Research Statement

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1. Introduction

My research lies at the intersection of algebraic combinatorics and representation theory. The interplay between these fields is strikingly powerful: combinatorics offers concrete and intuitive structures that are easy to manipulate and ideal for gathering experimental data, while algebra offers guidance about which manipulations are natural as well as elegant methods of proof.

Canonical bases originated in the investigations of Kazhdan and Lusztig on singularities of Schubert varieties. Since their introduction in the famous paper [25] of 1979, they have played a central role at the intersection of representation theory, algebraic geometry, and combinatorics. Much of my research involves canonical bases. Of particular importance for my work is their remarkable ability to connect combinatorics and representation theory. For instance, canonical bases beautifully connect the RSK correspondence with quantum Schur-Weyl duality (as will be explained in §3.1). In my thesis work [5, 7], I use the canonical basis of the type A extended affine Hecke algebra to explain the appearance of the cyclage and catabolism operations of Lascoux and Schützenberger [35, 34] in the graded characters of Garsia-Procesi modules.

For the last three years, my research has centered around geometric complexity theory (GCT), an approach to **P** vs. **NP** and related problems in complexity theory using algebraic geometry and representation theory. GCT originated in the 1999 paper [39], in which Mulmuley successfully applied algebraic geometry to prove a weakened version of the $\mathbf{NC} \neq \mathbf{P}$ conjecture. Geometric complexity theory has since been developed primarily by Mulmuley and Sohoni. In the last several years, it has attracted attention from researchers in algebraic geometry, complexity theory, and algebraic combinatorics including Landsberg, Weyman, Kumar, Bürgisser, and myself.

Much of algebraic combinatorics is motivated by the search for positive combinatorial formulae for quantities that are known to be nonnegative by geometry or representation theory [49]. The archetypal example of such a formula is the *Littlewood-Richardson rule*, which gives the multiplicities for decomposing a tensor product of two irreducible representations of GL_n. As will be explained in §4, GCT hints of a deep connection between complexity theory and positivity in algebraic combinatorics.

The positivity problem which Mulmuley and I believe to be the current most important step for GCT is the *Kronecker problem*, which asks for a positive combinatorial formula for the multiplicity $g_{\lambda\mu\nu}$ of an irreducible representation M_ν of the symmetric group $\mathcal{S}_r$ in the tensor product $M_\lambda \otimes M_\mu$. The bulk of my recent research has been to develop (with Mulmuley and Sohoni) an approach to this problem using quantum groups and canonical bases [3, 4, 6, 9]. This has resulted in a canonical basis-theoretic solution to the Kronecker problem in the case of two two-row shapes [3].

A few months ago, I happened upon an amazing computer experiment that could be a guide to a complete solution to the Kronecker problem. So far, it has led to a solution in the case of one hook shape and two arbitrary shapes [8]. This is an important breakthrough since all previously known formulae for Kronecker coefficients require two of the partitions to be restricted.

1.1. **Organization.** Section 2 describes the purely combinatorial approach to the Kronecker problem of [8], with future directions given in §2.3. Section 3 highlights some results from [3] on the basis-theoretic approach, with future directions given in §3.8–3.10. Section 4 explains the connection between complexity theory and positivity in algebraic combinatorics initiated in the GCT paper [41] and describes a plan to further explore this connection.

1.2. **Notation.** We fix some notation that will be needed later. We write $\lambda \vdash_l r$ for a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of size $r = \sum_{i=1}^{l} \lambda_i$. The conjugate partition λ' of λ is the partition whose diagram is the transpose of the diagram of λ. The set of standard Young tableaux is denoted SYT and the set of semistandard Young tableaux with entries in $[l]$ is denoted SSYT_l. For any set of tableaux ST, we write $\mathrm{ST}(\nu)$ to denote the subset of ST consisting of tableaux of shape ν. Let Z^*_λ be the SYT of shape λ with $1, \ldots, \lambda_1$ in the first row, $\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2$ in the second row, etc.

2. Kronecker coefficients for one hook shape

This section describes my rule for Kronecker coefficients for one hook shape and two arbitrary shapes [8] and the miraculous computer experiment that led to it. The rule uses *mixed insertion*, a generalization of Schensted insertion to colored words, developed by Haiman in [18]. Unlike almost all previous formulae for Kronecker coefficients, this formula is a count of certain tableaux. Thus there is now hope that the vast toolkit of tableau combinatorics will go a long way towards a complete solution to the Kronecker problem.

2.1. **A miraculous experiment.** This work began with the following experiment, first investigated by Lascoux in [33]: for any partition λ of r, let Γ_λ denote the set of permutations with insertion tableau Z^*_λ (see §1.2). Form the multiset of permutations

$$\Gamma_\lambda \circ \Gamma_\mu := \{u \circ v : u \in \Gamma_\lambda, v \in \Gamma_\mu\}, \tag{1}$$

where $\circ$ denotes multiplication in $\mathcal{S}_r$, i.e. composition of permutations. Then form the multisets of insertion and recording tableaux of these permutations; denote these by $P(\Gamma_\lambda \circ \Gamma_\mu)$ and $Q(\Gamma_\lambda \circ \Gamma_\mu)$.

The set Γ_λ naturally labels a basis of M_λ (for instance, it can be identified with the canonical basis of M_λ defined by Kazhdan-Lusztig in [25]). A nice solution to the Kronecker problem might assign labels to a basis of $M_\lambda \otimes M_\mu$ so that the decomposition of $M_\lambda \otimes M_\mu$ into irreducibles is apparent from these labels. The following two properties, if true for every partition ν of r, would make $\Gamma_\lambda \circ \Gamma_\mu$ a beautifully simple candidate for such labels.

(A) For every $T \in \mathrm{SYT}(\nu)$, the multiplicity of T in $P(\Gamma_\lambda \circ \Gamma_\mu)$ is $|\mathrm{SYT}(\nu)| g_{\lambda\mu\nu}$ or 0.

(B) For every $T \in \mathrm{SYT}(\nu)$, the multiplicity of T in $Q(\Gamma_\lambda \circ \Gamma_\mu)$ is $g_{\lambda\mu\nu}$.

Theorem 2.1 (Lascoux's Kronecker Rule [33])**.** *If λ and μ are hook shapes, then (A) and (B) hold for all ν.*

Lascoux [33] and Garsia-Remmel [15] both investigate the extent to which this rule generalizes to other shapes. They give examples showing that it does not extend beyond the hook hook case. However, our computations indicate that (B) is amazingly close to being true in general. We therefore believe that there is much more to be gained from this experiment, the Hook Kronecker Rule (below) being just the first example.

2.2. **The Hook Kronecker Rule.** By modifying the experiment above using colored words and mixed insertion, I obtained the following solution to the Kronecker problem for μ a hook shape.

A *colored word* is a word in the alphabet of barred letters $\{\overline{1}, \overline{2}, \cdots\}$ and unbarred letters $\{1, 2, \cdots\}$. Let w be a colored word. The *total color* of w is the number of barred letters in w. Define w^{blft} to be the ordinary word formed from w by shuffling the barred letters to the left and then removing their bars. We say that w is *Yamanouchi* of content λ if w^{blft} is Yamanouchi of content λ. For example, if $w = 1\,\overline{3}\,\overline{1}\,1\,\overline{2}\,\overline{2}\,2\,1$, then $w^{\mathrm{blft}} = 3\,1\,2\,2\,1\,1\,2\,1$, and these are Yamanouchi of content $(4, 3, 1)$. Set $\mu(d) := (r - d, 1^d)$. We define $\mathrm{CYW}_{\lambda,d}$ to be the set of colored Yamanouchi words of content λ and total color d, and this replaces the multiset of permutations $\Gamma_\lambda \circ \big(\Gamma_{\mu(d)} \sqcup \Gamma_{\mu(d-1)}\big)$ in the experiment above (why this is a replacement is explained in [8, §5.4]).

Mixed insertion is a generalization of Schensted insertion to colored words, developed by Haiman in [18]. Its chief advantage for this work is that it is "simultaneously compatible" with the orders $\overline{1} < 1 < \overline{2} < 2 \cdots$ and $\overline{1} \prec \overline{2} \prec \overline{3} \prec \cdots \prec 1 \prec 2 \cdots$. Let $\mathrm{CYT}_{\lambda,d}$ (resp. $\mathrm{CYT}^{\prec}_{\lambda,d}$) denote the set of mixed insertion tableaux of the words in $\mathrm{CYW}_{\lambda,d}$ using the order $<$ (resp. $\prec$).

A basic fact here is that $\mathrm{CYT}^{\prec}_{\lambda,d}(\nu)$ has size $g_{\lambda\,\mu(d)\,\nu} + g_{\lambda\,\mu(d-1)\,\nu}$. This is in some sense not new, as it can be deduced from results about hook Schur functions (which are characters of certain irreducible representations of the general linear Lie superalgebra) [2]. What is genuinely new here is the use of mixed insertion for both the orders $<$ and $\prec$. The miracle in this setup is that

Theorem 2.2 (Hook Kronecker Rule [8])**.** *The subset of $CYT_{\lambda,d}(\nu)$ consisting of those tableaux with unbarred southwest corner has cardinality $g_{\lambda\,\mu(d)\,\nu}$.*

2.3. **Towards a rule for Kronecker coefficients for one two-row shape.** I have recently been working on adapting the ideas of [8] to solve the Kronecker problem in the case that μ has two rows. This case is formidable and important because it contains as a special case the plethysm problem of decomposing a symmetric power of an $\mathfrak{sl}_2$-irreducible into irreducibles. This plethysm problem has been intensively studied since nineteenth-century invariant theory from algebraic, geometric, and combinatorial perspectives [16, 28, 45, 46, 50], yet no positive combinatorial formula for these coefficients is known.

To obtain a set of colored words suitable for the $\mu = (r-d, d)$ case of the Kronecker problem, replace w^{blft} with $w^{\mathrm{rev-\,blft}}$ in the definition of $\mathrm{CYW}_{\lambda,d}$, where $w^{\mathrm{rev-}}$ denotes the colored word obtained from w by reversing its subword of barred letters; denote the resulting set of words by $\mathrm{CYW}'_{\lambda,d}$. Define $\mathrm{CYT}'_{\lambda,d}$ to be the set of mixed insertion tableaux of the words in $\mathrm{CYW}'_{\lambda,d}$ using the order $<$. It is not hard to show that $g_{\lambda\,(r-d,d)\,\nu} = |\mathrm{CYT}'_{\lambda,d}(\nu)| - |\mathrm{CYT}'_{\lambda,d-1}(\nu)|$.

Obtaining this Kronecker coefficient as the difference in size of two sets of combinatorial objects is not new, but using mixed insertion in this way is. To obtain a positive formula for $g_{\lambda\,(r-d,d)\,\nu}$, we need an injection from $\mathrm{CYT}'_{\lambda,d-1}(\nu)$ to $\mathrm{CYT}'_{\lambda,d}(\nu)$, and this may be easier to find in this setup than others. So far I have obtained a conjecture for such an injection in the case λ and ν have two rows; it uses a more elaborate version of the parentheses matching rule described in §3.2. These initial investigations also suggest that this approach will connect to the Kronecker graphical calculus of [3] (see §3.7) and help to construct the conjectural basis $\check{B}^r$ discussed in §3.8.

3. Crystal bases for the Kronecker problem

Mulmuley, Sohoni, and I have been developing an approach to the Kronecker problem using quantum groups and crystal bases [3, 9, 4, 6] (canonical bases of quantum groups are also called *global crystal bases* and their $q = 0$ limits are *crystal bases*). In §3.1–3.2, we describe some of the background needed for this approach. This background also nicely illustrates the kind of interplay between algebra and combinatorics that is central to my research. We then highlight (§3.3–3.7) some of the results of [3] and describe three projects to extend this work (§3.8–3.10).

3.1. **Canonical bases connect quantum Schur-Weyl duality and RSK.** Let V and V_q be $\mathbb{Q}$- and $\mathbb{Q}(q)$-vector spaces, respectively, of dimension d_V. The canonical basis of $V_q^{\otimes r}$ beautifully connects quantum Schur-Weyl duality with the RSK correspondence.

Let $U_q(\mathfrak{g}_V)$ be the quantized enveloping algebra of $\mathfrak{g}_V := \mathfrak{gl}(V)$ and $\mathscr{H}_r$ the type A_{r-1} Hecke algebra over $\mathbb{Q}(q)$. The algebra $\mathscr{H}_r$ is a quantization of the group algebra $\mathbb{Q}\mathcal{S}_r$ and specializes to it at $q = 1$. The quantized enveloping algebra $U_q(\mathfrak{g}_V)$ is a deformation of the universal enveloping algebra $U(\mathfrak{g}_V)$ in the category of Hopf algebras. The most important properties of $U_q(\mathfrak{g}_V)$ for this work are (1) it specializes[1] to $U(\mathfrak{g}_V)$ at $q = 1$; (2) its representation theory is essentially the same as that of $U(\mathfrak{g}_V)$; (3) it is possible to define canonical bases of $U_q(\mathfrak{g}_V)$-modules (which cannot be done directly in the $q = 1$ limit) because "quantization removes degeneracy."

We let $V_{q,\lambda}$ (resp. $M_{q,\lambda}$) denote the irreducible $U_q(\mathfrak{g}_V)$-module (resp. $\mathscr{H}_r$-module) corresponding to $\lambda \vdash r$ and let V_λ (resp. M_λ) denote the corresponding $U(\mathfrak{g}_V)$-module (resp. $\mathbb{Q}\mathcal{S}_r$-module). Schur-Weyl duality generalizes nicely to the quantum setting:

Theorem 3.1 (Jimbo [20]). *As a $(U_q(\mathfrak{g}_V), \mathscr{H}_r)$-bimodule, $V_q^{\otimes r}$ decomposes into irreducibles as*

$$V_q^{\otimes r} \cong \bigoplus_{\lambda \vdash_{d_V} r} V_{q,\lambda} \otimes M_{q,\lambda}. \tag{2}$$

This algebraic decomposition has a combinatorial underpinning, which is the bijection

$$[d_V]^r \cong \bigsqcup_{\lambda \vdash_{d_V} r} \mathrm{SSYT}_{d_V}(\lambda) \times \mathrm{SYT}(\lambda), \ \mathbf{k} \mapsto (P(\mathbf{k}), Q(\mathbf{k})), \tag{3}$$

given by the RSK correspondence. These decompositions are unified by the *upper canonical basis* B_V^r of $V_q^{\otimes r}$, which has $U_q(\mathfrak{g}_V)$- and $\mathscr{H}_r$-cells corresponding to (2) and labels to (3) [6, 12]. This was first shown by Grojnowski and Lusztig in [17] using geometric methods.

3.2. **Graphical calculus for $U_q(\mathfrak{sl}_2)$-modules.** The graphical calculus for $U_q(\mathfrak{sl}_2)$-modules was developed by Kauffman-Lins [24], Kuperberg [29], Frenkel-Khovanov [14], and others. It allows for explicit computations in $U_q(\mathfrak{sl}_2)$ using topological arguments. Specifically, it associates to the element of B_V^r labeled by $\mathbf{k} \in [d_V]^r$ (assuming $d_V = 2$) the *diagram* obtained from $\mathbf{k}$ by pairing 2's and 1's as left and right parentheses and then drawing noncrossing arcs in the plane between matching pairs. These diagrams allow for explicit computations with the canonical basis. This $d_V = 2$ case is one of the only cases in which explicit computations of canonical basis elements are possible, so it is often the first case I explore when encountering a problem about canonical bases.

3.3. **Schur-Weyl duality for the Kronecker problem.** Let V_q, W_q be $\mathbb{Q}(q)$-vector spaces of dimensions d_V, d_W, respectively, considered as the natural representations of $U_q(\mathfrak{g}_V), U_q(\mathfrak{g}_W)$, and set $X_q = V_q \otimes W_q$. Let V, W and X be their $q = 1$ specializations. Schur-Weyl duality for $X^{\otimes r} = V^{\otimes r} \otimes W^{\otimes r}$ as a $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule and a $(U(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathbb{Q}\mathcal{S}_r \otimes \mathbb{Q}\mathcal{S}_r)$-bimodule yields

$$\bigoplus_{\nu} X_\nu \otimes M_\nu \cong X^{\otimes r} = V^{\otimes r} \otimes W^{\otimes r} \cong \bigoplus_{\lambda,\mu} V_\lambda \otimes W_\mu \otimes M_\lambda \otimes M_\mu. \tag{4}$$

This is a convenient setup for studying the Kronecker problem because the Kronecker coefficient $g_{\lambda\mu\nu}$ can be viewed simultaneously as the multiplicity of M_ν in $M_\lambda \otimes M_\mu$ and the multiplicity of $V_\lambda \otimes W_\mu$ in the restriction $\mathrm{Res}_{U(\mathfrak{g}_V \oplus \mathfrak{g}_W)} X_\nu$ of the $U(\mathfrak{g}_X)$-irreducible X_ν. Our main goal in [3] is to construct a basis of $X^{\otimes r}$ that is compatible with the decompositions in (4) and connects representation theory to combinatorics just as B_V^r does in the simpler setting of §3.1.

3.4. **The nonstandard quantum group and Hecke algebra.** In [3], we construct two quantum objects, the nonstandard Hecke algebra $\check{\mathscr{H}}_r$ and the nonstandard coordinate algebra $\mathscr{O}(GL_q(\check{X}))$, (we write $\check{X}$ in place of X_q when it is associated to a nonstandard object) that are compatible with the $(U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W), \mathscr{H}_r \otimes \mathscr{H}_r)$-bimodule structure on $\check{X}^{\otimes r}$ and quantize the $(U(\mathfrak{g}_X), \mathbb{Q}\mathcal{S}_r)$-bimodule structure on $\check{X}^{\otimes r}$ in a certain sense. We show that $\mathscr{O}(GL_q(\check{X}))$ is a cosemisimple Hopf algebra, that $\check{\mathscr{H}}_r$ is semisimple, and that a nonstandard analog of quantum Schur-Weyl duality holds.

[1] This can be made precise by defining an *integral form* $U_q(\mathfrak{g}_V)_{\mathbf{A}}$ of $U_q(\mathfrak{g}_V)$, which is a $\mathbb{Z}[q, q^{-1}]$-module such that $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} U_q(\mathfrak{g}_V)_{\mathbf{A}} = U_q(\mathfrak{g}_V)$.

These constructions are not straightforward because the Hecke algebra is not a Hopf algebra in any natural way and the homomorphism $U(\mathfrak{g}_V \oplus \mathfrak{g}_W) \to U(\mathfrak{g}_X)$ cannot be quantized in the category of Drinfel′d-Jimbo quantum groups [19]. Essentially because of these difficulties, $\check{\mathscr{H}}_r$ and $\mathscr{O}(GL_q(\check{X}))$ turn out to be quite complicated and have not been as helpful for the Kronecker problem as was originally hoped. They do, however, add *some* structure to the problem of finding a nice basis compatible with the decompositions in (4). They have also proved to be interesting algebraic objects in their own right. For example, Chebyshev polynomials evaluated at $\frac{1}{q+q^{-1}}$ appear rather mysteriously in the defining relations of a family of algebras generalizing $\check{\mathscr{H}}_3$ [4].

3.5. **Constructing $\check{X}_\nu$ in the two-row case.** Our strongest results are for the case $d_V = d_W = 2$, which corresponds to those Kronecker coefficients $g_{\lambda\mu\nu}$ for which λ and μ have at most two rows; we refer to this as *the two-row case*. In the two-row case, we construct a $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$-module $\check{X}_\nu$ that specializes to $\mathrm{Res}_{U(\mathfrak{g}_V \oplus \mathfrak{g}_W)} X_\nu$ at $q = 1$. Our construction of $\check{X}_\nu$ can be viewed as a quantum version of the robust characteristic-free definition of Schur modules due to Akin-Buchsbaum-Weyman [1] (see [51, §2.1]). In this construction, X_ν is realized as the quotient of the tensor product of exterior powers of X by certain "straightening-relations" in every two consecutive columns of ν.

Quantizing this construction and showing that it has the correct $q = 1$ specialization is not easy (in fact, it fails outside the two-row case—see §3.9), and here the additional structure of the nonstandard coordinate algebra is quite helpful. We define $\check{X}_\nu$ as a quotient of

$$\check{Y}_{\nu'} := \check{\Lambda}^{\nu'_1}\check{X} \otimes \check{\Lambda}^{\nu'_2}\check{X} \otimes \ldots \otimes \check{\Lambda}^{\nu'_l}\check{X},$$

where $\check{\Lambda}(\check{X}) = \bigoplus_r \check{\Lambda}^r \check{X}$ is an $\mathscr{O}(GL_q(\check{X}))$-comodule, called the *nonstandard exterior algebra*, which specializes to $\Lambda(X)$ at $q = 1$. Determining the analog of the straightening relations in this setting is aided by the additional structure of the nonstandard coordinate algebra.

3.6. **A global crystal basis for $\check{X}_\nu$.** We now explain a deep result in crystal basis theory obtained in this approach. To define a global crystal basis of $\check{X}_\nu$, we first define a global crystal basis of $\check{\Lambda}^r\check{X}$, whose elements are labeled by what we call *nonstandard columns of height* r (NSC^r). There are $\binom{4}{r}$ nonstandard columns of height r, several of which appear in Figure 1. We then define a canonical basis $\mathrm{NST}(\nu')$ of $\check{Y}_{\nu'}$ by putting the bases $\mathrm{NSC}^{\nu'_i}$ together using Lusztig's construction for tensoring based modules [38]. Here, NST stands for *nonstandard tabloid*, which is just a sequence of nonstandard columns. We identify a certain subset of (a rescaled version of) $\mathrm{NST}(\nu')$ such that its image $\mathrm{HNSTC}(\nu)$ in $\check{X}_\nu$ is a basis (HNSTC stands for *honest nonstandard tabloid class*).

Theorem 3.2 ([3])**.** *The set $HNSTC(\nu)$ is a global crystal basis of $\check{X}_\nu$ that solves the two-row Kronecker problem: the number of highest weight elements of $HNSTC(\nu)$ of weight (λ, μ) is the Kronecker coefficient $g_{\lambda\mu\nu}$.*

The deepest part of this theorem from the point of view of crystal basis theory is the rescaling of the NST. Since $\check{X}_\nu$ is a $U_q(\mathfrak{g}_V)$ ($\cong U_q(\mathfrak{gl}_2)$) module, each element of $\mathrm{NST}(\nu')$ has a graphical description (see §3.2) called its *diagram of V-arcs*. Similarly, the $U_q(\mathfrak{g}_W)$-module structure yields a *diagram of W-arcs*. To each NST, we associate the *diagram* in which its V-arcs and W-arcs are drawn as $\frown$ and $\smile$, respectively. The *degree* of T, denoted $\deg(T)$, is the number of closed loops in this diagram. The rescaled version of T is then $(\frac{-1}{q+q^{-1}})^{\deg(T)}T$. Counting loops is a familiar operation from the $U_q(\mathfrak{sl}_2)$ graphical calculus, however its appearance here and the integral form of the rescaled basis that results are genuinely new constructions in the theory of crystal bases.

3.7. **Explicit formulae for two-row Kronecker coefficients.** The action of $U_q(\mathfrak{g}_V \oplus \mathfrak{g}_W)$ on HNSTC is encoded by a *crystal graph*. NST diagrams make it is easy to determine the components of this graph and also help organize and count these components. Degree for NST gives rise to a well-defined notion of degree for these components. We show that the degree-zero components can be grouped into eight different one-parameter families depending on the heights of the columns that the arcs connect (see Figure 1), and counting components easily reduces to the degree-zero case. We hope this to be the beginnings of a new kind of *Kronecker graphical calculus* (see §3.8).

Formulae for two-row Kronecker coefficients are given in [47, 48, 10]. Although these are quite explicit, none is obviously positive. Theorem 3.2 and the Kronecker graphical calculus yield a fairly simple, positive formula for two-row Kronecker coefficients. From it, we easily recover the nice formulae for certain two-row Kronecker coefficients from [11] as well as the exact conditions for two-row Kronecker coefficients to vanish, from [10].

Our approach also yields a formula for the *symmetric* and *exterior Kronecker coefficients*, defined to be the multiplicities of M_ν in S^2M_λ, $\Lambda^2 M_\lambda$, respectively, and denoted $g_{+1\,\lambda\nu}$, $g_{-1\,\lambda\nu}$ (thus $g_{\lambda\lambda\nu} = g_{+1\,\lambda\nu} + g_{-1\,\lambda\nu}$).

Theorem 3.3 ([3]). *The symmetric and exterior Kronecker coefficients are given by equating coefficients of the following generating functions in x:*

$$\sum_{\lambda \vdash_2 r} g_{\varepsilon\lambda\nu} x^{\lambda_1-\lambda_2} = \begin{cases} [\![n_1]\!][\![n_2]\!][\![n_3]\!] & \text{if} \quad (-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-1]\!][\![n_3]\!]x & \text{if} \ -(-1)^{n_2} = \quad (-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1]\!][\![n_2-1]\!][\![n_3-1]\!]x & \text{if} \ -(-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1, \\ [\![n_1-1]\!][\![n_2-2]\!][\![n_3-1]\!]x^2 & \text{if} \quad (-1)^{n_2} = -(-1)^{n_3+n_4}\varepsilon = 1. \end{cases}$$

Here $\varepsilon \in \{-1, 1\}$, ν is any partition of r with at most 4 parts, n_i is the number of columns of height i in the diagram of ν, and $[\![2k]\!] := x^{2k} + x^{2k-2} + \cdots + x^0$, $[\![2k-1]\!] := x^{2k-1} + x^{2k-3} + \cdots + x^1$.

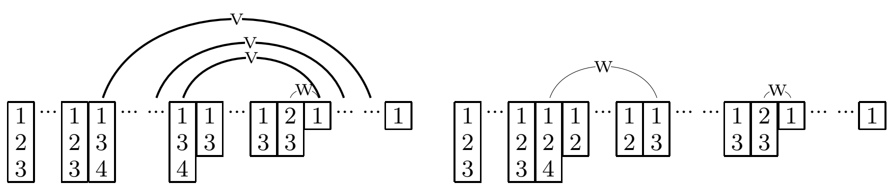

Figure 1: Representatives of two of the eight one-parameter families of degree-zero highest weight HNSTC. Two sets of dots indicates that there is at least one column of that type. The V- and W-arcs correspond to the graphical calculus for $U_q(\mathfrak{g}_V)$ and $U_q(\mathfrak{g}_W)$, respectively.

3.8. **A canonical basis of $\check{X}^{\otimes r}$.** We succeeded in constructing a canonical basis HNSTC(ν) of $\check{X}_\nu$ (§3.6) in the two-row case, but this is only the beginning of a theory of canonical bases for nonstandard objects. In [3] we conjecture the existence of a canonical basis $\check{B}^r$ of $\check{X}^{\otimes r}$ which contains the bases HNSTC(ν) as cellular subquotients (and satisfies several other conditions). We have succeeded in constructing such a basis for $r \leq 4$, and partially for $r = 5, 6, 7, 8$. We also have a promising candidate for the integral form $\mathbb{Z}[q, q^{-1}]\check{B}^r$ (which is easier to find than $\check{B}^r$ itself). Indeed, we can prove, using diagrammatic calculations involving the definition of degree (§3.6), that this integral form is part of a balanced triple in the sense of Kashiwara [22, 23].

This basis could yield a more complete Kronecker graphical calculus. Ideally, we would have a graphical description of each element of $\check{B}^r$, generalizing what we currently have for HNSTC(ν), and a graphical description of the action of the nonstandard Hecke algebra on $\check{B}_r$. Additionally, our computations suggest that this basis will yield bases for certain M_ν similar to the Kazhdan-Lusztig [25] and web bases [29]; this could yield insight into the difference between these two bases [26].

3.9. **Beyond the two-row case.** To advance our work in [3], we have begun to adapt constructions from multilinear algebra like symmetric and exterior bialgebras, Schur functors, and Schur complexes (as explained, for instance, in [51]) to the nonstandard setting. This is delicate as many definitions that work in the standard setting do not carry over.

For general d_V, d_W, $\check{X}_\nu$ can be constructed in a similar way to that described in §3.5. However, its $q = 1$ specialization is typically too small. In the case ν has two columns and in the case ν is a hook shape, we can show that $\check{X}_\nu$ does have the correct $q = 1$ specialization; the proof of this in the hook case involves an explicit calculation of the straightening maps in terms of canonical bases. The definition of NST carries over to these cases as well. The difficulty in mimicking Theorem 3.2 is determining the image of the NST in $\check{X}_\nu$. In the case ν is a hook shape, $d_W = 2$, d_V arbitrary, we conjecture that $\mathrm{NST}(\nu')$ determines a unique (up to a diagonal transformation) basis of $\check{X}_\nu$. This conjecture has led us to a conjecture about Kazhdan-Lusztig polynomials that needs to be proved first. Another part of extending the work of [3] to these cases is to generalize the definition of degree. In the special case $\nu = (r)$, we have a candidate definition of degree which may be related to Lusztig's a-invariant [36, 37].

3.10. **Tensor rank.** The *tensor decomposition problem* asks for a decomposition of a tensor into a sum of as few as possible simple tensors. This is an important problem in statistics and signal processing and has applications in fluorescence spectroscopy, computer vision, blind source separation, and fast matrix multiplication [30]. It has also been well studied from the algebro-geometric perspective [32, 31, 30], where it translates to the problem of finding equations for the k-th secant variety $\sigma_k(\mathrm{Seg})$ of the triple Segre variety $\mathrm{Seg} \subseteq \mathbb{P}(U \otimes V \otimes W)$. Kronecker coefficients appear in the decomposition of the homogeneous coordinate ring of $\mathbb{P}(U \otimes V \otimes W)$ into $G := GL(U) \times GL(V) \times GL(W)$-irreducibles. We are hopeful that our basis-theoretic approach to the Kronecker problem will help determine G-equivariant equations for the secant varieties $\sigma_k(\mathrm{Seg})$. As a first step in this direction, Luke Oeding and I plan to use results of [3] to give an explicit presentation for the subring of covariants of the homogenous coordinate ring of $\mathbb{P}(\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$.

4. Complexity theory and positivity

Complexity theory provides a new and intriguing motivation for positivity problems in algebraic combinatorics. A positive combinatorial interpretation of some quantity that is already known to be nonnegative is not *just* a more elementary proof of nonnegativity. It is also at the heart of efficient computation, though precisely how is somewhat subtle.

Computing Littlewood-Richardson coefficients is known to be #**P**-complete [13]. The class #**P** is a counting version of **NP**, so being complete for #**P** means that computing these coefficients is expected to be hard. However, deciding nonvanishing of Littlewood-Richardson coefficients can be done in polynomial time [44] by the saturation theorem of Knutson and Tao [27]. The *permanent vs. determinant problem*, which has been the main focus of GCT so far, is to show that the permanent of an $n \times n$ variable matrix cannot be expressed as the determinant of an $m \times m$ matrix with constant or single variable entries if $m = O(2^{\mathrm{polylog}\, n})$. Representation theoretic questions arising in the permanent vs. determinant problem, together with the Knutson-Tao result and the Flip Theorem [40], is the basis for a philosophy in GCT called the *flip* [43, 40, 41]. The flip suggests that separating **P** from **NP** will require solving a problem in representation theory in polynomial time, of the same flavor as deciding nonvanishing of Littlewood-Richardson coefficients, but much harder. As part of the flip, Mulmuley hypothesizes

Hypothesis 4.1 (Kronecker Flip Hypothesis (Kronecker FH) [41])**.**

(1) *There is a #**P**-formula for the Kronecker coefficients* $g_{\lambda\mu\nu}$*.*

(2) *There is a polynomial time algorithm to decide nonvanishing of Kronecker coefficients.*

The hypothesis (1) is a precise complexity-theoretic version of the problem of finding a positive combinatorial formula for Kronecker coefficients. Computing Kronecker coefficients is known to be in Gap**P**, meaning that Kronecker coefficients can be expressed as the difference of two #**P** quantities. The complexity classes Gap**P** and #**P** are both counting analogs of the class **NP**, but the distinction becomes important in light of Kronecker FH (2). It is expected that (1) is easier than (2) and needs to be solved first. Roughly, this is because if a quantity is only known as the difference of two quantities that are hard to compute, then deciding if their difference vanishes is as hard as computing each one.

I believe that Kronecker FH and related hypotheses in [41] are important guides to uncovering a deep connection between complexity theory and algebraic combinatorics. Therefore, in addition to investigating Kronecker FH (1) which has been my focus, I also hope to

Problem 4.2. (1) *Investigate Kronecker FH (2) in the case that μ is a hook shape using the Hook Kronecker Rule (see §2.2).*

(2) *Determine whether the nonvanishing of the Littlewood-Richardson coefficients $C^{\nu}_{\lambda\,\mu}$ in other types can be decided in polynomial time. In [42], it is noted that a polynomial time algorithm would follow from a conjecture of De Loera and McAllister stating that the associated stretching quasi-polynomials $C(n) := C^{n\nu}_{n\lambda\,n\mu}$ have nonnegative coefficients.*

(3) *Let G be a connected complex linear algebraic group acting irreducibly on a vector space Y with finitely many orbits. (Such pairs are classified in [21].) Determine the positivity indices of the stretching quasi-polynomials associated to the G-module $\mathbb{C}[Y]$, where the* positivity index *of a quasi-polynomial $f(n)$ is defined to be the smallest integer i such that $f(n+i)$ has nonnegative coefficients [41].*

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