^{\nu'}_{\alpha'\,\beta'}$ of Littlewood-Richardson coefficients.

Let $\alpha \oplus (\nu/\alpha)'$ be the skew shape consisting of the disjoint union of α and $(\nu/\alpha)'$, translated to lie in distinct rows and columns (with α to the southwest). By the Littlewood-Richardson rule, $\langle s_\lambda, s_\alpha s_{\nu'/\alpha'} \rangle$ is the number of semistandard tableaux U of shape $\alpha \oplus (\nu/\alpha)'$ and content λ whose (row) reading word is Yamanouchi. By transposing the part of U that lies in $(\nu/\alpha)'$, barring each letter within, and combining with the part of U lying in α, we arrive at a $\prec$-colored tableau $U^{\prec}$ of content λ, with d barred letters (whose barred letters form the shape ν/α), and shape ν such that $\mathrm{arwread}(U^{\prec})^{\mathrm{brgt}}$ is Yamanouchi; here we are using that the ordinary word $\mathrm{arwread}(U^{\prec})^{\mathrm{brgt}}$ is a reading word of U since it consists of a reading word of the part of U that lies in α followed by a reading word of the part of U that lies in $(\nu/\alpha)'$. Summing over all α yields

$$\sum_{\alpha \vdash n-d} \langle s_\lambda, s_\alpha s_{\nu'/\alpha'} \rangle = \left|\left\{T \in \mathrm{CT}_\nu^{\prec} \mid \mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}\right\}\right|,$$

as desired. □

4.2. **Word conversion.** Here we prove Proposition 2.9 using an operator called word conversion.

Definition 4.2 (Word conversion)**.** Let $\lessdot$ and $\lessdot'$ be two shuffle orders on $\mathcal{A}$ that are identical except for the order of $\bar{a}$ and b, say $b \lessdot \bar{a}$ and $\bar{a} \lessdot' b$ for some $a, b \in [N]$. For a colored word $\mathsf{w} = \mathsf{w}_1 \cdots \mathsf{w}_t$ consisting of $\bar{\mathsf{a}}$'s and b's, define $\mathsf{w}(\lessdot \to \lessdot')$ to be the result of cyclically rotating w once to the right, i.e.

$$\mathsf{w}(\lessdot \to \lessdot') := \mathsf{w}_t \mathsf{w}_1 \mathsf{w}_2 \cdots \mathsf{w}_{t-1}.$$

In general, for any colored word w, $\mathsf{w}(\lessdot \to \lessdot')$ fixes the subword consisting of letters that are not $\bar{\mathsf{a}}$ or b and rotates each subword consisting of $\bar{\mathsf{a}}$'s and b's once to the right.

Example 4.3. Let $N = 3$ and let $\prec$ and $\prec'$ be the following two shuffle orders on $\mathcal{A}$

$$1 \prec 2 \prec 3 \prec \bar{1} \prec \bar{2} \prec \bar{3}$$
$$1 \prec' 2 \prec' \bar{1} \prec' 3 \prec' \bar{2} \prec' \bar{3}.$$

Below are two examples of colored words and the result obtained by applying the word conversion operator.

$$\begin{array}{ll} \mathsf{w} & \bar{1}\bar{1}33\bar{1}\bar{1}\bar{1}3\bar{1} \\ \mathsf{w}(\prec \to \prec') & \bar{1}\bar{1}\bar{1}33\bar{1}\bar{1}\bar{1}3 \\ & \\ \mathsf{v} & 33\bar{1}\bar{1}3\bar{1}32\bar{2}1\bar{1}3 \\ \mathsf{v}(\prec \to \prec') & 333\bar{1}\bar{1}3\bar{1}2\bar{2}13\bar{1} \end{array}$$

Word conversion is defined essentially to make the following property true. Its proof is straightforward.

Lemma 4.4. *Word conversion respects descent sets in the following sense:*

$$\mathrm{Des}_{\lessdot'}(\mathsf{w}(\lessdot \to \lessdot')) = \mathrm{Des}_{\lessdot}(\mathsf{w}), \tag{27}$$

for shuffle orders $\lessdot$ and $\lessdot'$ on $\mathcal{A}$ as in Definition 4.2.

Recall that a set of colored words W is shuffle closed if for every $\mathsf{w} \in W$, any word obtained from w by swapping a barred letter with an adjacent unbarred letter also lies in W. Proposition 2.9 is a consequence of the following more general result.

Proposition 4.5. *For any shuffle closed set of colored words W, $F_W^{\prec}(\mathbf{x}) = F_W^{<}(\mathbf{x})$.*

Proof. Let $\lessdot$ and $\lessdot'$ be two shuffle orders on $\mathcal{A}$ that differ by swapping a single covering relation (as in Definition 4.2). The word conversion operator $\mathsf{w} \mapsto \mathsf{w}(\lessdot \to \lessdot')$ is an involution on colored words. In fact, it is an involution on any shuffle closed set. Together with Lemma 4.4, this yields

$$F_W^{\prec}(\mathbf{x}) = \sum_{\mathsf{w} \in W} Q_{\mathrm{Des}_{\lessdot'}(\mathsf{w})}(\mathbf{x}) = \sum_{\mathsf{w} \in W} Q_{\mathrm{Des}_{\lessdot}(\mathsf{w})}(\mathbf{x}) = F_W^{<}(\mathbf{x}).$$

Applying this repeatedly, converting from the big bar order $\prec$ to the natural order $<$ by swapping one covering relation at a time, gives the result. □

4.3. **Proof of Corollary 2.10.** Aside from the proof of the main theorem (Theorem 2.3), the following fact is all that remains to prove Corollary 2.10.

Proposition 4.6. *There holds* $\sum_{\mathsf{w} \in \mathrm{CYW}_{\lambda,d}} \mathsf{w} \in {I_{\mathrm{Kron}}}^{\perp}$.

Proof. Recall Remark 3.16 and let J be as in the remark. It suffices to show that $\mathrm{CYW}_{\lambda,d}$ is a union of equivalence classes modulo J. This amounts to showing that if w and w' differ by a single application of one of the relations (4)–(7), (9), (25), then $\mathsf{w} \in \mathrm{CYW}_{\lambda,d} \iff \mathsf{w}' \in \mathrm{CYW}_{\lambda,d}$. Since $\mathrm{CYW}_{\lambda,d}$ is shuffle closed, the necessary result holds for w and w' that differ by (25) or by (4)–(7), (9) in the case that exactly one of the two letters involved is barred. Since Knuth transformations preserve whether an ordinary word is Yamanouchi, the necessary result holds if w and w' differ by (4)–(7) and the two letters are both barred or both unbarred. Finally, an ordinary word's being Yamanouchi is unchanged by swapping two adjacent letters that differ by more than 1, which handles the case that w and w' differ by (9) and the two letters involved are both barred or both unbarred. □

4.4. **Proof of Corollary 2.11.** For convenience, we reproduce the argument from the proof of [22, Theorem 4.2] which shows that Corollary 2.11 follows from Corollary 2.10.

Let $\mathrm{CYT}_{\lambda,d}(\nu)$ denote the set $\{T \in \mathrm{CT}^{<}_{\nu} \mid \mathrm{arwread}(T) \in \mathrm{CYW}_{\lambda,d}\}$, which has size $g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu}$ by Corollary 2.10. Write $\mathrm{CYT}^{-}_{\lambda,d}(\nu)$ (resp. $\mathrm{CYT}^{+}_{\lambda,d}(\nu)$) for the subset of $\mathrm{CYT}_{\lambda,d}(\nu)$ with unbarred (resp. barred) northeast corner; to relate this to the informal proof in §2.5, this is the set of $T \in \mathrm{CYT}_{\lambda,d}(\nu)$ such that $\mathrm{arwread}(T)$ ends in an unbarred (resp. barred) letter.

Since $\bar{a}$ immediately follows a in $<$ for all unbarred letters a, the northeast corner of a $<$-colored tableau T can always be toggled from barred to unbarred or vice versa and remain a $<$-colored tableau. Moreover, this toggle does not change the word $\mathrm{arwread}(T)^{\mathrm{brgt}}$ since it simply moves the last letter of the unbarred subword of $\mathrm{arwread}(T)$ to be the last letter of the barred subword of $\mathrm{arwread}(T)$ or vice versa. Hence $|\mathrm{CYT}^{+}_{\lambda,d}(\nu)| = |\mathrm{CYT}^{-}_{\lambda,d-1}(\nu)|$.

The result now follows by induction on d: if $|\mathrm{CYT}^{-}_{\lambda,d-1}(\nu)| = g_{\lambda\,\mu(d-1)\,\nu}$, then

$$g_{\lambda\,\mu(d)\,\nu} = |\mathrm{CYT}_{\lambda,d}(\nu)| - |\mathrm{CYT}^{-}_{\lambda,d-1}(\nu)| = |\mathrm{CYT}_{\lambda,d}(\nu)| - |\mathrm{CYT}^{+}_{\lambda,d}(\nu)| = |\mathrm{CYT}^{-}_{\lambda,d}(\nu)|.$$

4.5. **Standardization.** We fill in the details of the standardization argument needed to prove Theorem 3.10. This material will also be used in §7.2.

In our discussion of standardization, we will work with an alphabet of ordinary (unbarred) letters and the ordinary plactic algebra. Let $\mathcal{V}_M$ be the free associative $\mathbb{Z}$-algebra in the noncommuting variables v_x, $x \in [M]$. We identify monomials of $\mathcal{V}_M$ with words in the alphabet $\mathsf{1}, \mathsf{2}, \dots, \mathsf{M}$. Let $\mathcal{V}_M/I_{\mathrm{plac}}$ be the *plactic algebra*, the quotient of $\mathcal{V}_M$ by the *plactic relations*

$$\mathsf{acb} = \mathsf{cab} \qquad \text{for } a, b, c \in [M],\ a < b < c, \tag{28}$$

$$\mathsf{bac} = \mathsf{bca} \qquad \text{for } a, b, c \in [M],\ a < b < c, \tag{29}$$

$$\mathsf{bba} = \mathsf{bab} \qquad \text{for } a, b \in [M],\ a < b, \tag{30}$$

$$\mathsf{cbb} = \mathsf{bcb} \qquad \text{for } b, c \in [M],\ b < c. \tag{31}$$

Equivalence classes of ordinary words modulo I_{plac} are known as *Knuth equivalence classes*. The equivalence classes of colored words modulo $I^{\lessdot}_{\text{plac}}$ we call *$\lessdot$-colored plactic equivalence classes*.

For a colored word w, the *colored content* of w is the pair of weak compositions $\beta = (\beta^{\varnothing}, \beta^{-})$ such that $\beta^{\varnothing}_a$ is the number of a's in w and β^{-}_a is the number of $\bar{\mathsf{a}}$'s in w for all $a \in [N]$. For example, if $N = 3$, then the colored content of $\mathsf{w} = \mathsf{2\bar{1}\bar{2}1\bar{3}\bar{1}21}$ is $((2,2,0),(2,1,1))$. For a colored content $\beta = (\beta^{\varnothing}, \beta^{-})$, we set $|\beta| = \sum_{a=1}^{N}(\beta^{\varnothing}_a + \beta^{-}_a)$. For a $\lessdot$-colored tableau T, the *colored content* of T is the colored content of $\text{colword}(T)$.

The relations of $\mathcal{U}/I^{\lessdot}_{\text{plac}}$ are colored content-preserving, so there is a $\mathbb{Z}$-module decomposition $\mathcal{U}/I^{\lessdot}_{\text{plac}} \cong \bigoplus_\beta (\mathcal{U}/I^{\lessdot}_{\text{plac}})_\beta$, where $(\mathcal{U}/I^{\lessdot}_{\text{plac}})_\beta$ denotes the $\mathbb{Z}$-span of the colored words of colored content β in the algebra $\mathcal{U}/I^{\lessdot}_{\text{plac}}$.

Definition 4.7 (Standardization)**.** For a colored word w, the *$\lessdot$-standardization of* w, denoted $\mathsf{w}^{\text{st}^{\lessdot}}$, is the permutation obtained from w by first relabeling, from left to right (resp. right to left), the occurrences of the smallest (for $\lessdot$) letter in w by $1, \dots, k$ if this letter is unbarred (resp. barred), then relabeling the occurrences of the next smallest letter of w by $k+1, \dots, k+k'$, and so on.

The standardization of a $\lessdot$-colored tableau T, denoted $T^{\text{st}^{\lessdot}}$, is defined as for colored words, except that barred letters are relabeled from top to bottom and unbarred letters from left to right.

For a colored content β, define the set of permutations

$$\mathcal{S}(\beta) := \big\{\mathsf{w}^{\text{st}^{\lessdot}} \mid \mathsf{w} \text{ has colored content } \beta\big\}. \tag{32}$$

There is a bijection

$$\big\{\text{colored words of colored content } \beta\big\} \xrightarrow{\cong} \mathcal{S}(\beta), \quad \mathsf{w} \mapsto \mathsf{w}^{\text{st}^{\lessdot}}. \tag{33}$$

Also, it is well known that

$$\text{the set } \mathcal{S}(\beta) \text{ is a union of Knuth equivalence classes.} \tag{34}$$

The colored plactic relations and the plactic relations are compatible with standardization in the following sense:

Proposition 4.8. *For every colored content β and $M \geq |\beta|$, the standardization map defines a $\mathbb{Z}$-module isomorphism*

$$(\mathcal{U}/I^{\lessdot}_{\text{plac}})_\beta \xrightarrow{\cong} (\mathcal{V}_M/I_{\text{plac}})[\beta], \quad \mathsf{w} \mapsto \mathsf{w}^{\text{st}^{\lessdot}}, \tag{35}$$

where $(\mathcal{V}_M/I_{\text{plac}})[\beta]$ denotes the $\mathbb{Z}$-submodule of $\mathcal{V}_M/I_{\text{plac}}$ spanned by the words $\mathcal{S}(\beta)$.

Proof. Fix a colored content β. It is straightforward to check that if two colored words differ by a single application of one of the relations (15)–(20), then their $\lessdot$-standardizations differ by a single application of one of the relations (28), (29); also, if two permutations differ by a single application of one of the relations (28), (29), then either they both do not belong to $\mathcal{S}(\beta)$, or they both belong to $\mathcal{S}(\beta)$ and the inverse of the bijection (33) takes these words to two colored words that differ by a single application of one of the relations (15)–(20). Hence the $\lessdot$-standardization of a $\lessdot$-colored plactic equivalence class is an (ordinary) Knuth equivalence class, and under this correspondence, the $\lessdot$-colored plactic equivalence classes of content β are in bijection with the Knuth equivalence classes that partition $\mathcal{S}(\beta)$. The result follows. □

Let SSYT_ν denote the set of semistandard Young tableaux (SSYT) of shape ν with entries in the alphabet $[M]$. We need the following fact relating standardization to colored tableaux.

Proposition 4.9. *The standardization map* $\mathsf{w} \mapsto \mathsf{w}^{\mathrm{st}^\lessdot}$ *defines a bijection*

$$\big\{\mathrm{colword}(T) \mid T \in \mathrm{CT}_\nu^\lessdot,\ T \textit{ has colored content } \beta\big\}$$
$$\xrightarrow{\cong}\big\{\mathrm{colword}(T) \mid T \in \mathrm{SSYT}_\nu,\ \mathrm{colword}(T) \in \mathcal{S}(\beta)\big\}.$$

Proof. It is clear that $\mathrm{colword}(T)^{\mathrm{st}^\lessdot} = \mathrm{colword}(T^{\mathrm{st}^\lessdot})$ and $T^{\mathrm{st}^\lessdot} \in \mathrm{SSYT}_\nu$ for any $T \in \mathrm{CT}_\nu^\lessdot$, hence the bijection (33) restricts to a map between the given sets. The inverse of (33) is computed on some $\mathrm{colword}(T) \in \mathcal{S}(\beta)$ by relabeling some of its decreasing subsequences with barred letters and some of its increasing subsequences with unbarred letters to obtain the colored word w ($\mathrm{colword}(T) = \mathsf{w}^{\mathrm{st}^\lessdot}$); since the columns (resp. rows) of T correspond to decreasing (resp. increasing) subsequences of $\mathrm{colword}(T)$, it follows that w is the column reading word of a $\lessdot$-colored tableau. The result follows. □

We need to recall the following definitions from [11].

Definition 4.10. The *noncommutative elementary symmetric functions* are given by

$$e_k(\mathbf{v}) = \sum_{M \geq a_1 > a_2 > \cdots > a_k \geq 1} v_{a_1} v_{a_2} \cdots v_{a_k} \ \in \mathcal{V}_M$$

for any positive integer k; set $e_0(\mathbf{v}) = 1$ and $e_k(\mathbf{v}) = 0$ for $k < 0$. Now define the *noncommutative Schur functions* $\mathfrak{J}_\nu(\mathbf{v})$ exactly as in Definition 2.1, except with $e_k(\mathbf{v})$ in place of $e_k(\mathbf{u})$.

Our proof of Theorem 3.10 is based on the following consequence of [11, Lemma 3.2].

Theorem 4.11 ([11])**.** *For any partition* ν*,*

$$\mathfrak{J}_\nu(\mathbf{v}) = \sum_{T \in \mathrm{SSYT}_\nu} \mathrm{colword}(T) \qquad \textit{in} \ \ \mathcal{V}_M/I_{\mathrm{plac}}.$$

Proof of Theorem 3.10. Let β be a colored content and $M = |\beta|$. Consider $\mathfrak{J}_\nu(\mathbf{v})$ as an element of $\mathcal{V}_M$, written as the signed sum of words obtained by expanding the sum of products of $e_k(\mathbf{v})$'s in its definition. Let $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ denote the result of restricting this signed sum to the permutations $\mathcal{S}(\beta)$ (defined in (32)). It follows from (34) that the image of $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ in $\mathcal{V}_M/I_{\mathrm{plac}}$ depends only on the image of $\mathfrak{J}_\nu(\mathbf{v})$ in $\mathcal{V}_M/I_{\mathrm{plac}}$. Moreover, fact (34) and Theorem 4.11 yield

$$\mathfrak{J}_\nu(\mathbf{v})[\beta] = \sum_{\substack{T \in \mathrm{SSYT}_\nu \\ \mathrm{colword}(T) \in \mathcal{S}(\beta)}} \mathrm{colword}(T) \qquad \text{in} \ \ \mathcal{V}_M/I_{\mathrm{plac}}. \tag{36}$$

By the decomposition $\mathcal{U}/I^{\lessdot}_{\text{plac}} \cong \bigoplus_\beta (\mathcal{U}/I^{\lessdot}_{\text{plac}})_\beta$, we can write the noncommutative super Schur function $\mathfrak{J}^{\lessdot}_\nu(\mathbf{u})$ uniquely as a sum $\mathfrak{J}^{\lessdot}_\nu(\mathbf{u}) = \sum_\beta (\mathfrak{J}^{\lessdot}_\nu(\mathbf{u}))_\beta$, for $(\mathfrak{J}^{\lessdot}_\nu(\mathbf{u}))_\beta \in (\mathcal{U}/I^{\lessdot}_{\text{plac}})_\beta$. Now note that a concatenation of colored words of the form $\mathsf{z_1} \cdots \mathsf{z_t}$, $z_1 \geq_{\text{col}} z_2 \geq_{\text{col}} \cdots \geq_{\text{col}} z_t$, standardizes to a word that is a concatenation of strictly decreasing (ordinary) words. Using this, one checks that the isomorphism (35) takes $(\mathfrak{J}^{\lessdot}_\nu(\mathbf{u}))_\beta$ to $\mathfrak{J}_\nu(\mathbf{v})[\beta]$. Applying the inverse of this isomorphism to both sides of (36) yields (by Proposition 4.9)

$$(\mathfrak{J}^{\lessdot}_\nu(\mathbf{u}))_\beta = \sum_{\substack{T \in \text{CT}^{\lessdot}_\nu \\ T \text{ has colored content } \beta}} \text{colword}(T) \qquad \text{in} \;\; \mathcal{U}/I^{\lessdot}_{\text{plac}}.$$

Summing over β, the theorem follows. □

5. Reading words for $\mathfrak{J}_\nu(\mathbf{u})$ in $\mathcal{U}/I_{\text{Kron}}$

Here we introduce new kinds of tableaux and reading words that arose naturally in our discovery of a monomial positive expression for $\mathfrak{J}_\nu(\mathbf{u})$ in $\mathcal{U}/I_{\text{Kron}}$. The main new feature of these objects is that they involve posets obtained from posets of diagrams by adding a small number of covering relations. This section has much in common with [5, §3], but there are also important differences.

5.1. **Diagrams and tableaux.**

A *diagram* or *shape* is a finite subset of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. A diagram is drawn as a set of boxes in the plane with the English (matrix-style) convention so that row (resp. column) labels start with 1 and increase from north to south (resp. west to east).

A partition λ of n is a weakly decreasing sequence $(\lambda_1, \dots, \lambda_l)$ of nonnegative integers that sum to n. The *shape* of λ is the subset $\{(r, c) \mid r \in [l],\ c \in [\lambda_r]\}$ of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. Write $\mu \subseteq \lambda$ if the shape of μ is contained in the shape of λ. If $\mu \subseteq \lambda$, then λ/μ denotes the *skew shape* obtained by removing the boxes of μ from the shape of λ. The *conjugate partition* λ' of λ is the partition whose shape is the transpose of the shape of λ.

We will make use of the following partial orders on $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$:

$$(r, c) \leq_{\searrow} (r', c') \text{ whenever } r \leq r' \text{ and } c \leq c',$$
$$(r, c) \leq_{\nearrow} (r', c') \text{ whenever } r \geq r' \text{ and } c \leq c'.$$

It will occasionally be useful to think of diagrams as posets for the order $<_{\searrow}$ or $<_{\nearrow}$.

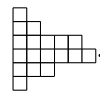
Let θ be a diagram. A *tableau of shape* θ is the diagram θ together with a colored letter (an element of $\mathcal{A}$) in each of its boxes. The *size* of a tableau T, denoted $|T|$, is the number of boxes of T, and $\text{sh}(T)$ denotes the shape of T. For a tableau T and a set of boxes S such that $S \subseteq \text{sh}(T)$, T_S denotes the subtableau of T obtained by restricting T to the diagram S. If β is a box of T, then T_β denotes the entry of T in β. When it is clear, we will occasionally identify a tableau entry with the box containing it.

If T is a tableau, $x \in \mathcal{A}$ a colored letter, and $\beta = (r, c)$ is a box not belonging to $\text{sh}(T)$, then $T \sqcup \boxed{x}_{\,r,c}$ denotes the result of adding the box β to T and filling it with x.

5.2. **Restricted shapes and restricted colored tableaux.**

Definition 5.1. A *restricted shape* is a lower order ideal of a partition diagram for the order $<_{\nearrow}$. We will typically specify a restricted shape as follows: for any weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$, let α' denote the diagram $\{(r, c) \mid c \in [l],\ r \in [\alpha_c]\}$. Now let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition and $\alpha = (\alpha_1, \dots, \alpha_l)$ a weak composition such that $0 \le \alpha_1 \le \cdots \le \alpha_{j'}$, $\alpha_1 < \lambda_1,\ \alpha_2 < \lambda_2, \dots, \alpha_{j'} < \lambda_{j'}$, and $\alpha_{j'+1} = \lambda_{j'+1}, \dots, \alpha_l = \lambda_l$ for some $j' \in \{0, 1, \dots, l\}$. Then the set difference of λ' by α', denoted $\lambda' \setminus \alpha'$, is a restricted shape and any restricted shape can be written in this way.

Note that, just as for skew shapes, different pairs λ, α may define the same restricted shape $\lambda' \setminus \alpha'$. An example of a restricted shape is

$$(655444221)' \setminus (012223221)' \;=$$  .

Definition 5.2. A *restricted tableau* is a tableau whose shape is a restricted shape. A *restricted colored tableau* (RCT) is a restricted tableau such that each row and column is weakly increasing with respect to the natural order $<$, while the unbarred letters in each column and the barred letters in each row are strictly increasing.

Hence a restricted colored tableau of partition shape is the same as a $<$-colored tableau (defined in §2.3).

For example,

$\bar{1}$					
2	$\bar{2}$				
$\bar{2}$	3	3	3	$\bar{3}$	
3	$\bar{3}$	4	4	4	$\bar{4}$
$\bar{3}$	4	5			
$\bar{3}$					

is an RCT of shape $(655444)' \setminus (012223)'$.

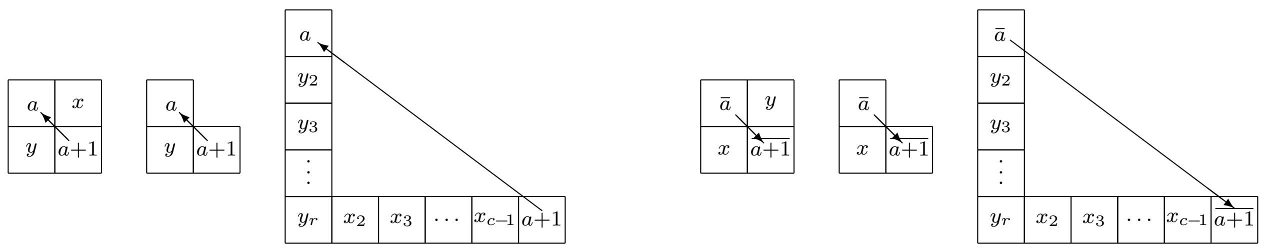
Recall that $x \le_{\text{col}} y$ means that $x < y$ or x and y are equal barred letters. We write $x \le_{\text{row}} y$ to mean that $x < y$ or x and y are equal unbarred letters. Hence the condition for a restricted tableau to be a restricted colored tableau is exactly that if the entry x lies immediately west of y, then $x \le_{\text{row}} y$, and if the entry x lies immediately north of y, then $x \le_{\text{col}} y$. Be aware that $\le_{\text{col}}$ and $\le_{\text{row}}$ are not partial orders.

The following basic fact will be used frequently in the proof of Theorem 2.3.

Proposition 5.3. *If $x \le_{col} y \le_{row} z$, then $x < z$ and ($x = a$ is unbarred $\implies$ $z \ge a + 1$) and ($z = \overline{a+1}$ is barred $\implies$ $x \le \bar{a}$). In particular, if $d_1, d_2, \dots, d_t$ is the diagonal of an RCT read in the $\searrow$ direction, then $d_i < d_{i+1}$ and ($d_i = a$ is unbarred $\implies$ $d_{i+1} \ge a + 1$) and ($d_{i+1} = \overline{a+1}$ is barred $\implies$ $d_i \le \bar{a}$).*

5.3. **Arrow respecting reading words.**

Definition 5.4. An *arrow subtableau* S of an RCT R is a subtableau of R such that $\mathrm{sh}(S)$ is the intersection of $\mathrm{sh}(R)$ with a rectangular shape, and S is of the form



where $x, y,\ x_2, \dots, x_{c-1},\ y_2, \dots, y_r \in \mathcal{A}$ and $a \in [N-1]$; also, $r > 2$, $c \geq 2$ in the third tableau and $r \geq 2$, $c > 2$ in the last tableau. The first three of these are called $\nwarrow$ *arrow subtableaux* and the last three are $\searrow$ *arrow subtableaux*. Also, *an arrow* of R is a directed edge between the two boxes of an arrow subtableau, as indicated in the picture; we think of the arrows of R as the edges of a directed graph with vertex set the boxes of R.

Definition 5.5. A *reading word* of a tableau R is a colored word w consisting of the entries of R such that for any two boxes β and β' of R such that $\beta <_{\nearrow} \beta'$, R_β appears to the left of $\mathsf{R}_{\beta'}$ in w.

An *arrow respecting reading word* w of an RCT R is a reading word of R such that for each arrow of R, the tail of the arrow appears to the left of the head of the arrow in w.

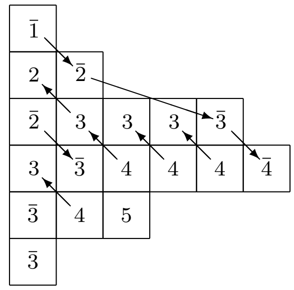
There is no reason to prefer one arrow respecting reading word over another (see Theorem 5.10), but it is useful to have notation for one such word. The definition of the colored word $\mathrm{arwread}(T)$ (Definition 2.2) extends verbatim to the case T is an RCT. Though $\mathrm{arwread}(T)$ is not an arrow respecting reading word of T in general, it is in the case T has partition shape and in all cases encountered in the proof of Theorem 2.3 (see Remark 6.7 (b)).

Remark 5.6. Suppose S is the intersection of an RCT R with a rectangular subtableau such that (1) the northwesternmost box of S is a ($a \in [N-1]$), (2) the southeasternmost box of S is $a+1$, and (3) the rectangular subtableau has more than two rows. Then S has the form of the third tableau in Definition 5.4 because if not, there would be a box β immediately north of the box containing x_2 and $\bar{a} = y_{r-1} \leq_{\mathrm{row}} R_\beta \leq_{\mathrm{col}} x_2 = a+1$ yields a contradiction. Note that this also shows that if an RCT R contains an S satisfying (1)–(3), then R cannot be completed to a $<$-colored tableau. (However, such tableaux still need to be considered in intermediate stages in the proofs of Theorems 5.10 and 2.3.)

Similarly, suppose S is the intersection of an RCT R with a rectangular subtableau such that (1′) the northwesternmost box of S is $\bar{a}$, (2′) the southeasternmost box of S is $\overline{a+1}$, and (3′) the rectangular subtableau has more than two columns. Then S has the form of the last tableau in Definition 5.4. Just as above, it is also the case that if an RCT contains an S satisfying (1′)–(3′), then it cannot be completed to a $<$-colored tableau.

Also note that a given box is the tail of at most one arrow, but a single box can be the head of more than one arrow.

Example 5.7. Here is an RCT drawn with its arrows:

$R =$ 

Of the following three reading words of R, the first two are arrow respecting, but the last is not.

$$\begin{aligned} \text{arwread}(R) = {} & \mathsf{\bar{3}\,\bar{3}\,4\,3\,5\,\bar{2}\,\bar{3}\,4\,3\,2\,4\,3\,\bar{1}\,\bar{2}\,4\,3\,\bar{3}\,\bar{4}} && \text{(arrow respecting)} \\ & \mathsf{\bar{3}\,\bar{3}\,4\,3\,5\,\bar{2}\,\bar{3}\,4\,3\,4\,3\,2\,\bar{1}\,\bar{2}\,4\,3\,\bar{3}\,\bar{4}} && \text{(arrow respecting)} \\ & \mathsf{\bar{3}\,\bar{3}\,4\,3\,5\,\bar{2}\,\bar{3}\,3\,2\,\bar{1}\,\bar{2}\,4\,4\,3\,4\,3\,\bar{3}\,\bar{4}} && \text{(not arrow respecting)} \end{aligned}$$

A *$\nearrow$-maximal box* of a diagram is a box that is maximal for the order $<_{\nearrow}$. A *nontail removable box* of an RCT R is a $\nearrow$-maximal box of R that is not the tail of an arrow of R. The last letter of an arrow respecting reading word of R must lie in a nontail removable box of R. For example, the $\nearrow$-maximal boxes of the R in Example 5.7 are $(1,1),(2,2),(3,5),(4,6)$, which contain the entries $\bar{1},\bar{2},\bar{3},\bar{4}$, respectively. The RCT R has only the single nontail removable box $(4,6)$.

5.4. **Arrow respecting reading words in $\mathcal{U}/I_{\text{Kron}}$.** We assemble some basic results about arrow respecting reading words and the images of these words in $\mathcal{U}/I_{\text{Kron}}$. These are needed for the proof of Theorem 2.3.

Lemma 5.8. *Every RCT has a nontail removable box.*

Proof. Consider the $\nearrow$-maximal boxes of an RCT R. The northernmost such box containing an unbarred letter (if it exists) is a nontail removable box, and the southernmost such box containing a barred letter (if it exists) is a nontail removable box. Hence R contains a nontail removable box. □

Corollary 5.9. *If β is a nontail removable box of an RCT R, then there is an arrow respecting reading word of R that ends in R_β.*

Proof. This follows by induction on $|R|$ using Lemma 5.8. □

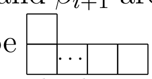
The next result gives a natural way to associate an element of $\mathcal{U}/I_{\text{Kron}}$ to any RCT.

Theorem 5.10. *Any two arrow respecting reading words of an RCT are equal in $\mathcal{U}/I_{\text{Kron}}$.*

Proof. Throughout the proof, we write $f \equiv g$ to mean that f and g are congruent modulo I_{Kron} (for $f, g \in \mathcal{U}$).

Let R be an RCT and let $\mathscr{R}_R$ denote the graph with vertex set the arrow respecting reading words of R and an edge between any two such words that differ by a single application of one of the relations (9), (4), (6). We prove that $\mathscr{R}_R$ is connected by induction on $|R|$. Let $\beta_1, \ldots, \beta_t$ denote the nontail removable boxes of R, labeled so that $\beta_1 <_{\searrow} \beta_2 <_{\searrow} \cdots <_{\searrow} \beta_t$ ($t \geq 1$ by Lemma 5.8). By induction, each $\mathscr{R}_{R-\beta_i}$ is connected. Hence the induced subgraph of $\mathscr{R}_R$ with vertex set consisting of those words that end by reading the box β_i, call it $\mathscr{R}_{R-\beta_i}\mathsf{R}_{\beta_i}$, is connected. Let H be the graph (with t vertices) obtained from $\mathscr{R}_R$ by contracting the subgraphs $\mathscr{R}_{R-\beta_i}\mathsf{R}_{\beta_i}$. We must show that H is connected.

To prove that H is connected, we will show that there is an edge between vertices i and $i+1$ of H for each i. It follows from Proposition 5.3 that $R_{\beta_i} < R_{\beta_{i+1}}\downarrow$. We first consider the case $R_{\beta_i} < R_{\beta_{i+1}}\downarrow\downarrow$. Let w be an arrow respecting reading word of $R - \beta_i - \beta_{i+1}$. Then $\mathsf{w}\mathsf{R}_{\beta_i}\mathsf{R}_{\beta_{i+1}} \equiv \mathsf{w}\mathsf{R}_{\beta_{i+1}}\mathsf{R}_{\beta_i}$ by the far commutation relations (9), hence there is an edge between vertices i and $i+1$ of H.

Next consider the case $R_{\beta_i} = a$, $R_{\beta_{i+1}} = a+1$ for some $a \in [N-1]$. By Proposition 5.3, β_i and β_{i+1} are consecutive $\nearrow$-maximal boxes of R i.e. there is no $\nearrow$-maximal box β of R such that $\beta_i <_{\searrow} \beta <_{\searrow} \beta_{i+1}$. Let S be the subtableau of R consisting of the boxes $\geq_{\searrow} \beta_i$ and $\leq_{\searrow} \beta_{i+1}$. Since there is no arrow from β_{i+1} to β_i and β_i and β_{i+1} are $\nearrow$-maximal boxes of R, by Definition 5.4 the only possibility is that S has shape  (S has 2 rows and more than 2 columns). Let $R' = R - \beta_i - \beta_{i+1}$ and let β denote the box immediately west of β_{i+1}. By the strictness conventions of RCT, we must have $R_\beta = a+1$. We next claim that β is not an arrow tail in R' (there could be an arrow from β to β_i in R). By the forms of the first three arrow subtableaux in Definition 5.4 and since $\beta_i \notin R'$, if there is an arrow from β to some β' in R', then β' must be strictly north of β_i; but this implies $R_{\beta'} \leq_{\text{col}} R_{\beta_i} = a$, so there cannot be an arrow from β to β' in R'. Corollary 5.9 then implies that there is an arrow respecting reading word of R' that ends in R_β; let $\mathsf{w}'\mathsf{R}_\beta$ be one such word. Hence the congruence

$$\mathsf{w}'\mathsf{R}_\beta\mathsf{R}_{\beta_i}\mathsf{R}_{\beta_{i+1}} = \mathsf{w}'(\mathsf{a}+1)\mathsf{a}(\mathsf{a}+1) \equiv \mathsf{w}'(\mathsf{a}+1)(\mathsf{a}+1)\mathsf{a} = \mathsf{w}'\mathsf{R}_\beta\mathsf{R}_{\beta_{i+1}}\mathsf{R}_{\beta_i}$$

obtained by applying the relation (4) exhibits an edge between vertices i and $i+1$ of H. The case $R_{\beta_i} = \bar{a}$, $R_{\beta_{i+1}} = \overline{a+1}$ is handled in a similar way. $\square$

6. Proof of Theorem 2.3

Here we prove Theorem 2.3 by an inductive computation that involves flagged versions of the noncommutative super Schur functions. Lemma 6.4 is the key computation in the algebra $\mathcal{U}/I_{\text{Kron}}$ that makes the proof possible and is the main cause for the words $\text{arwread}(T)$ appearing in the statement of the theorem. This section has much in common with [5, §4.3], but there are also important differences.

Here is some notation we will use throughout this section: for $f, g \in \mathcal{U}$, we write $f \equiv g$ to mean that f and g are congruent modulo I_{Kron}. We will work with the poset $\{\bar{0}\} \sqcup \mathcal{A}$ which extends the natural order $<$ on $\mathcal{A}$ and where $\bar{0} < x$ for all $x \in \mathcal{A}$. The notation $x\downarrow$ was introduced in (1); we recall this and introduce the similar notation $x\rfloor$. For any $x \in \mathcal{A}$, define

$$x\downarrow = \begin{cases} \overline{a-1} & \text{if } x = a, \quad a \in [N], \\ a & \text{if } x = \bar{a}, \quad a \in [N]. \end{cases} \qquad x\rfloor = \begin{cases} \overline{a-1} & \text{if } x = a, \quad a \in [N], \\ \bar{a} & \text{if } x = \bar{a}, \quad a \in [N]. \end{cases}$$

6.1. **Noncommutative column-flagged super Schur functions.** Here we introduce a flagged generalization of the noncommutative super Schur functions. This will be important for our inductive computation of $\mathfrak{J}_\nu(\mathbf{u})$ in the proof of Theorem 2.3.

For any subset $S \subseteq \mathcal{A}$ and positive integer k, there is a natural generalization of the functions $e_k(\mathbf{u})$ given by

$$e_k(S) = \sum_{\substack{z_1 \geq_{\text{col}} z_2 \geq_{\text{col}} \cdots \geq_{\text{col}} z_k \\ z_1,\dots,z_k \in S}} u_{z_1} u_{z_2} \cdots u_{z_k} \in \mathcal{U};$$

set $e_0(S) = 1$ and $e_k(S) = 0$ for $k < 0$. By the proof of Proposition 8.1, $e_k(S)e_l(S) = e_l(S)e_k(S)$ in $\mathcal{U}/I_{\text{Kron-K}}$ (and therefore in $\mathcal{U}/I_{\text{Kron}}$) for all k and l.

For $x \in \{\bar{0}\} \sqcup \mathcal{A}$, define $\mathcal{A}_{\leq x} = \{y \in \mathcal{A} \mid y \leq x\}$ (thus $\mathcal{A}_{\leq \bar{0}} = \{\}$). Given a weak composition $\alpha = (\alpha_1, \dots, \alpha_l)$ and an l-tuple $\mathbf{n} = (n_1, n_2, \dots, n_l) \in (\{\bar{0}\} \sqcup \mathcal{A})^l$, define the *noncommutative column-flagged super Schur function* by

$$\begin{aligned} J_\alpha(\mathbf{n}) &= J_\alpha(n_1, n_2, \dots, n_l) \\ &:= \sum_{\pi \in \mathcal{S}_l} \operatorname{sgn}(\pi)\, e_{\alpha_1+\pi(1)-1}(\mathcal{A}_{\leq n_1}) e_{\alpha_2+\pi(2)-2}(\mathcal{A}_{\leq n_2}) \cdots e_{\alpha_l+\pi(l)-l}(\mathcal{A}_{\leq n_l}) \in \mathcal{U}. \end{aligned} \tag{37}$$

These functions are related to the $\mathfrak{J}_\nu(\mathbf{u})$ of Definition 2.1 by $\mathfrak{J}_\nu(\mathbf{u}) = J_{\nu'}(\overline{N}, \overline{N}, \dots, \overline{N})$.

For colored words $\mathsf{w}^1, \dots, \mathsf{w}^{l-1} \in \mathcal{U}$, we will also make use of the *augmented noncommutative column-flagged super Schur functions*, given by

$$\begin{aligned} &J_\alpha(\mathbf{n} : \underset{1}{\mathsf{w}^1}, \underset{2}{\mathsf{w}^2}, \dots, \underset{l-1}{\mathsf{w}^{l-1}}) = J_\alpha(n_1, n_2, \dots, n_l : \underset{1}{\mathsf{w}^1}, \underset{2}{\mathsf{w}^2}, \dots, \underset{l-1}{\mathsf{w}^{l-1}}) \\ &:= \sum_{\pi \in \mathcal{S}_l} \operatorname{sgn}(\pi)\, e_{\alpha_1+\pi(1)-1}(\mathcal{A}_{\leq n_1}) \mathsf{w}^1 e_{\alpha_2+\pi(2)-2}(\mathcal{A}_{\leq n_2}) \mathsf{w}^2 \cdots \mathsf{w}^{l-1} e_{\alpha_l+\pi(l)-l}(\mathcal{A}_{\leq n_l}) \in \mathcal{U}. \end{aligned}$$

We omit $\underset{i}{\mathsf{w}^i}$ from the notation if the word w^i is empty.

Note that because the noncommutative super elementary symmetric functions commute in $\mathcal{U}/I_{\text{Kron}}$,

$$J_\alpha(\mathbf{n}) \equiv -J_{\alpha_1,\dots,\alpha_{j-1},\alpha_{j+1}-1,\alpha_j+1,\dots,\alpha_l}(\mathbf{n}) \quad \text{whenever } n_j = n_{j+1}. \tag{38}$$

In particular,

$$J_\alpha(\mathbf{n}) \equiv 0 \qquad \text{whenever } \alpha_j = \alpha_{j+1} - 1 \text{ and } n_j = n_{j+1}. \tag{39}$$

More generally, (38) and (39) hold for the augmented case provided w^j is empty.

We will make frequent use of the following fact (keeping in mind $\mathcal{A}_{\leq \bar{0}} = \{\}$):

$$e_k(\mathcal{A}_{\leq x}) = \mathsf{x} e_{k-1}(\mathcal{A}_{\leq x\downarrow}) + e_k(\mathcal{A}_{\leq x\downarrow}) \text{ for } x \in \mathcal{A} \text{ and any integer } k. \tag{40}$$

Note that

$$e_k(\mathcal{A}_{\leq \bar{0}}) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We will often apply the identity (40) to $J_\alpha(\mathbf{n})$ and its variants by expanding $e_{\alpha_j+\pi(j)-j}(\mathcal{A}_{\leq n_j})$ in (37) using (40) (so that (40) is applied once to each of the $l!$ terms in the sum). We refer to this as a *j-expansion of $J_\alpha(\mathbf{n})$* or simply a *j-expansion*.

6.2. **Proof of Theorem 2.3.** After three preliminary results, we state and prove a more technical version of Theorem 2.3, which involves computing $J_\nu(\mathbf{n})$ inductively by peeling off diagonals from the shape ν'.

For $x, y \in \mathcal{A}$, let $[\mathsf{x}, \mathsf{y}]$ denote the commutator $\mathsf{xy} - \mathsf{yx} \in \mathcal{U}$.

Lemma 6.1. *Let $y, z, x_1, x_2, \dots, x_t \in \mathcal{A}$ be letters satisfying $y \geq_{col} x_1 \geq_{col} x_2 \cdots \geq_{col} x_t$ and $y \leq z$. Suppose that neither ($y = z{\downarrow}$) nor ($y = z$, $y, z \in \mathcal{A}_-$) is true. Then*

$$\mathsf{y x_1 x_2 \cdots x_t z} = \mathsf{y z x_1 x_2 \cdots x_t} \quad \text{in } \mathcal{U}/I_{\text{Kron}}.$$

Proof. First suppose that z is unbarred. If no x_i is equal to $z-1$, then the relations (9), (4), and (6) imply

$$\mathsf{y x_1 x_2 \cdots x_t z} \equiv \mathsf{y x_1 x_2 \cdots x_{t-1} z x_t} \equiv \mathsf{y x_1 x_2 \cdots x_{t-2} z x_{t-1} x_t} \equiv \cdots \equiv \mathsf{y z x_1 x_2 \cdots x_t}.$$

If $x_i = z - 1$, then the argument just given shows

$$\mathsf{y x_1 x_2 \cdots x_t z} \equiv \mathsf{y x_1 \cdots x_i z x_{i+1} \cdots x_t}.$$

Since $z \geq y \geq x_1 \geq x_{i-1} > x_i$, we must have $x_1 = x_2 = \cdots = x_{i-1} = \overline{z-1}$ and $y \in \{\overline{z-1}, z\}$. The case $y = \overline{z-1} = z{\downarrow}$ is excluded, so $y = z$. Now we compute

$$\begin{aligned} & \mathsf{y x_1 \cdots x_i z x_{i+1} \cdots x_t} \\ = \;& \mathsf{y x_1 \cdots x_{i-1} [x_i, z] x_{i+1} \cdots x_t} + \mathsf{y x_1 \cdots x_{i-1} z x_i x_{i+1} \cdots x_t} \\ \equiv \;& \mathsf{y [x_i, z] x_1 \cdots x_{i-1} x_{i+1} \cdots x_t} + \mathsf{y x_1 \cdots x_{i-1} z x_i x_{i+1} \cdots x_t} \\ \equiv \;& \mathsf{y x_1 \cdots x_{i-1} z x_i x_{i+1} \cdots x_t} \\ \equiv \;& \mathsf{y z x_1 x_2 \cdots x_{i-1} x_i x_{i+1} \cdots x_t}, \end{aligned}$$

where the first congruence is by $i-1$ applications of the relation (8), the second is by the relation (4) (which yields $\mathsf{y[x_i, z]} = \mathsf{z[z-1, z]} \equiv 0$), and the third is by the relations (4) and (6).

If $z = \overline{a+1}$ is a barred letter, then by the assumptions of the lemma, $y \leq \bar{a}$, hence the relations (9) and (6) imply

$$\mathsf{y x_1 x_2 \cdots x_t z} \equiv \mathsf{y x_1 x_2 \cdots x_{t-1} z x_t} \equiv \mathsf{y x_1 x_2 \cdots x_{t-2} z x_{t-1} x_t} \equiv \cdots \equiv \mathsf{y z x_1 x_2 \cdots x_t},$$

as desired. $\square$

Example 6.2. In the case $\mathsf{y} = \mathsf{z} = \mathsf{4}$, $\mathsf{x_1 x_2 \cdots x_t} = \bar{\mathsf{3}}\bar{\mathsf{3}}\mathsf{3}$, Lemma 6.1 and its proof yield

$$\mathsf{4\bar{3}\bar{3}34} = \mathsf{4\bar{3}\bar{3}[3,4]} + \mathsf{4\bar{3}\bar{3}43} \equiv \mathsf{4[3,4]\bar{3}\bar{3}} + \mathsf{4\bar{3}\bar{3}43} \equiv \mathsf{4\bar{3}\bar{3}43} \equiv \mathsf{4\bar{3}4\bar{3}3} \equiv \mathsf{44\bar{3}\bar{3}3},$$

where the congruences are by (8), (4), (6), and (4), respectively.

Remark 6.3. The assumptions on y and z in Lemma 6.1 are necessary: the lemma does not extend to the case $y = a, z = \bar{a}$ since $\mathsf{2\bar{1}\bar{2}} \not\equiv \mathsf{2\bar{2}\bar{1}}$, nor to the case $y = \bar{a}, z = a+1$ since $\bar{\mathsf{2}}\mathsf{23} \not\equiv \bar{\mathsf{2}}\mathsf{32}$, nor to the case $y = \bar{a}, z = \bar{a}$ since $\bar{\mathsf{2}}\mathsf{2}\bar{\mathsf{2}} \not\equiv \bar{\mathsf{2}}\bar{\mathsf{2}}\mathsf{2}$.

The following lemma is key to the proof of Theorem 2.3.

Lemma 6.4. *Let $m \in [N-1]$ and set $x = m+1$ (an unbarred letter). Then*

$$J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) = \mathsf{x} J_{(a,a)}(\overline{m}, \overline{m}) \qquad \text{in } \mathcal{U}/I_{\mathrm{Kron}}.$$

More generally, if α is a weak composition satisfying $\alpha_j = \alpha_{j+1}$ and $\mathbf{n} = (n_1, \dots, n_l)$ with $n_j = n_{j+1} = \overline{m}$, then

$$J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j}) = J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}) \qquad \text{in } \mathcal{U}/I_{\mathrm{Kron}}.$$

Proof. Since the proofs of both statements are essentially the same, we prove only the first to avoid extra notation. We compute

$$\begin{aligned}
0 &\equiv J_{(a,a+1)}(m+1, m+1) \\
&= \mathsf{x} J_{(a-1,a+1)}(\overline{m}, m+1) + J_{(a,a+1)}(\overline{m}, m+1) \\
&= \mathsf{x} J_{(a-1,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) + \mathsf{x} J_{(a-1,a+1)}(\overline{m}, \overline{m}) + J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) + J_{(a,a+1)}(\overline{m}, \overline{m}) \\
&\equiv \mathsf{x} J_{(a-1,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) + \mathsf{x} J_{(a-1,a+1)}(\overline{m}, \overline{m}) + J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) \\
&\equiv \mathsf{x}\mathsf{x} J_{(a-1,a)}(\overline{m}, \overline{m}) + \mathsf{x} J_{(a-1,a+1)}(\overline{m}, \overline{m}) + J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) \\
&\equiv \mathsf{x} J_{(a-1,a+1)}(\overline{m}, \overline{m}) + J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}) \\
&\equiv -\mathsf{x} J_{(a,a)}(\overline{m}, \overline{m}) + J_{(a,a)}(\overline{m}, \overline{m} : {}^{\mathsf{x}}_{1}).
\end{aligned}$$

The first, second, and fourth congruences are by (39). The equalities are a 1-expansion and a 2-expansion (see §6.1 for notation). Lemma 6.1 implies that $\mathsf{x} e_k(\mathcal{A}_{\leq \overline{m}})\mathsf{x} \equiv \mathsf{x}\mathsf{x} e_k(\mathcal{A}_{\leq \overline{m}})$ for any k, hence the third congruence. The last congruence is by (38). □

Corollary 6.5. *Maintain the notation of Lemma 6.4 and assume in addition that*

$$\mathsf{w} \text{ is a colored word such that } \mathsf{w} = \mathsf{x}\mathsf{w}' \text{ and the letters of the word } \mathsf{w}' \text{ are } \geq m+2. \quad (41)$$

Then

$$J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}, {}^{\mathsf{w}'}_{j}) \qquad \text{in } \mathcal{U}/I_{\mathrm{Kron}}.$$

Proof. We compute (modulo I_{Kron})

$$J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) \equiv J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j}, {}^{\mathsf{w}'}_{j+1}) \equiv J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}, {}^{\mathsf{w}'}_{j+1}) \equiv J_\alpha(\mathbf{n} : {}^{\mathsf{x}}_{j-1}, {}^{\mathsf{w}'}_{j}).$$

The first and third congruences are by the far commutation relations (9) and the second is by Lemma 6.4 (the proof of the lemma still works with the word w' present). □

For a diagram θ contained in columns $1, \dots, l$ and a tuple $\mathbf{n} = (n_1, n_2, \dots, n_l)$ with $n_1, \dots, n_l \in \{\overline{0}\} \sqcup \mathcal{A}$, let $\mathrm{Tab}_\theta(\mathbf{n})$ denote the set of tableaux of shape θ such that the entries in column c lie in $\mathcal{A}_{\leq n_c}$.

For a weak composition or partition $\alpha = (\alpha_1, \dots, \alpha_l)$, define $\alpha_{l+1} = 0$.

Theorem 6.6. *Let $\nu' \setminus \alpha'$ be a restricted shape and let l be the number of parts of ν. Set $j = \min(\{i \mid \alpha_i > 0,\ \alpha_i \geq \alpha_{i+1}\} \cup \{l+1\})$ and $j' = \max(\{i \mid \alpha_i < \nu_i\} \cup \{0\})$ (see Figure 5 and the discussion following Remark 6.7). Suppose*

(i) *α is of the form*

$$(0, \dots, 0, 1, 2, \dots, a-1, a, a, a+1, a+2, \dots, \alpha_{j'}, \nu_{j'+1}, \nu_{j'+2}, \dots), \qquad \text{or}$$
$$(0, \dots, 0, 1, 2, \dots, \alpha_{j'}-1, \alpha_{j'}, \nu_{j'+1}, \nu_{j'+2}, \dots),$$

where $\alpha_{j'} \geq \alpha_{j'+1} - 1 = \nu_{j'+1} - 1$, the initial run of 0's may be empty, and on the top line (resp. bottom line) j is $< j'$ and is the position of the first a (resp. j is j' or $j'+1$);

(ii) *R is an RCT of shape $\nu' \setminus \alpha'$ with entries $r_1 = R_{\alpha_1+1,1}, \ldots, r_{j'} = R_{\alpha_{j'}+1,j'}$ along its northern border;*

(iii) *vw is an arrow respecting reading word of R such that w is a subsequence of $\mathsf{r}_{\mathsf{j+1}} \cdots \mathsf{r}_{\mathsf{j'}}$ which contains $\mathsf{r}_{\mathsf{j+1}}$ if $\mathsf{r}_{\mathsf{j+1}}$ is barred;*

(iv) *$\mathbf{n} = (n_1, \ldots, n_l)$, where $n_i \in \{\bar{0}\} \sqcup \mathcal{A}$ and $\bar{0} \leq n_1 \leq n_2 \leq \cdots \leq n_l$;*

(v) *$n_c = r_c\downharpoonleft$ for $c \in [j'] \setminus \{j\}$, and $n_j \leq r_j\downharpoonleft$;*

(vi) *it is not the case that $w_1 = r_{j+1} = a+1$ and $n_j = a$ (for some $a \in [N-1]$).*

Then

$$\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = \sum_{\substack{T \in \mathrm{RCT}_{\nu'},\ T_{\nu' \setminus \alpha'} = R \\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})}} \mathrm{arwread}(T) \quad \textit{in } \mathcal{U}/I_{\mathrm{Kron}}. \tag{42}$$

To parse this statement, it is instructive to first understand the case when $j \geq j'$ (which implies w is empty) and $\alpha_2 > 0$. In this case the theorem becomes

> Suppose $\nu' \setminus \alpha'$ is a restricted shape with $\alpha_1 = \alpha_2 - 1 = \cdots = \alpha_j - j + 1$ and $\alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_l > \alpha_{l+1} = 0$. Let R be an RCT of shape $\nu' \setminus \alpha'$ with entries $r_1, \ldots, r_{j'}$ along its northern border. Suppose $\mathbf{n} = (n_1, \ldots, n_l)$ satisfies $\bar{0} \leq n_1 \leq n_2 \leq \cdots \leq n_l$, $n_c = r_c\downharpoonleft$ for $c \in [j-1]$, and $n_j \leq r_j\downharpoonleft$. Then
>
> $$\mathrm{arwread}(R) J_\alpha(\mathbf{n}) = \sum_{\substack{T \in \mathrm{RCT}_{\nu'},\ T_{\nu' \setminus \alpha'} = R \\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})}} \mathrm{arwread}(T) \quad \text{in } \mathcal{U}/I_{\mathrm{Kron}}.$$

Remark 6.7.

(a) Theorem 2.3 is the special case of Theorem 6.6 when $\alpha = \nu$ and $\mathbf{n} = (\overline{N}, \overline{N}, \ldots, \overline{N})$ (the ν and ν' of Theorem 2.3 must be interchanged to match the notation here).

(b) For any RCT R as in (ii), all arrow subtableaux of R have two rows and two columns, assuming $n_{j-1} < n_j$ (this is an innocent assumption since $n_{j-1} = n_j$ implies both sides of (42) are 0, by the proof of Theorem 6.6 below). To see this, it suffices to show that there is no arrow between the boxes containing r_{j-1} and r_{j+1}. Such an arrow would imply $r_{j-1} = \bar{a}$, $r_{j+1} = \overline{a+1}$, which in turn would imply by (v) that $n_{j-1} = \bar{a}$, $n_j \leq r_j\downharpoonleft = (a+1)\downharpoonleft = \bar{a}$, contradicting $n_{j-1} < n_j$.

(c) A reading word vw as in (iii) always exists—for instance take $\mathsf{vw} = \mathrm{arwread}(R)$ with w equal to the subsequence of barred letters of $\mathsf{r}_{\mathsf{j+1}} \cdots \mathsf{r}_{\mathsf{j'}}$.

(d) It follows from the theorem that $\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) \equiv 0$ if R cannot be completed to an RCT of shape ν', however we purposely do not make this assumption so that it does not have to be verified at the inductive step.

The assumptions on j, j', and α look complicated, but their purpose is simply to peel off the entries of a tableau of shape ν' one diagonal at a time, starting with the southwesternmost diagonal, and reading each diagonal in the $\nwarrow$ direction (see Figure 5). The proof goes by induction, peeling off one letter at a time from $J_\alpha(\mathbf{n})$, in the order just specified,

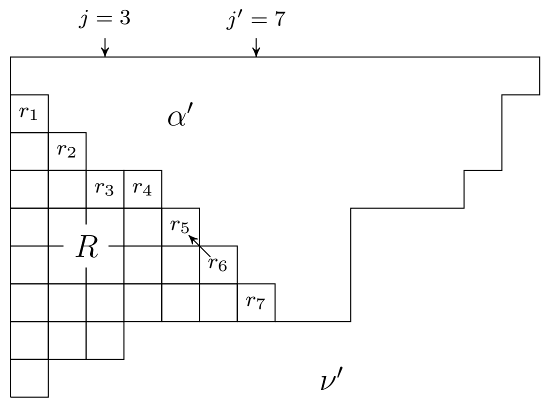


Figure 5: The setup of the proof of Theorem 6.6 for $\nu = (9,8,8,7,7,7,7,7,7,4,4,4,3,1)$, $\alpha = (1,2,3,3,4,5,6,7,7,4,4,4,3,1)$. A possibility for the arrows of R is shown. A possibility for w is $\mathsf{r}_4\mathsf{r}_5\mathsf{r}_7$.

and incorporating them into R. The index j indicates the column of the next letter to be removed.

The reader is encouraged to follow along the proof below with Example 6.8.

Proof of Theorem 6.6. The proof is by induction on $|\alpha|$ and the n_i. Throughout the proof, a denotes an element of $[N-1]$, and $a, a+1$ will often be regarded as unbarred letters and $\bar{a}, \overline{a+1}$ as the corresponding barred letters.

If $|\alpha| = 0$, $J_\alpha(\mathbf{n})$ is a noncommutative version of the determinant of an upper unitriangular matrix, hence $J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = \mathsf{w}$. The theorem then states that $\mathsf{vw} \equiv \text{arwread}(R)$, which holds by Theorem 5.10.

Next consider the case $n_1 = \bar{0}$ and $\alpha_1 > 0$. This implies that $J_\alpha(\mathbf{n})$ is a noncommutative version of the determinant of a matrix whose first row is all 0's, hence $\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = 0$. The right side of (42) is also 0 because $\text{Tab}_{\alpha'}(\mathbf{n})$ is empty for $n_1 = \bar{0}$ and $\alpha_1 > 0$.

If $n_i = n_{i+1}$ for any $i \neq j$ such that $\alpha_i = \alpha_{i+1} - 1$, then $\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = 0$ by (39). To see that the right side of (42) is also 0 in this case, observe that if T is an RCT from this sum, then

$$n_i < T_{\alpha_{i+1},i+1} \leq n_{i+1} = n_i,$$

hence the sum is empty. Here, the inequality $n_i < T_{\alpha_{i+1},i+1}$ follows from Proposition 5.3 with $x = n_i$, $y = r_i$, $z = T_{\alpha_{i+1},i+1}$ (note that this argument works in the case $\alpha_i = 0$).

We proceed to the main body of the proof. By what has been said so far, we may assume $|\alpha| > 0$, $\alpha_j > 0$, and $\bar{0} \leq n_{j-1} < n_j$ if $j > 1$ and $\bar{0} < n_j$ if $j = 1$. Since $n_j \leq_{\text{col}} r_j \leq_{\text{row}} r_{j+1}$, if r_{j+1} is barred, then $n_j < r_{j+1}\!\downarrow$ by Proposition 5.3. If r_{j+1} is unbarred then either ($r_{j+1} = a+1$, $n_j = \bar{a}$) or ($n_j < r_{j+1}\!\downarrow$). Hence exactly one of the following occurs: ($n_j < r_{j+1}\!\downarrow$) or ($r_{j+1} = a+1$, $n_j = \bar{a}$). It follows that exactly one of the following occurs: ($n_j < r_{j+1}\!\downarrow$ or w is empty or $w_1 \neq r_{j+1}$) or ($w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$), which we consider as two separate cases in remainder of the proof. Set $\mathbf{n}_{\downharpoonleft} = (n_1, n_2, \cdots, n_{j-1}, n_j\!\downharpoonleft, n_{j+1}, \cdots, n_l)$, $\mathbf{n}_{\downarrow} = (n_1, n_2, \cdots, n_{j-1}, n_j\!\downarrow, n_{j+1}, \cdots, n_l)$, and $\alpha_- = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots)$.

Case $n_j < r_{j+1}\!\downarrow$ or w is empty or $w_1 \neq r_{j+1}$.

There are three subcases depending on whether $j = 1$, ($j > 1$ and $\alpha_j = 1$), or ($j > 1$ and $\alpha_j > 1$). We argue only the last, as the first two are similar and easier. A j-expansion yields

$$\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) = \mathsf{v}J_{\alpha_-}(\mathbf{n}_\rfloor : {}^{\mathsf{n_j}}_{j-1}, {}^{\mathsf{w}}_{j}) + \mathsf{v}J_\alpha(\mathbf{n}_\downarrow : {}^{\mathsf{w}}_{j}) \tag{43}$$

$$\equiv \mathsf{v}J_{\alpha_-}(\mathbf{n}_\rfloor : {}^{\mathsf{n_j w}}_{j-1}) + \mathsf{v}J_\alpha(\mathbf{n}_\downarrow : {}^{\mathsf{w}}_{j}) \tag{44}$$

$$\equiv \sum_{\substack{T\in \mathrm{RCT}_{\nu'},\ T_{\nu'\setminus\alpha'_-} = R\sqcup \boxed{n_j}_{\alpha_j,j} \\ T_{\alpha'_-}\in \mathrm{Tab}_{\alpha'_-}(\mathbf{n}_\rfloor)}} \mathrm{arwread}(T) + \sum_{\substack{T\in \mathrm{RCT}_{\nu'},\ T_{\nu'\setminus\alpha'} = R \\ T'_\alpha\in \mathrm{Tab}_{\alpha'}(\mathbf{n}_\downarrow)}} \mathrm{arwread}(T). \tag{45}$$

The first congruence is by Lemma 6.1; the conditions of the lemma are satisfied because $n_j < r_{j+1}\!\downarrow$ if n_j is unbarred (by Proposition 5.3), and $w_1 \geq r_{j+1}$; if $n_j = \bar{a}$ is barred, then $w_1 > a+1$ because we are not in the case ($w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$). The second term of (44) is congruent (modulo I_{Kron}) to the second sum of (45) by induction. The conditions of the theorem are satisfied here: (i)–(iii) are clear, (iv) holds since $n_{j-1} < n_j$, (v) holds since $(n_\downarrow)_j = n_j\!\downarrow < n_j \leq r_j\!\rfloor$, and (vi) holds since we are not in the case $w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$.

We next claim that the first term of (44) satisfies conditions (i)–(vi) of the theorem (with α_- in place of α, $\mathbf{n}_\rfloor$ in place of $\mathbf{n}$, $R\sqcup \boxed{n_j}_{\alpha_j,j}$ in place of R, $\mathsf{n_j w}$ in place of w, $j-1$ in place of j), hence the first term of (44) is congruent to the first sum of (45) by induction. Conditions (i) and (v) are clear, (iv) holds since $n_{j-1} < n_j$, and (vi) follows from the fact that $n_{j-1} = r_{j-1}\!\rfloor$ is barred. Since $n_{j-1} < n_j$ and $n_{j-1} = r_{j-1}\!\rfloor$, it follows that $r_{j-1} \leq_{\mathrm{row}} n_j$, hence $R\sqcup \boxed{n_j}_{\alpha_j,j}$ is an RCT; this verifies condition (ii).

Finally we check (iii), which requires showing that $\mathsf{n_j w}$ is the end of an arrow respecting reading word of $R\sqcup \boxed{n_j}_{\alpha_j,j}$ satisfying an extra condition. There are two ways this can fail: either (I) $R\sqcup \boxed{n_j}_{\alpha_j,j}$ has a $\searrow$ arrow from n_j to r_{j+1} and $w_1 \neq r_{j+1}$, or (II) $R\sqcup \boxed{n_j}_{\alpha_j,j}$ has a $\nwarrow$ arrow from r_{j+1} to n_j and $r_{j+1} = w_1$; (I) implies r_{j+1} is barred by the definition of a $\searrow$ arrow, so by assumption (iii) of the theorem $w_1 = r_{j+1}$, thus (I) cannot occur; (II) implies $n_j = a$, $r_{j+1} = a+1$ by the definition of a $\nwarrow$ arrow, which combined with $r_{j+1} = w_1$ contradicts assumption (vi) of the theorem, thus (II) cannot occur. (By Remark 6.7 (b), we can assume $R\sqcup \boxed{n_j}_{\alpha_j,j}$ does not have an arrow between r_{j-2} and n_j; in any case, this is not an issue because such an arrow must go from r_{j-2} to n_j and r_{j-2} is a letter in v.)

Now (45) is equal to the right side of (42) because (45) is simply the result of partitioning the set $\{T \in \mathrm{RCT}_{\nu'} \mid T_{\nu'\setminus\alpha'} = R,\ T_{\alpha'} \in \mathrm{Tab}_{\alpha'}(\mathbf{n})\}$ into two, depending on whether or not $T_{\alpha_j,j}$ is equal to or less than n_j. This completes the case $n_j < r_{j+1}\!\downarrow$ or w is empty or $w_1 \neq r_{j+1}$.

Case $w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$.

Corollary 6.5 yields the first congruence below; the hypotheses of the corollary are satisfied since $n_{j+1} = r_{j+1}\!\rfloor$ implies $n_{j+1} = \bar{a} = n_j$, and Proposition 5.3 implies $r_{j+2} \geq a+2$.

$$\mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w}}_{j}) \equiv \mathsf{v}J_\alpha(\mathbf{n} : {}^{\mathsf{w_1}}_{j-1}, {}^{\mathsf{w_2\cdots w_t}}_{j})$$

$$\equiv \mathsf{v}\mathsf{w_1}J_\alpha(\mathbf{n} : {}^{\mathsf{w_2\cdots w_t}}_{j}). \tag{46}$$

Here, t denotes the length of w. The last congruence is by the far commutation relations if $j > 1$ since $n_{j-1} < a$ by the facts that $n_{j-1} = r_{j-1}\downarrow$ is a barred letter and $n_{j-1} < n_j = \bar{a}$ (if $j = 1$ there is nothing to prove).

Finally, observe that the case ($n_j < r_{j+1}\downarrow$ or w is empty or $w_1 \neq r_{j+1}$) applies to (46) (with $\mathsf{v}\mathsf{w}_1$ in place of v, $\mathsf{w}_2 \cdots \mathsf{w}_t$ in place of w). Since the right side of (42) depends only on R and not directly on v and w, (46) is congruent to this right side, and this gives the desired statement in the present case $w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$. $\square$

Example 6.8. Let $\nu = (4,4,4,4)$. Let $R = \begin{array}{|c|c|c|} \hline 2 \\ \hline \bar{2} & 3 \\ \hline 3 & \bar{3} & \bar{4} \\ \hline \end{array}$; then $\mathrm{arwread}(R) = \mathsf{3\bar{2}\bar{3}32\bar{4}}$. We illustrate several steps of the inductive computation of $\mathsf{3\bar{2}\bar{3}32\bar{4}}J_{(1,2,3,4)}(\bar{1},\bar{2},\bar{4},5)$ from the proof of Theorem 6.6. After each step in which we add an entry to R, we record the new values of R, j, and j'. We first apply the proof of the theorem to $\mathsf{3\bar{2}\bar{3}32\bar{4}}J_{(1,2,3,4)}(\bar{1},\bar{2},\bar{4},5)$ ($\mathsf{v} = \mathsf{3\bar{2}\bar{3}32\bar{4}}$, w empty, R as above, $j = 4$, $j' = 3$) and expand as in (43):

$$\begin{aligned} &\mathsf{3\bar{2}\bar{3}32\bar{4}}J_{(1,2,3,4)}(\bar{1},\bar{2},\bar{4},5) \\ &= \mathsf{3\bar{2}\bar{3}32\bar{4}}\Big(J_{(1,2,3,3)}(\bar{1},\bar{2},\bar{4},\bar{4} : {}^{\mathsf{5}}_{3}) + J_{(1,2,3,4)}(\bar{1},\bar{2},\bar{4},\bar{4})\Big) \end{aligned} \tag{47}$$

Next, apply the theorem to the first term of (47) $\big(\mathsf{v} = \mathsf{3\bar{2}\bar{3}32\bar{4}}$, $\mathsf{w} = \mathsf{5}$, $R = \begin{array}{|c|c|c|c|} \hline 2 \\ \hline \bar{2} & 3 \\ \hline 3 & \bar{3} & \bar{4} & 5 \\ \hline \end{array}$, $j = 3$, $j' = 4\big)$. This is handled by the case $w_1 = r_{j+1} = a+1$, $n_j = \bar{a}$, hence the computation proceeds by applying (46) and then a 3-expansion:

$$\begin{aligned} &\mathsf{3\bar{2}\bar{3}32\bar{4}}J_{(1,2,3,3)}(\bar{1},\bar{2},\bar{4},\bar{4} : {}^{\mathsf{5}}_{3}) \\ &\equiv \mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,2,3,3)}(\bar{1},\bar{2},\bar{4},\bar{4}) \\ &= \mathsf{3\bar{2}\bar{3}32\bar{4}5}\Big(J_{(1,2,2,3)}(\bar{1},\bar{2},\bar{4},\bar{4} : {}^{\bar{\mathsf{4}}}_{2}) + J_{(1,2,3,3)}(\bar{1},\bar{2},4,\bar{4})\Big). \end{aligned}$$

The first term is $\equiv 0$ by (39). We expand the second term as in (43) ($j = 3$):

$$\begin{aligned} &\mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,2,3,3)}(\bar{1},\bar{2},4,\bar{4}) \\ &= \mathsf{3\bar{2}\bar{3}32\bar{4}5}\Big(J_{(1,2,2,3)}(\bar{1},\bar{2},\bar{3},\bar{4} : {}^{\mathsf{4}}_{2}) + J_{(1,2,3,3)}(\bar{1},\bar{2},\bar{3},\bar{4})\Big). \end{aligned}$$

Next, we proceed with the inductive computation of the first term above $\big(R = \begin{array}{|c|c|c|c|} \hline 2 \\ \hline \bar{2} & 3 & 4 \\ \hline 3 & \bar{3} & \bar{4} & 5 \\ \hline \end{array}$, $j = 2$, $j' = 4\big)$

$$\begin{aligned} &\mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,2,2,3)}(\bar{1},\bar{2},\bar{3},\bar{4} : {}^{\mathsf{4}}_{2}) \\ &= \mathsf{3\bar{2}\bar{3}32\bar{4}5}\Big(J_{(1,1,2,3)}(\bar{1},\bar{2},\bar{3},\bar{4} : {}^{\bar{\mathsf{2}}}_{1}, {}^{\mathsf{4}}_{2}) + J_{(1,2,2,3)}(\bar{1},2,\bar{3},\bar{4} : {}^{\mathsf{4}}_{2})\Big). \end{aligned}$$

We continue the computation with the second term by applying a 2-expansion (as in (43)) and then applying (44):

$$\begin{aligned}
&\mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,2,2,3)}(\bar{1},2,\bar{3},\bar{4} : {}^{4}_{2}) \\
&= \mathsf{3\bar{2}\bar{3}32\bar{4}5}\Big(J_{(1,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{2}_{1}, {}^{4}_{2}) + J_{(1,2,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{4}_{2})\Big) \\
&\equiv \mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{24}_{1}).
\end{aligned} \tag{48}$$

Note that the second term of the second line is $\equiv 0$ by (39). The input data to the theorem for the term (48) is $\mathsf{v} = \mathsf{3\bar{2}\bar{3}32\bar{4}5}$, $\mathsf{w} = \mathsf{24}$, $R = \begin{array}{|c|c|c|c|}\hline 2&2&&\\ \hline \bar{2}&3&4&\\ \hline 3&\bar{3}&\bar{4}&5\\ \hline\end{array}$, $j = 1$, $j' = 4$. This is handled by the case $w_1 = r_{j+1} = a + 1$, $n_j = \bar{a}$, hence the computation proceeds by applying (46) and then a 1-expansion:

$$\begin{aligned}
&\mathsf{3\bar{2}\bar{3}32\bar{4}5}J_{(1,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{24}_{1}) \\
&\equiv \mathsf{3\bar{2}\bar{3}32\bar{4}52}J_{(1,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{4}_{1}) \\
&= \mathsf{3\bar{2}\bar{3}32\bar{4}52}\Big(\mathsf{\bar{1}}J_{(0,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4} : {}^{4}_{1}) + J_{(1,1,2,3)}(1,\bar{1},\bar{3},\bar{4} : {}^{4}_{1})\Big) \\
&\equiv \mathsf{3\bar{2}\bar{3}32\bar{4}52}\Big(\mathsf{\bar{1}4}J_{(0,1,2,3)}(\bar{1},\bar{1},\bar{3},\bar{4}) + J_{(1,1,2,3)}(1,\bar{1},\bar{3},\bar{4} : {}^{4}_{1})\Big)
\end{aligned} \tag{49}$$

The last congruence is by (44). The first term of (49) is $\equiv 0$ by (39). We continue with the inductive computation of the second term of (49) $\Big(\mathsf{v} = \mathsf{3\bar{2}\bar{3}32\bar{4}52}$, $\mathsf{w} = \mathsf{4}$, $R = \begin{array}{|c|c|c|c|}\hline 2&2&&\\ \hline \bar{2}&3&4&\\ \hline 3&\bar{3}&\bar{4}&5\\ \hline\end{array}$, $j = 1$, $j' = 4\Big)$:

$$\begin{aligned}
&\mathsf{3\bar{2}\bar{3}32\bar{4}52}J_{(1,1,2,3)}(1,\bar{1},\bar{3},\bar{4} : {}^{4}_{1}) \\
&= \mathsf{3\bar{2}\bar{3}32\bar{4}52}\Big(\mathsf{1}J_{(0,1,2,3)}(\bar{0},\bar{1},\bar{3},\bar{4} : {}^{4}_{1}) + J_{(1,1,2,3)}(\bar{0},\bar{1},\bar{3},\bar{4} : {}^{4}_{1})\Big) \\
&\equiv \mathsf{3\bar{2}\bar{3}32\bar{4}52}\Big(\mathsf{14}J_{(0,1,2,3)}(\bar{0},\bar{1},\bar{3},\bar{4}) + J_{(1,1,2,3)}(\bar{0},\bar{1},\bar{3},\bar{4} : {}^{4}_{1})\Big).
\end{aligned}$$

The input data to the theorem for this first term is $\mathsf{v} = \mathsf{3\bar{2}\bar{3}32\bar{4}5214}$, w empty, $R = \begin{array}{|c|c|c|c|}\hline 1&&&\\ \hline 2&2&&\\ \hline \bar{2}&3&4&\\ \hline 3&\bar{3}&\bar{4}&5\\ \hline\end{array}$, $j = 4$, $j' = 4$, $\alpha = (0,1,2,3)$, $\mathbf{n} = (\bar{0},\bar{1},\bar{3},\bar{4})$.

7. A strengthening of the main theorem and comparison to [5]

We conjecture a strengthening of the main theorem and compare it to a similar conjecture in [5].

7.1. A conjectured strengthening.

We conjecture that Theorem 2.3 can be strengthened to hold in $\mathcal{U}/I_{\text{Kron-K}}$.

Conjecture 7.1. *For any partition ν,*

$$\mathfrak{J}_\nu(\mathbf{u}) = \sum_{T \in \mathrm{CT}^{<}_{\nu}} \mathrm{arwread}(T) \qquad \textit{in } \mathcal{U}/I_{\text{Kron-K}}.$$

In fact, we conjecture that Theorem 6.6 holds exactly as stated with $\mathcal{U}/I_{\text{Kron-K}}$ in place of $\mathcal{U}/I_{\text{Kron}}$, and we believe that the same inductive structure of the proof works in this setting, except some of the steps are much more difficult to justify. Specifically, Lemmas 6.1 and 6.4 are easily shown to hold with $\mathcal{U}/I_{\text{Kron-K}}$ in place of $\mathcal{U}/I_{\text{Kron}}$, whereas proving the correct analog of Corollary 6.5 for $\mathcal{U}/I_{\text{Kron-K}}$ seems to be the main difficulty to overcome to adapt the proof.

The conjectured strengthening of Theorem 6.6 to $\mathcal{U}/I_{\text{Kron-K}}$ was checked by computer in the following cases:

- $\nu = \alpha$ is a partition with at most 4 rows and at most 4 columns and size at most 10, and $\bar{0} \le n_1 \le \cdots \le n_l \le \bar{3}$,
- $\nu = \alpha$ is a partition with at most 6 rows and at most 6 columns and size at most 12, and $\bar{0} \le n_1 \le \cdots \le n_l \le \bar{2}$.

7.2. **Comparison with results of** [5]**.** The statement and proof of Theorem 6.6 have much in common with that of [5, Theorem 4.8], but there are also important differences. It is reasonable to expect that one result can be obtained from the other by a standardization argument, but we believe this cannot be done. It is quite possible that there is a more general monomial positivity result for noncommutative Schur functions that yields both results as a special case.

Instead of comparing Theorem 6.6 and [5, Theorem 4.8], we believe it more natural to compare Conjecture 7.1 with a conjecture in [5] similar to [5, Theorem 4.8]. We state this conjecture below, after introducing some notation.

Recall that $\mathcal{V}_M$ denotes the free $\mathbb{Z}$-algebra generated by $v_1, v_2, \ldots, v_M$ and its monomials are identified with words in the alphabet of ordinary letters $\mathsf{1}, \mathsf{2}, \ldots, \mathsf{M}$. The noncommutative Schur function $\mathfrak{J}_\nu(\mathbf{v}) \in \mathcal{V}_M$ is defined in Definition 4.10.

Let $I^{\text{st}}_{\text{KR},\le k}$ be the two-sided ideal of $\mathcal{V}_M$ corresponding to the relations

$$\mathsf{acb} = \mathsf{cab} \qquad \text{for } c - a > k \text{ and } a < b < c,\ a, b, c \in [M], \tag{50}$$

$$\mathsf{bac} = \mathsf{bca} \qquad \text{for } c - a > k \text{ and } a < b < c,\ a, b, c \in [M], \tag{51}$$

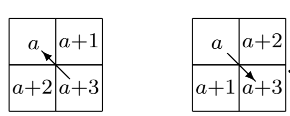
$$(\mathsf{ac} - \mathsf{ca})\mathsf{b} = \mathsf{b}(\mathsf{ac} - \mathsf{ca}) \qquad \text{for } c - a \le k \text{ and } a < b < c,\ a, b, c \in [M], \tag{52}$$

$$\mathsf{v} \qquad \text{for ordinary words } \mathsf{v} \text{ with a repeated letter.} \tag{53}$$

Conjecture 7.2 ([5, §5.2])**.** *For any partition ν,*

$$\mathfrak{J}_\nu(\mathbf{v}) = \sum_{T \in \mathrm{SYT}'_\nu} \mathrm{sqread}(T) \quad \textit{in } \mathcal{V}_M / I^{\text{st}}_{\text{KR},\le 3}. \tag{54}$$

Here, SYT'_ν denotes the set of semistandard Young tableaux of shape ν with entries in $[M]$, having no repeated letter. To define $\mathrm{sqread}(T)$, first draw arrows as shown between entries of T for each of its 2×2 subtableaux of the following forms:



The word $\mathrm{sqread}(T)$ is now defined as follows: let $D^1, D^2, \ldots, D^t$ be the diagonals of T, starting from the southwest. Let v^{i} be the result of reading, in the $\nwarrow$ direction, the entries of D^i that are $\nwarrow$ arrow tails followed by the remaining entries of D^i, read in the $\searrow$ direction. Set $\text{sqread}(T) = \mathsf{v}^1\mathsf{v}^2\cdots\mathsf{v}^\mathsf{t}$. For example,

$$\text{sqread}\left(\begin{array}{|c|c|c|c|}\hline 1&2&5&7\\\hline 3&4&6&8\\\hline 9\\\cline{1-1}\end{array}\right) = \mathsf{934126587}.$$

It is natural to wonder whether Conjecture 7.1 can be obtained from Conjecture 7.2 by a standardization argument or vice versa. We now argue informally that this is not possible in either direction. We will freely use the notation from §4.5. Also, by the *standardization map* we will mean the $\mathbb{Z}$-linear map from the degree M part of $\mathcal{U}$ to $\mathcal{V}_M$ extending $\mathsf{w} \mapsto \mathsf{w}^{\text{st}^<}$.

Since the relations of $\mathcal{U}/I_{\text{Kron-K}}$ preserve colored content, there is a $\mathbb{Z}$-module decomposition $\mathcal{U}/I_{\text{Kron-K}} \cong \bigoplus_\beta (\mathcal{U}/I_{\text{Kron-K}})_\beta$, where $(\mathcal{U}/I_{\text{Kron-K}})_\beta$ denotes the $\mathbb{Z}$-span of the colored words of colored content β in the algebra $\mathcal{U}/I_{\text{Kron-K}}$. Hence we can write $\mathfrak{J}_\nu(\mathbf{u})$ uniquely as a sum $\mathfrak{J}_\nu(\mathbf{u}) = \sum_\beta (\mathfrak{J}_\nu(\mathbf{u}))_\beta$, for $(\mathfrak{J}_\nu(\mathbf{u}))_\beta \in (\mathcal{U}/I_{\text{Kron-K}})_\beta$. Conjecture 7.2 then implies that for any colored content β,

$$(\mathfrak{J}_\nu(\mathbf{u}))_\beta = \sum_{\substack{T \in \text{CT}_\nu^< \\ T \text{ has colored content } \beta}} \text{arwread}(T) \qquad \text{in} \quad \mathcal{U}/I_{\text{Kron-K}}. \tag{55}$$

We can consider $\mathfrak{J}_\nu(\mathbf{v})$ as an element of $\mathcal{V}_M$, written as the signed sum of words obtained by directly expanding its definition as a sum of products of $e_k(\mathbf{v})$'s. Let $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ denote the restriction of the signed sum $\mathfrak{J}_\nu(\mathbf{v})$ to the permutations $\mathcal{S}(\beta)$. Similarly, $(\mathfrak{J}_\nu(\mathbf{u}))_\beta$ can be considered as an element of $\mathcal{U}$ obtained by restricting the expression for $\mathfrak{J}_\nu(\mathbf{u}) \in \mathcal{U}$ to the colored words of colored content β. It is true, then, that the image of $(\mathfrak{J}_\nu(\mathbf{u}))_\beta$ under the standardization map is $\mathfrak{J}_\nu(\mathbf{v})[\beta]$. However, as the examples below show, the relations of $\mathcal{U}/I_{\text{Kron-K}}$ and those of $\mathcal{V}_M/I^{\text{st}}_{\text{KR},\leq 3}$ are not compatible in a way required to relate the images of $(\mathfrak{J}_\nu(\mathbf{u}))_\beta$ and $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ in these algebras.

Example 7.3. For $\nu = (2, 2)$ and $\beta = ((1, 1), (2, 0))$, there holds

$$(\mathfrak{J}_\nu(\mathbf{u}))_\beta = \bar{\mathsf{1}}\mathsf{21}\bar{\mathsf{1}} \qquad \text{in} \quad \mathcal{U}/I_{\text{Kron-K}}.$$

On the other hand,

$$\mathfrak{J}_\nu(\mathbf{v})[\beta] = \mathsf{3412} + \mathsf{4132} - \mathsf{4312} \quad \text{in} \;\; \mathcal{V}_M/I^{\text{st}}_{\text{KR},\leq 3}$$

and this quantity is not equal to a positive sum of monomials in $\mathcal{V}_M/I^{\text{st}}_{\text{KR},\leq 3}$. Thus applying the standardization map to both sides of the identity (55) cannot be used to deduce the identity (54), at least not without writing $\mathfrak{J}_\nu(\mathbf{v})$ as a sum over $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ for some carefully selected colored contents β. And even if these carefully selected $\mathfrak{J}_\nu(\mathbf{v})[\beta]$ are monomial positive, this may not be enough to prove (54) from (55) since elements of $I_{\text{Kron-K}}$ often standardize to elements which do not belong to $I^{\text{st}}_{\text{KR},\leq 3}$; for example, $\bar{\mathsf{2}}\mathsf{1}\bar{\mathsf{2}}\bar{\mathsf{1}} - \bar{\mathsf{2}}\bar{\mathsf{2}}\mathsf{1}\bar{\mathsf{1}} \in I_{\text{Kron-K}}$ but $\mathsf{4132} - \mathsf{4312} \notin I^{\text{st}}_{\text{KR},\leq 3}$.

Example 7.4. We now argue that inverse standardization cannot be used to deduce (55) from (54). Let $\nu = (2, 2, 1)$ and $\beta = ((1, 1), (3, 0))$. There holds

$$(\mathfrak{J}_\nu(\mathbf{u}))_\beta = \bar{\mathsf{1}}\bar{\mathsf{1}}\mathsf{21}\bar{\mathsf{1}} \quad \text{in} \;\; \mathcal{U}/I_{\text{Kron-K}}, \tag{56}$$

$$\mathfrak{J}_\nu(\mathbf{v})[\beta] = \mathsf{43512} \quad \text{in} \;\; \mathcal{V}_M/I^{\text{st}}_{\text{KR},\leq 3}. \tag{57}$$

Even though $(\bar{1}\bar{1}21\bar{1})^{\mathrm{st}^<} = \mathsf{43512}$, we cannot deduce (56) (directly) from (57) because the element

$$\mathsf{43215} - \mathsf{43251} \in I^{\mathrm{st}}_{\mathrm{KR},\leq 3},$$

but its inverse image under the standardization map, restricted to colored content β, is

$$\bar{1}\bar{1}\bar{1}\mathsf{12} - \bar{1}\bar{1}\bar{1}\mathsf{21} \notin I_{\text{Kron-K}}.$$

8. Commuting super elementary symmetric functions

We adapt the theory of noncommutative Schur functions to the super setting and thereby prove Theorems 2.5 and 3.12.

8.1. Elementary functions commute in $\mathcal{U}/I_{\text{Kron-K}}$ and $\mathcal{U}/I^{\lessdot}_{\mathrm{plac}}$.

Proposition 8.1. *The noncommutative super elementary symmetric functions commute in $\mathcal{U}/I_{\text{Kron-K}}$, i.e., $e_k(\mathbf{u})e_l(\mathbf{u}) = e_l(\mathbf{u})e_k(\mathbf{u})$ in $\mathcal{U}/I_{\text{Kron-K}}$ for all k, l.*

Proof. Let s, t be formal variables that commute with each other and with all the u_z, $z \in \mathcal{A}$. Throughout the proof, we write $f \equiv g$ to mean that f and g are congruent modulo $I_{\text{Kron-K}}[[s,t]]$, for any $f, g \in \mathcal{U}[[s,t]]$. We also let $[f,g]$ denote the commutator $fg - gf$ for any $f, g \in \mathcal{U}[[s,t]]$.

Define

$$f_z = \begin{cases} 1 + u_z s & \text{if } z \text{ is unbarred,} \\ (1 - u_z s)^{-1} & \text{if } z \text{ is barred.} \end{cases} \qquad g_z = \begin{cases} 1 + u_z t & \text{if } z \text{ is unbarred,} \\ (1 - u_z t)^{-1} & \text{if } z \text{ is barred.} \end{cases}$$

We require the following key lemma.

Lemma 8.2. *For any $x, y, z \in \mathcal{A}$ such that $x < y < z$, $\mathsf{y} = u_y$ commutes with $g_z^{-1} f_x g_z f_x^{-1}$ in the algebra $(\mathcal{U}/I_{\text{Kron-K}})[[s,t]]$.*

Proof of Lemma 8.2. There are four cases depending on whether or not x and z are barred or unbarred. We first handle the case that x and z are unbarred. First note that y commutes with $g_z^{-1} f_x g_z f_x^{-1}$ in $(\mathcal{U}/I_{\text{Kron-K}})[[s,t]]$ if and only if y commutes with $g_z^{-1}[f_x, g_z] f_x^{-1}$ in $(\mathcal{U}/I_{\text{Kron-K}})[[s,t]]$. Further, $[f_x, g_z] = [\mathsf{x}, \mathsf{z}]st$. By relations (4) and (5), $\mathsf{z}[\mathsf{x},\mathsf{z}] \equiv 0$ and $[\mathsf{x},\mathsf{z}]\mathsf{x} \equiv 0$. Hence

$$g_z^{-1}[f_x, g_z] f_x^{-1} \equiv [\mathsf{x},\mathsf{z}]st.$$

Thus it suffices to show that $\mathsf{y}[\mathsf{x},\mathsf{z}] \equiv [\mathsf{x},\mathsf{z}]\mathsf{y}$; this holds by (8) if $x = z{\downarrow\downarrow}$ and by (13) and (14) if $x < z{\downarrow\downarrow}$.

If x and z are barred, then we instead show that y commutes with $f_x g_z^{-1} f_x^{-1} g_z$ in $(\mathcal{U}/I_{\text{Kron-K}})[[s,t]]$. This is equivalent to showing that y commutes with $f_x[g_z^{-1}, f_x^{-1}]g_z$ in $(\mathcal{U}/I_{\text{Kron-K}})[[s,t]]$. There holds $[g_z^{-1}, f_x^{-1}] = [\mathsf{z},\mathsf{x}]st$. By relations (6) and (7), $\mathsf{x}[\mathsf{z},\mathsf{x}] \equiv 0$ and $[\mathsf{z},\mathsf{x}]\mathsf{z} \equiv 0$. Hence

$$f_x[g_z^{-1}, f_x^{-1}]g_z \equiv [\mathsf{z},\mathsf{x}]st.$$

Thus it suffices to show that $\mathsf{y}[\mathsf{z},\mathsf{x}] \equiv [\mathsf{z},\mathsf{x}]\mathsf{y}$; this holds by (8) if $x = z{\downarrow\downarrow}$ and by (13) and (14) if $x < z{\downarrow\downarrow}$.

We now handle the case x is barred and z is unbarred. We compute

$$[f_x, g_z]f_x^{-1} = \sum_{d=1}^{\infty}[\mathsf{x}^d s^d, \mathsf{z}t](1-\mathsf{x}s) = [\mathsf{x},\mathsf{z}]st + \sum_{d=2}^{\infty}([\mathsf{x}^d,\mathsf{z}] - [\mathsf{x}^{d-1},\mathsf{z}]\mathsf{x})s^d t \equiv [\mathsf{x},\mathsf{z}]st,$$

where the congruence follows from (6). Combining this with the fact $\mathsf{z}[\mathsf{x},\mathsf{z}] \equiv 0$ yields

$$g_z^{-1}[f_x, g_z]f_x^{-1} \equiv g_z^{-1}[\mathsf{x},\mathsf{z}]st \equiv [\mathsf{x},\mathsf{z}]st. \tag{58}$$

Since $\mathsf{y}[\mathsf{x},\mathsf{z}] \equiv [\mathsf{x},\mathsf{z}]\mathsf{y}$ by the relations (13) and (14), this case of the lemma is proved. The case x is unbarred and z is barred is similar. □

We continue with the main thread of the proof of Proposition 8.1. It is clear that $f_x g_x = g_x f_x$ for all $x \in \mathcal{A}$. We next claim that for any $x, y \in \mathcal{A}$, $x < y$,

$$f_y f_x g_y g_x \equiv g_y g_x f_y f_x. \tag{59}$$

Note that this is equivalent to

$$g_y^{-1} f_x g_y f_x^{-1} \equiv f_y^{-1} g_x f_y g_x^{-1}. \tag{60}$$

Further, the congruence (60) is equivalent to $g_y^{-1}[f_x, g_y]f_x^{-1} \equiv f_y^{-1}[g_x, f_y]g_x^{-1}$. This follows from computations similar to but easier than those used to prove Lemma 8.2. For example, in the case x is barred and y is unbarred, (58) yields

$$g_y^{-1}[f_x, g_y]f_x^{-1} \equiv [\mathsf{x},\mathsf{y}]st \equiv f_y^{-1}[g_x, f_y]g_x^{-1}.$$

Now let $z_1 < z_2 < \cdots < z_{2N}$ denote the elements of $\mathcal{A}$ in natural order, and set $F_{j,i} = f_{z_j} f_{z_{j-1}} \cdots f_{z_i}$ and $G_{j,i} = g_{z_j} g_{z_{j-1}} \cdots g_{z_i}$. The proposition is equivalent to the statement $F_{2N,1}G_{2N,1} \equiv G_{2N,1}F_{2N,1}$. We will prove that $F_{j,i}G_{j,i} \equiv G_{j,i}F_{j,i}$ by induction on $j-i$. By the previous paragraph, this holds for the base cases $j-i=0$ and $j-i=1$. Now, for $j-i>1$, we obtain

$$\begin{aligned}
&F_{j,i}G_{j,i}\\
&= F_{j,i+1} g_{z_j} g_{z_j}^{-1} f_{z_i} g_{z_j} f_{z_i}^{-1} f_{z_i} G_{j-1,i}\\
&= F_{j,i+1} G_{j,i+1} G_{j-1,i+1}^{-1} \big(g_{z_j}^{-1} f_{z_i} g_{z_j} f_{z_i}^{-1}\big) F_{j-1,i+1}^{-1} F_{j-1,i} G_{j-1,i}\\
&\equiv G_{j,i+1} F_{j,i+1} G_{j-1,i+1}^{-1} \big(g_{z_j}^{-1} f_{z_i} g_{z_j} f_{z_i}^{-1}\big) F_{j-1,i+1}^{-1} G_{j-1,i} F_{j-1,i} && \text{by induction}\\
&\equiv G_{j,i+1} F_{j,i+1} F_{j-1,i+1}^{-1} \big(g_{z_j}^{-1} f_{z_i} g_{z_j} f_{z_i}^{-1}\big) G_{j-1,i+1}^{-1} G_{j-1,i} F_{j-1,i} && \text{by Lemma 8.2 and induction}\\
&= G_{j,i+1} f_{z_j} \big(g_{z_j}^{-1} f_{z_i} g_{z_j} f_{z_i}^{-1}\big) g_{z_i} F_{j-1,i}\\
&\equiv G_{j,i+1} f_{z_j} \big(f_{z_j}^{-1} g_{z_i} f_{z_j} g_{z_i}^{-1}\big) g_{z_i} F_{j-1,i} && \text{by (60)}\\
&= G_{j,i} F_{j,i},
\end{aligned}$$

as desired. □

Proposition 8.3. *In the algebra* $\mathcal{U}/I_{\text{plac}}^{\lessdot}$*, there holds* $e_k^{\lessdot}(\mathbf{u})e_l^{\lessdot}(\mathbf{u}) = e_l^{\lessdot}(\mathbf{u})e_k^{\lessdot}(\mathbf{u})$ *for all* k, l.

Proof. This follows from a similar result for the ordinary plactic algebra [20] and a standardization argument using Proposition 4.8. □

8.2. **Noncommutative super Schur function basics.** Here we adapt the basic setup of [11, 8] to the super setting. Throughout this section, fix a shuffle order $\lessdot$ on $\mathcal{A}$.

Let $x_1, x_2, \ldots$ be formal variables that commute with each other and with all the u_z, $z \in \mathcal{A}$. Define

$$c_z(x_j) = \begin{cases} (1 - u_z x_j)^{-1} & \text{if } z \text{ is unbarred,} \\ 1 + u_z x_j & \text{if } z \text{ is barred.} \end{cases}$$

Define the "noncommutative Cauchy product"

$$\Omega^{\lessdot}(\mathbf{x}, \mathbf{u}) = \prod_{j=1}^{\infty} \big(c_{z_1}(x_j) c_{z_2}(x_j) \cdots c_{z_{2N}}(x_j)\big), \tag{61}$$

where $z_1 \lessdot z_2 \lessdot \cdots \lessdot z_{2N}$ are the elements of $\mathcal{A}$.

The noncommutative Cauchy product $\Omega^{\lessdot}(\mathbf{x}, \mathbf{u})$ is naturally expressed in terms of the *noncommutative super homogeneous symmetric functions*, which are given by

$$h_k^{\lessdot}(\mathbf{u}) = \sum_{\substack{z_1 \leq_{\text{row}} z_2 \leq_{\text{row}} \cdots \leq_{\text{row}} z_k \\ z_1, \ldots, z_k \in \mathcal{A}}} u_{z_1} u_{z_2} \cdots u_{z_k} \ \in \mathcal{U}$$

for any positive integer k; set $h_0^{\lessdot}(\mathbf{u}) = 1$ and $h_k^{\lessdot}(\mathbf{u}) = 0$ for $k < 0$. Here, for $y, z \in \mathcal{A}$, the notation $y \leq_{\text{row}} z$ means that either $y \lessdot z$ or y and z are equal unbarred letters.

Recall that the noncommutative super Schur functions for the order $\lessdot$ are denoted $\mathfrak{J}_\nu^{\lessdot}(\mathbf{u})$ (Definition 3.9). Also recall that $\langle \cdot, \cdot \rangle$ denotes the symmetric bilinear form on $\mathcal{U}$ for which the monomials (colored words) form an orthonormal basis. We have the following generalization of the setup of [11, 8] to the super setting.

Theorem 8.4. *Let I be a two-sided ideal of $\mathcal{U}$.*

(i) *The noncommutative Cauchy product can be written in the following two ways (in $\mathcal{U}$)*

$$\Omega^{\lessdot}(\mathbf{x}, \mathbf{u}) = \sum_{\text{colored words } \mathsf{w}} Q_{\mathrm{Des}_{\lessdot}(\mathsf{w})}(\mathbf{x}) \mathsf{w} = \sum_{\text{weak compositions } \alpha} \mathbf{x}^\alpha h_\alpha^{\lessdot}(\mathbf{u}). \tag{62}$$

Hence for any $\gamma \in \mathcal{U}$, $F_\gamma^{\lessdot}(\mathbf{x}) = \big\langle \Omega^{\lessdot}(\mathbf{x}, \mathbf{u}), \gamma \big\rangle$. Here, $h_\alpha^{\lessdot}(\mathbf{u}) := h_{\alpha_1}^{\lessdot}(\mathbf{u}) h_{\alpha_2}^{\lessdot}(\mathbf{u}) \cdots$.

(ii) *There holds $e_k^{\lessdot}(\mathbf{u}) e_l^{\lessdot}(\mathbf{u}) = e_l^{\lessdot}(\mathbf{u}) e_k^{\lessdot}(\mathbf{u})$ in $\mathcal{U}/I$ for all k, l if and only if $h_k^{\lessdot}(\mathbf{u}) h_l^{\lessdot}(\mathbf{u}) = h_l^{\lessdot}(\mathbf{u}) h_k^{\lessdot}(\mathbf{u})$ in $\mathcal{U}/I$ for all k, l.*

(iii) *If $h_k^{\lessdot}(\mathbf{u}) h_l^{\lessdot}(\mathbf{u}) = h_l^{\lessdot}(\mathbf{u}) h_k^{\lessdot}(\mathbf{u})$ in $\mathcal{U}/I$ for all k, l, then*

$$\Omega^{\lessdot}(\mathbf{x}, \mathbf{u}) = \sum_\nu m_\nu(\mathbf{x}) h_\nu^{\lessdot}(\mathbf{u}) = \sum_\nu s_\nu(\mathbf{x}) \mathfrak{J}_\nu^{\lessdot}(\mathbf{u}) \quad \text{in } \mathcal{U}/I, \tag{63}$$

where the sums are over all partitions ν. Hence for any $\gamma \in I^\perp$,

$$F_\gamma^{\lessdot}(\mathbf{x}) = \big\langle \Omega^{\lessdot}(\mathbf{x}, \mathbf{u}), \gamma \big\rangle = \sum_\nu s_\nu(\mathbf{x}) \langle \mathfrak{J}_\nu^{\lessdot}(\mathbf{u}), \gamma \rangle.$$

Proof. Formula (62) is seen by fixing a colored word w and collecting all monomials in the x_i that appear with it in the noncommutative Cauchy product, or by fixing a monomial

$\mathbf{x}^\alpha$ and collecting all colored words that appear with it in the noncommutative Cauchy product. The second part of (i) is immediate from (62) and Definition 2.4.

We next prove (ii). Let f_z, g_z be as in the proof of Proposition 8.1, and let $z_1 \lessdot z_2 \lessdot \cdots \lessdot z_{2N}$ be the elements of $\mathcal{A}$. Then the $e_k^{\lessdot}(\mathbf{u})$ commuting is equivalent to $f_{z_{2N}} f_{z_{2N-1}} \cdots f_{z_1}$ commuting with $g_{z_{2N}} g_{z_{2N-1}} \cdots g_{z_1}$ and the $h_k^{\lessdot}(\mathbf{u})$ commuting is equivalent to $f_{z_1}^{-1} f_{z_2}^{-1} \cdots f_{z_{2N}}^{-1}$ commuting with $g_{z_1}^{-1} g_{z_2}^{-1} \cdots g_{z_{2N}}^{-1}$ (in the algebra $(\mathcal{U}/I)[[s,t]]$). Statement (ii) follows.

For (iii), the first equality of (63) is immediate from (i) and the hypothesis that the $h_k^{\lessdot}(\mathbf{u})$ commute. For the second equality of (63), simply note that when the $h_k^{\lessdot}(\mathbf{u})$ commute, the subalgebra they generate is the surjective image of the ring of symmetric functions in commuting variables and hence all the usual identities hold. □

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